Lower Bounds for Eigenvalues of Elliptic Operators by Overlapping Domain Decomposition

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Summary. In this paper, we consider a new approach to estimation from below of the lowest eigenvalues of symmetric positive definite elliptic operators. The approach is based on the overlapping domain decomposition procedure and on the replacement of subdomain operators by special low rank perturbed scalar operators. The algorithm is illustrated by applications to model problems with mixed boundary conditions and strongly discontinuous coefficients.

1 Introduction

In this paper, we propose a new approach for estimations from below of the lowest eigenvalues of a symmetric elliptic operator

$$\mathcal{L} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} + c. \tag{1}$$

Here, $a = (a_{ij})$ is a symmetric uniformly positive definite $d \times d$ matrix with piecewise smooth bounded entries a_{ij}, $i, j = \overline{1,\ d}$, c is a nonnegative piecewise smooth bounded function, and $d = 2, 3$. Without loss of generality, we assume that the matrix $a = a(x)$ and the coefficient $c = c(x)$, $x \in \mathbb{R}^d$ are piecewise constant.

We consider the eigenvalue problem

$$\mathcal{L}w = \lambda w \tag{2}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ with the boundary $\partial\Omega$ subject to the boundary conditions

$$\begin{aligned} w &= 0 \quad \text{on } \Gamma_D, \\ \mathbf{u} \cdot \mathbf{n} - \sigma w &= 0 \quad \text{on } \Gamma_R, \end{aligned} \tag{3}$$

where $\mathbf{u} = -a\nabla w$ is the flux vector-function, $\Gamma_D = \overline{\Gamma}_D$ is a Dirichlet part of $\partial\Omega$, Γ_R is a Robin part of $\partial\Omega$, $\sigma = \sigma(x)$, $x \in \Gamma_R$, is a nonnegative piecewise constant function, and $\mathbf{n}$ is the outward unit normal to Γ_R. In the case $\sigma \equiv 0$ the Robin boundary condition becomes the Neumann boundary condition. We assume that $\Gamma_D \cup \overline{\Gamma}_R = \partial\Omega$. For the sake of simplicity, we assume that Ω is either a polygon ($d = 2$), or a polyhedron ($d = 3$).

It is well known that all the eigenvalues λ in (2), (3) are real, nonnegative, and the lowest eigenvalue λ_1 is the solution of the minimization problem

$$\lambda_1 = \inf_{v \in V, ||v||_2 = 1} \Phi(v). \tag{4}$$

Here,

$$\Phi(v) = \int_\Omega \Big[\sum_{i,j=1}^{d} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} + cv^2 \Big] \,\mathrm{dx} + \int_{\Gamma_R} \sigma v^2 \,\mathrm{ds}. \tag{5}$$

and

$$V = \{ v : v \in H^1,\ v = 0 \text{ on } \Gamma_D \}, \tag{6}$$

where $H^1 \equiv H^1(\Omega)$ is the Sobolev space and $\|v\|_2$ is the $L_2(\Omega)$ norm of v.

In the only case $\sigma \equiv 0$ on Γ_R, $\overline{\Gamma}_R = \partial\Omega$, and $c \equiv 0$ in Ω (the Neumann problem) the minimal eigenvalue λ_1 is equal to zero. Otherwise, λ_1 is positive. In any case, λ_1 is a single eigenvalue, and an eigenfunction $w_1 = w_1(x)$ does not change its sign in Ω (for instance, $w_1(x) > 0$ for all $x \in \Omega$). For the Neumann problem, we denote the minimal nonzero eigenvalue by λ_2. This eigenvalue may be multiple.

Estimations from above for the minimal (or the minimal nonzero) eigenvalue in (2), (3) can be obtained by the Ritz method, in particular, by using the P_1 finite element method. In many practical applications, estimations from below are much more important. In particular situations (see [1]), the estimates from below can be obtained by using the finite difference discretization of (2), (3). Another method is described in [4]. The latter method is rather limited and very complicated for implementation.

In this paper, we propose a new method to derive estimations from below for the minimal eigenvalue λ_1 (minimal nonzero eigenvalue λ_2) in (2), (3). The method is based on a partitioning of the domain Ω into simpler shaped subdomains. We assume that we are able to derive explicit estimates from below of the lowest eigenvalues of the eigenvalue problems in subdomains. The accuracy of the estimates depends on a partitioning into subdomains. Thus, the method does not always provide sufficiently reliable (or practically acceptable) estimates from below of the lowest eigenvalues.

The paper is organized as follows. In Section 2, we describe the new method on the functional level. The finite element justification of the method is given in Section 3.

2 Description of the Method

Let Ω be partitioned into $m \geq 1$ polygonal, $d = 2$ (polyhedral, $d = 3$), open overlapping subdomains Ω_k, $k = \overline{1, m}$, i.e. $\Omega = \bigcup_{k=1}^{m} \Omega_k$. We define m quadratic functionals

$$\Phi_k(v) = \int_\Omega \sum_{i,j=1}^{d} a_{ij}^{(k)} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\,\mathrm{dx} + \int_{\Gamma_R} \sigma^{(k)} v^2\,\mathrm{ds} \tag{7}$$

where $a^{(k)} = (a_{ij}^{(k)})$ are symmetric $d \times d$ matrices with piecewise constant entries $a_{ij}^{(k)}$, $i, j = \overline{1,\, d}$, $\sigma^{(k)}$ are nonnegative piecewise constant functions defined on Γ_R, and $v \in V$. We assume that the matrices $a^{(k)}$ are positive definite in Ω_k and $a^{(k)} = 0$ in $\Omega \setminus \overline{\Omega}_k$, and that the functions $\sigma^{(k)}$ are zero on $\Gamma_R \setminus \partial\Omega_k$, $k = \overline{1,\, m}$. To this end, the formulae

$$\Phi_k(v) = \int_{\Omega_k} \sum_{i,j=1}^{d} a_{ij}^{(k)} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\,\mathrm{dx} + \int_{\Gamma_{R,k}} \sigma^{(k)} v^2\,\mathrm{ds}, \tag{8}$$

where $\Gamma_{R,k} = \Gamma_R \cap \partial\Omega_k$, gives an alternative definition for $\Phi_k(v)$, $k = \overline{1,\, m}$.

We assume that

$$a = \sum_{k=1}^{m} a^{(k)} \quad \text{in } \Omega \tag{9}$$

$$\sigma = \sum_{k=1}^{m} \sigma_k \quad \text{on } \Gamma_R. \tag{10}$$

Then, under the latter assumptions we get

$$\Phi(v) = \sum_{k=1}^{m} \Phi_k(v) + \int_\Omega c v^2\,\mathrm{dx}. \tag{11}$$

Let us consider the eigenvalue problems

$$\begin{aligned} \mathcal{L}_k w &= \mu w && \text{in } \Omega_k, \\ w &= 0 && \text{on } \Gamma_D \cap \partial\Omega_k, \\ \mathbf{u}^{(k)} \cdot \mathbf{n}_k &= 0 && \text{on } \partial\Omega_k \setminus \partial\Omega, \\ \mathbf{u}^{(k)} \cdot \mathbf{n}_k - \sigma^{(k)} w &= 0 && \text{on } \Gamma_{R,k}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \mathcal{L}_k &= -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} a_{ij}^{(k)} \frac{\partial}{\partial x_j} \\ \mathbf{u}^{(k)} &= -a^{(k)} \nabla w, \end{aligned} \tag{13}$$

and $\mathbf{n}_k$ is the outward unit normal to $\partial\Omega_k$, $k = \overline{1,\, m}$.

We partition the subdomains Ω_k, $k = \overline{1,\, m}$, into two groups. For the subdomains Ω_k in the first group we assume that $\Gamma_D \cap \partial\Omega_k = \emptyset$ and $\sigma^{(k)} = 0$ on $\Gamma_{R,k}$ (or $\Gamma_{R,k} = \emptyset$), $1 \le k \le m$. For the subdomains Ω_k in the first group, the minimal eigenvalue $\mu_0^{(k)}$ in (12) is equal to zero and

$$w_0 = \frac{1}{|\Omega_k|^{1/2}} \quad \text{in } \Omega_k, \tag{14}$$

where $|\Omega_k|$ is the area of Ω_k, $d = 2$ (volume of Ω_k, $d = 3$), is the corresponding L_2-normalized positive eigenfunction. We denote the minimal (lowest) nonzero eigenvalue in (12) by $\mu_1^{(k)}$, $1 \le k \le m$.

All other subdomains Ω_k, $1 \le k \le m$, we put into the second group. For a subdomain Ω_k in the second group the minimal eigenvalue in (12) is positive. We denote this eigenvalue also by $\mu_1^{(k)}$, $1 \le k \le m$.

It is obvious (see [3]) that for any subdomain Ω_k in the first group the inequality

$$\Phi_k(v) \ge \mu_1^{(k)} (P_k v,\ v) \equiv \mu_1^{(k)} \int_\Omega (P_k v)\, v \,\mathrm{dx} \tag{15}$$

holds for any $v \in V$ where the operator P_k is defined by

$$(P_k v)(x) = \begin{cases} 0, & x \in \Omega \setminus \overline{\Omega}_k, \\ v(x) - \frac{1}{|\Omega_k|} \int_{\Omega_k} v(x')\,\mathrm{dx}', & x \in \Omega_k. \end{cases} \tag{16}$$

For the subdomains Ω_k in the second group the inequality (15) also holds with the operator P_k defined by

$$(P_k v)(x) = \begin{cases} 0, & x \in \Omega \setminus \overline{\Omega}_k, \\ v(x), & x \in \Omega_k. \end{cases} \tag{17}$$

In both cases, the operator P_k is an orthogonal L_2-projector, i.e. $P_k = P_k^*$ and $P_k^2 = P_k$.

Let us assume that we have a set of positive numbers μ_k which estimate from below the eigenvalues $\mu_1^{(k)}$ in (15), i.e. $\mu_1^{(k)} \ge \mu_k > 0$, $k = \overline{1,\ m}$, and define the operator

$$P = \sum_{k=1}^{m} \mu_k P_k. \tag{18}$$

By the definition,

$$\Phi(v) \ge (P v,\ v) + (cv,\ v) \quad \text{for all } v \in V. \tag{19}$$

Thus, in the case of a positive definite operator $\mathcal{L}$ we obtain

$$\lambda_1 = \min_{v \in V, ||v||_2 = 1} \Phi(v) \ge \nu_1 = \min_{v \in L_2, ||v||_2 = 1} [(P v,\ v) + (cv,\ v)] \tag{20}$$

where ν_1 is the minimal eigenvalue of the eigenvalue problem

$$P v + cv = \nu v \quad \text{in } \Omega. \tag{21}$$

In the case of the Neumann problem, the minimal nonzero eigenvalue λ_2 in (2), (3) is estimated from below by the minimal nonzero eigenvalue ν_2 in (21), i.e.

$$\lambda_2 \geq \nu_2 = \min_{v \in L_2, \|v\|_2 = 1, (v,1)=0} (Pv,\ v). \tag{22}$$

The set $\widehat{\Gamma} = \bigcup_{k=1}^{m} \partial\Omega_k$ partitions Ω into $\widehat{n}$ polygonal, $d = 2$ (polyhedral, $d = 3$), subdomains $\widehat{G}_s$, $s = \overline{1,\ \widehat{n}}$. We impose an additional partitioning of Ω into n, $n \geq \widehat{n}$, nonoverlapping subdomains G_s with boundaries ∂G_s, $s = \overline{1,\ n}$, such that the set $\widehat{\Gamma}$ belongs to the set $\Gamma = \bigcup_{s=1}^{n} \partial G_s$. We assume that each of the subdomains G_s, $s = \overline{1,\ n}$, is simply connected and does not coincide with any of the subdomains Ω_k, $k = \overline{1,\ m}$. We also assume that the coefficient $c = c(x)$ is constant in each of the subdomains G_s, $s = \overline{1,\ n}$.

Define the set of orthogonal projectors Q_k by

$$(Q_k v)(x) = \begin{cases} 0, & x \in \Omega \setminus \overline{G}_k, \\ \frac{1}{|G_k|} \int_{G_k} v(x')\,\mathrm{d}x', & x \in G_k, \end{cases} \tag{23}$$

where $v \in L_2(\Omega)$. Then, the mean values v_s in G_s of a function $v \in L_2(\Omega)$ are defined by

$$v_s = Q_s v, \quad s = \overline{1,\ n}. \tag{24}$$

Assume that a subdomain Ω_k, $1 \leq k \leq m$, belongs to the first group and

$$\overline{\Omega}_k = \bigcup_{s=1}^{t} \overline{G}_s. \tag{25}$$

Then, it is obvious that

$$P_k v = v - \frac{1}{|\Omega_k|} \sum_{s=1}^{t} |G_s|\, v_s, \tag{26}$$

where $v_s = Q_s v$, $s = \overline{1,\ t}$.

Let ν be an eigenvalue in (21) and W be the set of all the eigenfunctions corresponding to this eigenvalue. A simple analysis of equations (21) in subdomains G_s, $s = \overline{1,\ n}$, shows that W always contains a function which is a constant in each of the subdomains G_s, $s = \overline{1,\ n}$.

It follows that in (20) and (21) we can replace the space L_2 by the space V_h of functions which are constant in each of the subdomains G_s, $s = \overline{1,\ n}$, i.e. the definitions of ν_1 in (20) and ν_2 in (22) can be replaced by

$$\nu_1 = \min_{v \in V_h,\ \|v\|_2 = 1} [(Pv,\ v) + (cv,\ v)], \tag{27}$$

$$\nu_2 = \min_{v \in V_h,\ \|v\|_2 = 1,\ (v,\ 1)=0} (Pv,\ v), \tag{28}$$

respectively.

The variational problems (27) and (28) result in the algebraic eigenvalue problems

$$K\bar{w} = \nu M \bar{w}, \quad \bar{w} \in \mathbb{R}^n, \tag{29}$$

with the diagonal $n \times n$ matrix

$$M = \mathrm{diag}\{|G_1|,\ \dots\ ,\ |G_n|\}\ .$$

The matrix K for problem (27) is symmetric and positive definite. The matrix K for problem (28) is symmetric and positive semidefinite with the explicitly known one-dimensional null-space.

Remark 1. The replacement of the space V by the space $L_2(\Omega)$ in (20) and (22) can be justified by using the convergence results for the P_1 finite element method for eigenvalue problem (4)–(6) on quasiuniform regular shaped triangular mesh/tetrahedral meshes. To prove the latter statement, we have to apply the proposed method to the P_1 discretization of (2)–(3) on the meshes which are conforming with respect to the partitioning of Ω into subdomains G_s, $s = \overline{1,\ n}$.

Remark 2. The requirement $\Omega = \bigcup_{k=1}^{m} \Omega_k$ in the beginning of Section 2 can be replaced by the following weaker requirement. Namely, we may require that each two points in Ω should be connected by a curve γ in $\bigcup_{k=1}^{m} \Omega_k$. For instance, the partitioning of the unit square $\Omega = (0;\ 1) \times (0;\ 1)$ into rectangles $\Omega_1 = (0;\ 0.5) \times (0;\ 1)$, $\Omega_2 = (0.5;\ 1) \times (0;\ 1)$, and $\Omega_3 = (0;\ 1) \times (0;\ 0.5)$ is admissible (see Example 2 in the next section).

3 Two Simple Examples

Example 1. Let Ω be the unit square and ω be a simply connected subdomain in Ω. We denote by δ the area of ω and assume that $\mathcal{L} = -\triangle + c$ where $\triangle$ denotes the Laplace operator, $\partial\Omega = \overline{\Gamma}_N$ and the coefficient c in (1) equals to a positive constant c_ω in ω and zero in $\Omega \setminus \overline{\omega}$. We choose $m = 1$, i.e. $\Omega_1 = \Omega$, and partition Ω into subdomains $G_1 = \omega$ and $G_2 = \Omega \setminus \overline{\omega}$, i.e. $n = 2$. Applying the algorithm described in the previous section with $\mu^{(1)} = \pi^2$ we get $K = M\widehat{K}$ where

$$\widehat{K} = \begin{pmatrix} \pi^2(1-\delta) + c_\omega & -1+\delta \\ -\delta & \delta \end{pmatrix} \tag{30}$$

and $M = \mathrm{diag}\{\delta;\ 1-\delta\}$. Computing the minimal eigenvalue ν_1 of (20), we get the estimate

$$\lambda_1 \geq \nu_1 > \frac{c_\omega \delta}{2(\pi^2 + c_\omega)}\ . \tag{31}$$

Example 2. Let Ω be the unit square partitioned into three subdomains $\Omega_1 = (0;\ 1) \times (0;\ \delta)$, $\Omega_2 = (\delta;\ 1) \times (0;\ 1)$, and $\Omega_3 = (0;\ 1) \times (\delta;\ 1)$ as shown in Fig. 1 where $\delta \in (0;\ 1)$. We assume that $\mathcal{L} = -\triangle$ and $\Gamma_D = \{(x_1, x_2)\ :\ x_1 = 0,\ x_2 \in (0;\ \delta)\}$. In Figure 1, we show the partitioning of Ω into rectangles G_i, $i = \overline{1,\ 4}$.

We define the operators $\mathcal{L}_k$ by setting $a^{(k)} = a_k I_2$, $k = 1, 2, 3$. Here, I_2 denotes the identity 2×2 matrix and the functions a_k, $k = 1, 2, 3$, are defined as follows:

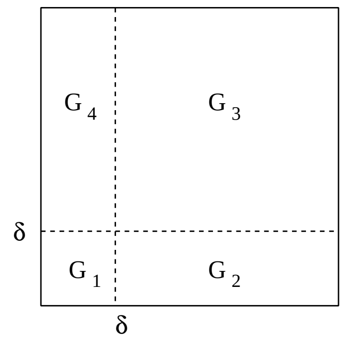

Fig. 1. Partitioning of Ω into rectangles G_i, $i = \overline{1,\,4}$

$$a_1 = \begin{cases} 1 & \text{in } G_1, \\ 0.5 & \text{in } G_2, \end{cases}$$
$$a_2 = 0.5 \quad \text{in } G_2 \bigcup G_3, \tag{32}$$
$$a_3 = \begin{cases} 1 & \text{in } G_3, \\ 0.5 & \text{in } G_4. \end{cases}$$

Applying the algorithm described in the previous section with $\mu^{(1)} = \mu^{(2)} = \mu^{(3)} = \pi^2/2$, we get $K = M\widehat{K}$ where

$$\widehat{K} = \frac{\pi^2}{2} \begin{pmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 2-\delta & -1+\delta & 0 \\ 0 & -\delta & 2\delta & -\delta \\ 0 & 0 & -1+\delta & 1-\delta \end{pmatrix}. \tag{33}$$

By using the straightforward calculations, we derive the estimate

$$\lambda_1 \geq \nu_1 \geq \|\widehat{K}\|_\infty^{-1} = \frac{\pi^2}{2} \cdot \frac{\delta(1-\delta)(2-\delta)}{(1+\delta)(3-\delta)} \,. \tag{34}$$

Thus, in the case $\delta \ll 1$ we get the asymptotic estimate

$$\lambda_1 \geq \frac{\pi^2}{3}\delta \,. \tag{35}$$

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