A Domain Decomposition Method Based on Augmented Lagrangian with a Penalty Term

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Summary. An iterative substructuring method with Lagrange multipliers is considered for the second order elliptic problem, which is a variant of the FETI-DP method. The standard FETI-DP formulation is associated with a saddle-point problem which is induced from the minimization problem with a constraint for imposing the continuity across the interface. Starting from the slightly changed saddle-point problem by addition of a penalty term with a positive penalization parameter η, we propose a dual substructuring method which is implemented iteratively by the conjugate gradient method. In spite of the absence of any preconditioners, it is shown that the proposed method is numerically scalable in the sense that for a large value of η, the condition number of the resultant dual problem is bounded by a constant independent of both the subdomain size H and the mesh size h. We discuss computational issues and present numerical results.

1 Introduction

Let us consider the following Poisson model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$ and f is a given function in $L^2(\Omega)$. For simplicity, we assume that Ω is partitioned into two nonoverlapping subdomains $\{\Omega_i\}_{i=1}^2$ such that $\overline{\Omega} = \bigcup_{i=1}^2 \overline{\Omega}_i$. It is well-known that problem (1) is equivalent to the constrained minimization

$$\min_{\substack{v_i \in H^1(\Omega_i) \\ v_i = 0 \text{ on } \partial\Omega\cap\partial\Omega_i \\ v_1 = v_2 \text{ on } \partial\Omega_1\cap\partial\Omega_2}} \sum_{i=1}^{2} \left(\frac{1}{2}\int_{\Omega_i} |\nabla v_i|^2\, dx - \int_{\Omega_i} f v_i\, dx \right). \tag{2}$$

In a domain-decomposition approach, a key point is how to convert the constrained minimization problem (2) into an unconstrained one. The most popular methods, developed for different purposes are the Lagrangian method, the method of penalty functions, and the augmented Lagrangian method. Such various ideas have been introduced for handling constraints as the continuity across the interface in (2) (see [4, 6, 8]). The FETI-DP method is one of the most advanced dual substructuring methods, which introduces Lagrange multipliers to enforce the continuity constraint by following the Lagrangian method. In this paper, we propose a dual iterative substructuring algorithm which deals with the continuity constraint across the interface using the augmented Lagrangian method. Many studies of the augmented Lagrangian method have been done in the frame of domain-decomposition techniques which belong to families of nonoverlapping Schwarz alternating methods, variants of FETI method, etc. (cf. [1, 3, 8, 11])

This paper is organized as follows. In Section 2, we introduce a saddle-point formulation for an augmented Lagrangian with a penalty term. Section 3 provides a dual iterative substructuring method and presents algebraic condition number estimates. In Section 4, we mainly deal with computational issues in view of implementation of the proposed method and show the numerical results. For details omitted here due to space restrictions, we refer the reader to [9].

2 Saddle-Point Formulation

Let $\mathcal{T}_h$ denote a quasi-uniform triangulation on Ω. We consider the discretized variational problem for (1): find $u_h \in X_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in X_h, \tag{3}$$

where $a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h \, dx$ and $(f, v_h) = \int_\Omega f v_h \, dx$. Here, X_h is the standard $\mathcal{P}_1$-conforming finite element space.

Before proposing a constrained minimization problem whose minimizer has a connection with the solution of (3), we introduce some commonly-used notations. We assume that Ω is decomposed into N non-overlapping subdomains $\{\Omega_k\}_{k=1}^N$ such that

(i) Ω_k is a polygonally shaped open subset of Ω.
(ii) the decomposition $\{\Omega_k\}_{k=1}^N$ of Ω is geometrically conforming.
(iii) Γ_{kl} denotes the common interface of two adjacent subdomains Ω_k and Ω_l.

Let us use $\mathcal{T}_{h_k}$ to denote a quasi-uniform triangulation of Ω_k, where we have matching grids on the boundaries of neighboring subdomains across the interfaces. On each Ω_k, we set a finite-dimensional subspace X_h^k of $H^1(\Omega_k)$:

$$X_h^k = \{v_h^k \in \mathcal{C}^0(\overline{\Omega}_k) \mid \forall \tau \in \mathcal{T}_{h_k},\ v_h^k|_\tau \in \mathcal{P}_1(\tau),\ v_h^k|_{\partial\Omega \cap \partial\Omega_k} = 0\}.$$

Next, we define a bilinear form on $X_h^c \times X_h^c$:

$$a_h(u, v) = \sum_{k=1}^N \int_{\Omega_k} \nabla u \cdot \nabla v \, dx.$$

where $X_h^c = \{v = (v_h^k)_k \in \prod_{k=1}^N X_h^k \mid v$ is continuous at each corner$\}$.

It is well-known that solving the finite element problem (3) is equivalent to solving the saddle-point formulation: find a saddle point $(u_h, \lambda_h) \in X_h^c \times \mathbb{R}^E$ such that

$$\mathcal{L}(u_h, \lambda_h) = \max_{\mu_h \in \mathbb{R}^E} \min_{v_h \in X_h^c} \mathcal{L}(v_h, \mu_h) = \min_{v_h \in X_h^c} \max_{\mu_h \in \mathbb{R}^E} \mathcal{L}(v_h, \mu_h), \tag{4}$$

where the Lagrangian $\mathcal{L} : X_h^c \times \mathbb{R}^E \to \mathbb{R}$ is defined by

$$\mathcal{L}(v, \mu) = \mathcal{J}(v) + \langle Bv, \mu \rangle = \frac{1}{2} a_h(v, v) - (f, v) + \langle Bv, \mu \rangle.$$

Here, B is a signed Boolean matrix such that for any $v \in X_h^c$, $Bv = 0$ which enforces the continuity of v across the interface.

Now, we shall slightly change the saddle-point formulation (4) by addition of a penalty term to the Lagrangian $\mathcal{L}$. Let J_η be a bilinear form on $X_h^c \times X_h^c$ defined as

$$J_\eta(u, v) = \sum_{k<l} \frac{\eta}{h} \int_{\Gamma_{kl}} (u^k - u^l)(v^k - v^l)\, ds, \quad \eta > 0,$$

where $h = \max_{k=1,\cdots,N} h_k$. Given the augmented Lagrangian $\mathcal{L}_\eta$ defined by

$$\mathcal{L}_\eta(v, \mu) = \mathcal{L}(v, \mu) + \frac{1}{2} J_\eta(v, v),$$

we consider the following saddle-point problem:

$$\mathcal{L}_\eta(u_h, \lambda_h) = \max_{\mu_h \in \mathbb{R}^E} \min_{v_h \in X_h^c} \mathcal{L}_\eta(v_h, \mu_h) = \min_{v_h \in X_h^c} \max_{\mu_h \in \mathbb{R}^E} \mathcal{L}_\eta(v_h, \mu_h). \tag{5}$$

Based on the characterization of a saddle-point formulation like problem (5) by a variational problem in [7], it can be shown that the saddle-point of (5) is equivalent to the solution of the following variational problem: find $(u_h, \lambda_h) \in X_h^c \times \mathbb{R}^E$ such that

$$\begin{aligned} a_\eta(u_h, v_h) + \langle v_h, B^T \lambda_h \rangle &= (f, v_h) \quad && \forall v_h \in X_h^c, \\ \langle Bu_h, \mu_h \rangle &= 0 && \forall \mu_h \in \mathbb{R}^E. \end{aligned} \tag{6}$$

Moreover, the primal solution u_h of (6) is exactly equal to the solution of the variational problem (3).

3 Iterative Substructuring Method

The saddle-point formulation (6) is expressed in the following algebraic form

$$\begin{bmatrix} A_{\Pi\Pi} & A_{\Pi e} & 0 \\ A_{\Pi e}^T & A_{ee}^\eta & B_e^T \\ 0 & B_e & 0 \end{bmatrix} \begin{bmatrix} u_\Pi \\ u_e \\ \lambda \end{bmatrix} = \begin{bmatrix} f_\Pi \\ f_e \\ 0 \end{bmatrix}, \tag{7}$$

where u_Π denotes the degrees of freedom (dof) at the interior nodes and the corners, u_e those on the edge nodes on the interface except at the corners. After eliminating u_Π and u_e in (7), we have the following system for the Lagrange multipliers:

$$F_\eta \lambda = d_\eta \tag{8}$$

where

$$F_\eta = B_e S_\eta^{-1} B_e^T, \quad d_\eta = B_e S_\eta^{-1}(f_e - A_{\Pi e}^T A_{\Pi\Pi}^{-1} f_\Pi).$$

Here, $S_\eta = S + \eta J = (A_{ee} - A_{\Pi e}^T A_{\Pi\Pi}^{-1} A_{\Pi e}) + \eta J$. Noting that F_η is symmetric positive definite, we solve the resultant dual system (8) iteratively by the conjugate gradient method (CGM). Hence, the key issue is to provide a sharp estimate for the condition number of F_η.

Note that J in S_η is represented as $J = B_e^T D(J_B) B_e$ where $D(J_B)$ is a block diagonal matrix such that the diagonal block J_B is a positive definite matrix induced from

$$\frac{1}{h}\int_{\Gamma_{ij}} \varphi\psi\, ds \quad \forall \varphi, \psi \in X_h^c|_{\Gamma_{ij}}.$$

Let us denote by Λ the space of vectors of dof associated with the Lagrange multipliers where the norm $\|\cdot\|_\Lambda$ and the dual norm $\|\cdot\|_{\Lambda'}$ are defined by

$$\|\mu\|_\Lambda^2 = \mu^T D(J_B)\mu \ \ \forall \mu \in \Lambda \quad \text{and} \quad \|\lambda\|_{\Lambda'} = \max_{\mu\in\Lambda} \frac{|\langle \lambda, \mu\rangle|}{\|\mu\|_\Lambda} \ \ \forall \lambda \in \Lambda.$$

In order to derive bounds on the extreme eigenvalues of F_η, we first mention some useful properties.

Lemma 1. *For $S = A_{ee} - A_{\Pi e}^T A_{\Pi\Pi}^{-1} A_{\Pi e}$, there exists a constant $C > 0$ such that*

$$v_e^T S v_e \le C v_e^T J v_e \quad \forall v_e \perp \mathrm{Ker} B_e.$$

Proposition 1. *Let $\|\cdot\|_{S_\eta}$ be the norm induced by the symmetric positive definite matrix S_η. For any $\lambda \in \mathbb{R}^E$,*

$$\lambda^T F_\eta \lambda = \max_{v_e \neq 0} \frac{|v_e^T B_e^T \lambda|^2}{\|v_e\|_{S_\eta}^2}.$$

From Lemma 1 and Proposition 1, we have

Theorem 1. *For any $\lambda \in \Lambda$, we have that*

$$\frac{1}{C+\eta}\|\lambda\|_{\Lambda'}^2 \le \lambda^T F_\eta \lambda \le \frac{1}{\eta}\|\lambda\|_{\Lambda'}^2.$$

where C is the constant estimated in Lemma 1.

Using Theorem 1 based on Lemma 3.1 in [10], we now give the estimate of the condition number $\kappa(F_\eta)$.

Corollary 1. *We have the condition number estimate of the dual system (8)*

$$\kappa(F_\eta) \le \left(\frac{C}{\eta} + 1\right)\kappa(J_B), \quad C = \frac{\lambda_{\max}^S}{2\lambda_{\min}^{J_B}},$$

where $\lambda_{\max}^S$ and $\lambda_{\min}^{J_B}$ are the maximum eigenvalue of S and the minimum eigenvalue of J_B, respectively. Furthermore, the constant C is independent of the subdomain size H and the mesh size h.

Corollary 2. *For a sufficiently large η, there exists a constant C^* independent of h and H such that*

$$\kappa(F_\eta) \le C^*.$$

In particular, assuming that each triangulation $\mathcal{T}_{h_k}$ on Ω_k is uniform, $C^* = 3$.

Remark 1. To the best of our knowledge, the algorithm with such a constant bound of the condition number is unprecedented in the field of domain decomposition. Adding the penalization term J_η to the FETI-DP formulation results in a strongly scalable algorithm without any domain-decomposition-based preconditioners even if it is redundant in view of equivalence relations among the concerned minimization problems.

4 Computational Issues and Numerical Results

4.1 Computational Issues

In focusing on the implementation of the proposed algorithm, the saddle-point formulation in form of (7) is rewritten as follows

$$\begin{bmatrix} K_{rr}^\eta & K_{rc} & B_r^T \\ K_{rc}^T & K_{cc} & 0 \\ B_r & 0 & 0 \end{bmatrix} \begin{bmatrix} u_r \\ u_c \\ \lambda \end{bmatrix} = \begin{bmatrix} f_r \\ f_c \\ 0 \end{bmatrix}, \tag{9}$$

where u_c denotes the dof at the corners and u_r the remaining of dof. Eliminating u_r and u_c in (9) yields

$$F_\eta \lambda = d_\eta \tag{10}$$

where

$$F_\eta = F_{rr} + F_{rc}F_{cc}^{-1}F_{rc}^T, \quad d_\eta = d_r - F_{rc}F_{cc}^{-1}d_c.$$

In view of implementation, the difference with the FETI-DP method ([4]) is that we invert K_{rr}^η that contains the penalization parameter η. To compare our algorithm with the FETI-DP method, we need to make a more careful observation of behavior of $(K_{rr}^\eta)^{-1}$. Note that

$$K_{rr}^\eta = K_{rr} + \eta\tilde{J} = \begin{bmatrix} A_{ii} & A_{ie} \\ A_{ie}^T & A_{ee} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \eta J \end{bmatrix}$$

where $J = B_e^T D(J_B) B_e$. Thanks to the specific type of discrete Sobolev inequality in Lemma 3.4 of [2], we get the following estimate.

Theorem 2. *For each $\eta > 0$, we have that*

$$\kappa(K_{rr}^{\eta}) \lesssim \left(\frac{H}{h}\right)^2 \left(1+\log\frac{H}{h}\right)(1+\eta).$$

Theorem 2 shows how severely η damages the property of K_{rr}^{η} as η is increased. Since K_{rr}^{η} is solved iteratively, it might be expected that the large condition number of K_{rr}^{η} shown above may cause the computational cost relevant to K_{rr}^{η} to be more expensive. We shall establish a good preconditioner for K_{rr}^{η} in order to remove a bad effect of η. We introduce the preconditioner M as follows

$$M = \overline{K}_{rr} + \eta\tilde{J} = \begin{bmatrix} A_{ii} & 0 \\ 0 & A_{ee} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \eta J \end{bmatrix}.$$

Theorem 3. *The condition number of the preconditioned problem grows asymptotically as*

$$\kappa(M^{-1}K_{rr}^{\eta}) := \frac{\lambda_{\max}(M^{-1}K_{rr}^{\eta})}{\lambda_{\min}(M^{-1}K_{rr}^{\eta})} \lesssim \frac{H}{h}\left(1+\log\frac{H}{h}\right).$$

4.2 Numerical Results

Let Ω be $[0,1]^2 \subset \mathbb{R}^2$. We consider the Poisson problem with the exact solution

$$u(x,y) = y(1-y)\sin(\pi x).$$

The reduced dual problem (10) is solved iteratively by CGM. We monitor the convergence of CGM with the stopping criterion $\frac{\|r_k\|}{\|r_0\|} \leq$ TOL, where r_k is the dual residual error on the k-th CG iteration and TOL=10^{-8}. We decompose Ω into N_s square subdomains with $N_s = 1/H \times 1/H$, where each subdomain is partitioned into $2 \times H/h \times H/h$ uniform triangular elements.

First, we make a comparison between our proposed method and the FETI-DP method from the viewpoint of the conditioning of the related matrices F_η and F. Table 1 shows that the condition number $\kappa(F_\eta)$ and the CG iteration number remain almost constant when the mesh is refined and the number N_s of subdomains is increased while keeping the ratio H/h constant. Moreover, we observe numerically that the condition number of F_η is bounded by the constant 3 independently of h and H, while the condition number in the FETI-DP method grows with increasing H/h (cf. [4, 5]). In addition, it is shown in Table 1 that the proposed method is superior to the FETI-DP method in the number of CG iterations for convergence. In Table 2, the condition number of K_{rr}^{η} and $M^{-1}K_{rr}^{\eta}$ are listed to show how well the designed preconditioner M for $(K_{rr}^{\eta})^{-1}$ performs. It confirms that the influence of η on $\kappa(K_{rr}^{\eta})$ is completely removed after adopting M.

Table 1. Comparison between the proposed method ($\eta = 10^6$) and the FETI-DP method ($\eta = 0$)

N_s	$\frac{H}{h}$	$\eta = 10^6$		$\eta = 0$	
		iter. no	$\kappa(F_\eta)$	iter. no	$\kappa(F)$
4×4	4	3	2.0938	14	7.2033
	8	7	2.7170	23	2.2901e+1
	16	13	2.9243	33	5.9553e+1
	32	14	2.9771	48	1.4707e+2
8×8	4	3	2.0938	18	7.9241
	8	7	2.7170	32	2.5668e+1
	16	12	2.9245	48	6.7409e+1
16×16	4	3	2.0938	19	7.9461
	8	7	2.7170	34	2.6324e+1

Table 2. Performance of preconditioner M for $(K_{rr}^{\eta})^{-1}$ where $N_s = 4\times4$

η	$\frac{H}{h}=4$		$\frac{H}{h}=8$		$\frac{H}{h}=16$	
	$\kappa(K_{rr}^{\eta})$	$\kappa(M^{-1}K_{rr}^{\eta})$	$\kappa(K_{rr}^{\eta})$	$\kappa(M^{-1}K_{rr}^{\eta})$	$\kappa(K_{rr}^{\eta})$	$\kappa(M^{-1}K_{rr}^{\eta})$
0	43.2794	14.8532	228.0254	40.0332	1.1070e+3	104.3459
1	34.5773	11.8232	161.1716	28.7437	7.0562e+2	68.3468
10^1	91.3072	11.4010	420.1058	28.1835	1.8390e+3	67.6093
10^2	8.5119e+2	11.3525	3.9824e+3	28.1232	1.7513e+4	67.5325
10^3	8.4538e+3	11.3475	3.9616e+4	28.1170	1.7430e+5	67.5247
10^4	8.4480e+4	11.3470	3.9596e+5	28.1164	1.7421e+6	67.5240
10^5	8.4474e+5	11.3469	3.9593e+6	28.1164	1.7420e+7	67.5239
10^6	8.4473e+6	11.3469	3.9593e+7	28.1164	1.7420e+8	67.5239
10^7	8.4473e+7	11.3469	3.9593e+8	28.1164	1.7420e+9	67.5238

5 Conclusions

In this paper, we have proposed a dual substructuring method based on an augmented Lagrangian with a penalty term. Unlike other substructuring methods, it is shown that without any preconditioners, the designed method is scalable in the sense that for a large penalty parameter η, the condition number of the relevant dual system has a constant bound independent of H and h. In addition, we dealt with an implementational issue. An optimal preconditioner with respect to η is established in order to increase the ease of use and the practical efficiency of the presented method.

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