An Additive Neumann-Neumann Method for Mortar Finite Element for 4th Order Problems

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Summary. In this paper, we present an additive Neumann-Neumann type parallel method for solving the system of algebraic equations arising from the mortar finite element discretization of a plate problem on a nonconforming mesh. Locally, we use a conforming Hsieh-Clough-Tocher macro element in the subdomains. The proposed method is almost optimal i.e. the condition number of the preconditioned problem grows poly-logarithmically with respect to the parametes of the local triangulations.

1 Introduction

Many real life phenomenas and technical problems are modelled by partial differential equations. A way of constructing an effective approximation of the differential problem is to introduce one global conforming mesh and then to set an approximate discrete problem. However often it is required to use different approximation methods or independent local meshes in some subregions of the original domains. This may allow us to make an adaptive changes of the local mesh in a substructure without modifying meshes in other subdomains. A mortar method is an effective method of constructing approximation on nonconforming triangulations, cf. [1, 13].

There are many works for iterative solvers for mortar method for second order problem, see e.g. [2, 3, 6, 7] and references therein. But there is only a limited number of papers investigating fast solvers for mortar discretizations of fourth order elliptic problems, cf. [8, 10, 14].

In this paper, we focus on a Neumann-Neumann type of algorithm for solving a discrete problem arising from a mortar type discretization of a fourth order model elliptic problem with discontinuous coefficients in 2D. We consider a mortar discretization which use Hsieh-Clough-Tocher (HCT) elements locally in subdomains. Our method of solving system of equations is a Neumann-Neumann type of algorithm constructed with the help of Additive Schwarz Method (ASM) abstract framework. The obtained results are almost optimal i.e. it shown that the number of CG iteration applied to the preconditioned system grows only logarithmically with the ratio $H/\underline{h}$ and is independent of the jumps of the coefficients.

2 Discrete Problem

In this section, we introduce a model problem and discuss its mortar discretization.

We consider a polygonal domain Ω in the plane which is partitioned into disjoint polygonal subdomains Ω_k such that $\overline{\Omega} = \bigcup_{k=1}^N \overline{\Omega}_k$ with $\overline{\Omega}_k \cap \overline{\Omega}_l$ being an empty set, an edge or a vertex (crosspoint). We assume that these subdomains form a coarse triangulation of the domain which is shape regular the sense of [5].

The model differential problem is to find $u^* \in H_0^2(\Omega)$ such that

$$a(u^*, v) = \int_\Omega f v \, dx \quad \forall v \in H_0^2(\Omega), \tag{1}$$

where $f \in L^2(\Omega)$,

$$H_0^2(\Omega) = \{u \in H^2(\Omega) : u = \partial_n u = 0 \text{ on } \partial\Omega\}$$

and

$$a(u,v) = \sum_{k=1}^N \int_{\Omega_k} \rho_k [u_{x_1x_1} v_{x_1x_1} + 2\, u_{x_1x_2} v_{x_1x_2} + u_{x_2x_2} v_{x_2x_2}] \, dx.$$

Here ρ_k are any positive constant and ∂_n is a normal unit normal derivative.

A quasiuniform triangulation $T_h(\Omega_k)$ made of triangles is introduced in each subdomain Ω_k, and let $h_k = \max_{\tau \in T_h(\Omega_k)} \operatorname{diam}(\tau)$ be the parameter of this triangulation, cf. e.g. [4].

Let Γ_{ij} denote the interface between two subdomains Ω_i and Ω_j i.e. the open edge that is common to these subdomains, i.e. $\overline{\Gamma}_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$. We also introduce a global interface $\Gamma = \bigcup_i \overline{\partial\Omega_i \setminus \partial\Omega}$.

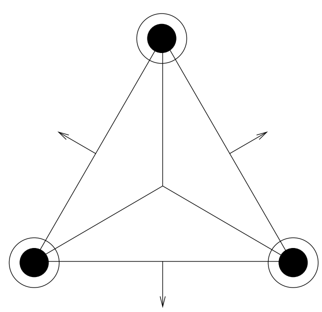

Fig. 1. HCT element.

We can now introduce local finite element spaces. Let $X_h(\Omega_k)$, be the finite element space defined as follows, cf. Fig. 1:

$$\begin{aligned} X_h(\Omega_k) = \{u \in C^1(\Omega_k) : \; & u \in P_3(\tau_i),\ \tau_i \in T_h(\Omega_k), \text{ for triangles } \tau_i, \\ & i = 1,2,3, \text{ formed by connecting the vertices of} \\ & \text{any } \tau \in T_h(\Omega_k) \text{ to its centroid, and} \\ & u = \partial_n u = 0 \text{ on } \partial\Omega_k \cap \partial\Omega\}, \end{aligned}$$

where $P_3(\tau_i)$ is the function space of cubic polynomials defined over τ_i.

Next a global space $X_h(\Omega)$ is defined as $X_h(\Omega) = \prod_{i=1}^N X_h(\Omega_k)$.

Each edge Γ_{ij} inherits two 1D triangulations made of segments that are edges of elements of the triangulations of Ω_i and Ω_j, respectively. In this way, each Γ_{ij} is provided with two independent and different 1D meshes which are denoted by $T_{h,i}(\Gamma_{ij})$ and $T_{h,j}(\Gamma_{ij})$, cf. Fig. 2.

One of the sides of Γ_{ij} is defined as a mortar (master) one, denoted by γ_{ij} and the other as a nonmortar (slave) one denoted by δ_{ji}. Let the mortar side of Γ_{ij} be chosen by the condition: $\rho_j \leq \rho_i$, (i.e. here, the mortar side is the i-th one).

For each interface Γ_{ij} two test spaces are defined: $M_t^h(\delta_{ji})$ the space formed by C^1 smooth piecewise cubic functions on the slave h_j triangulation of δ_{ji}, i.e $T_{h,j}(\Gamma_{ji})$, which are piecewise linear in the two end elements, and $M_n^h(\delta_{ji})$ the space of continuous piecewise quadratic functions on the elements of triangulation of $T_{h,j}(\Gamma_{ji})$, which are piecewise linear in the two end elements of this triangulation.

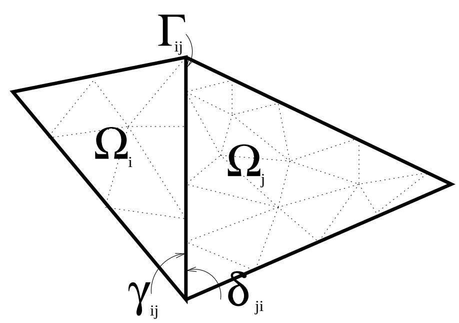

Fig. 2. Independent meshes on an interface.

The discrete space V^h is defined as the space formed by all function in $X_h(\Omega)$, which are continuous at the crosspoints, i.e. the common vertices of substructures, and satisfy the following mortar condition on each interface $\Gamma_{ij} = \delta_{ji} = \gamma_{ij} \subset \Gamma$:

$$\int_{\delta_{ji}} (u_i - u_j)\varphi \, ds = 0 \qquad \forall \varphi \in M_t^h(\delta_{ji}), \tag{2}$$

$$\int_{\delta_{ji}} (\partial_n u_i - \partial_n u_j)\psi \, ds = 0 \qquad \forall \psi \in M_n^h(\delta_{ji}).$$

It is worth mentioning that $u \in V^h$ has discontinuous ∇u at a crosspoint c_r, i.e. ∇u has as many values as the number of substructures with this crosspoint c_r.

Our discrete problem is to find $u_h^* \in V^h$ such that

$$a_h(u_h^*, v) = \int_\Omega f v \, dx \qquad \forall v \in V^h, \tag{3}$$

where $a_h(u,v) = \sum_{k=1}^N a_k(u,v)$ for

$$a_k(u,v) = \int_{\Omega_k} \rho_k [u_{x_1x_1} v_{x_1x_1} + 2\, u_{x_1x_2} v_{x_1x_2} + u_{x_2x_2} v_{x_2x_2}]\, dx.$$

This problem has a unique solution and for error estimates, we refer to [9].

3 Neumann-Neumann Method

In this section, we introduce our Neumann-Neumann method.

For the simplicity of presentation, we assume that our subdomains Ω_k are triangles which form a coarse triangulation of Ω.

We introduce a splitting of $u \in X_k(\Omega_k)$ into two $a_k(\cdot,\cdot)$ orthogonal parts: $u = P_k u$ and discrete biharmonic part $H_h = u - P_k u$, where $P_k u \in X_{h,0}(\Omega_k)$ is defined by

$$a_k(P_k u, v) = a_k(u,v) \qquad \forall v \in X_{h,0}(\Omega_k)$$

with $X_{h,0}(\Omega_k) = X_h(\Omega_k) \cap H_0^2(\Omega_k)$. The discrete biharmonic part of u: $H_k u = u - P_k u \in X_h(\Omega_k)$ satisfies

$$\begin{cases} a_k(H_k u, v) = 0 & \forall v \in X_{h,0}(\Omega_k), \\ Tr\, H_k u = Tr\, u & \text{on } \partial\Omega_k, \end{cases} \tag{4}$$

where $Tr\, u = (u, \nabla u)$. Let $Hu = (H_1 u, \ldots, H_N u)$ denotes the part of $u \in X_h(\Omega)$ which is discrete biharmonic in all subdomains. We also set

$$\tilde{V}^h = HV^h = \{u \in V^h : u \text{ is discrete biharmonic in all } \Omega_k\} \tag{5}$$

Each function in $\tilde{V}^h$ is uniquely defined by the values of all degree of freedoms associated with all HCT nodal points i.e. the vertices and the midpoints, which are on masters and at crosspoints since the values of the degrees of freedom corresponding to the HCT nodes in the interior of a nonmortar (slave) are defined by the mortar conditions (2) and that the values of the degrees of freedom of the nodes interior to the subdomains are defined by (4).

3.1 Local Subspaces

For each subdomain Ω_i, we introduce an extension operator $E_i : X_h(\Omega_i) \to V^h$ as $E_i u = \tilde{E}_i u + \widehat{P}_i u$, where $\widehat{P}_i u = (0, \ldots, 0, P_i u, 0, \ldots, 0)$ and $\tilde{E}_i : X_h(\Omega_i) \to \tilde{V}^h$ is defined as follows:

- $\tilde{E}_i u(x) \in \tilde{V}^h$, i.e. it is discrete biharmonic in all subdomains,
- $\tilde{E}_i u(x) = u(x)$ and $\nabla \tilde{E}_i u(x) = \nabla u(x)$ and $\partial_n \tilde{E}_i u(m) = \partial_n u(m)$ for an x a nodal point (vertex) and a midpoint m of an element of $T_{h,i}(\Gamma_{ij})$ for any mortar $\gamma_{ij} \subset \partial\Omega_i$,
- $\nabla \tilde{E}_i u(v) = \nabla u(v)$ for any vertex v of substructure Ω_i,
- $\tilde{E}_i u(c_r) = \frac{1}{N(c_r)} u(c_r)$ for any crosspoint c_r which is a vertex of $\partial\Omega_i$. Here, $N(c_r)$ is the number of domains which have c_r as a vertex.

- $Tr\,\tilde{E}_i u = 0$ on remaining masters and at the crosspoints which are not on $\partial\Omega_i$.

The values of the degrees of freedom of $E_i u$ on a slave are defined by the mortar conditions (2) and in subdomains $\Omega_j,\ j \neq i$ by (4).

We also have

$$\sum_{k=1}^{N} E_k u_k = u$$

for any $u = (u_1, \ldots, u_N) \in V^h$.

We next define local spaces $V_k = E_k V_c^h(\Omega_k)$, where $V_c^h(\Omega_k)$ is the subspace of $X_h(\Omega_k)$ of functions that have zero values at all vertices of Ω_k. Local bilinear forms are defined over $V_c^h(\Omega_k)$ as $b_k(u,v) = a_k(u,v)$. Note that $u \in V_k$ can be nonzero only in Ω_i and Ω_j for a j such that Γ_{ij} is a common edge of Ω_i and Ω_j and its master side is associated with Ω_i.

3.2 Coarse Space

For any $u = (u_1, \ldots, u_N) \in X_h(\Omega)$, we introduce $I_0 u \in V^h$ which is defined solely by the values of u at crosspoints, i.e. the common vertices of substructures in Ω as

$$I_0 u = \sum_{k=1}^{N} E_k(I_{H,k} u_k), \tag{6}$$

where $I_{H,k} u_k \in X_h(\Omega_k)$ is a linear interpolant of u_k at the three vertices of a triangular substructure Ω_k.

Next let us define a coarse space as

$$V_0 = I_0 V^h,$$

and a coarse bilinear form

$$b_0(u,v) = \left(1 + \log\left(\frac{H}{\underline{h}}\right)\right)^{-1} a_h(u,v),$$

where $H = \max_k H_k$ for $H_k = \mathrm{diam}(\Omega_k)$, and $\underline{h} = \min_k h_k$. Note that the dimension of V_0 equals to the number of crosspoints.

We see that $V^h = V_0 + \sum_{k=1}^{N} V_k$.

Next following the Additive Schwarz Method (ASM) abstract scheme, special projection-like operators are introduced: $T_k : V_k \to V^h$ for $k = 0, \ldots, N$ by

$$b_0(T_0 u, v) = a_h(u,v) \qquad \forall v \in V_0 \tag{7}$$

and let $T_k u = E_k \widehat{T}_k u$ for $k = 1, \ldots, N$, where $\widehat{T}_k u \in V_c^h(\Omega_k)$ is defined by

$$b_k(\widehat{T}_k u, v) = a_h(u, E_k v) \qquad \forall v \in V_c^h(\Omega_k). \tag{8}$$

The operator T_k is symmetric and nonnegative definite over V^h in the terms of the form $a_h(u,v)$.

Finally, an ASM operator $T : V_h \to V_h$ is defined by

$$T = T_0 + \sum_{k=1}^{N} T_k.$$

We then replace problem (3) by a new equivalent one:

$$T u_h^* = g, \tag{9}$$

where $g = \sum_{k=0}^{N} g_i$ and $g_i = T_i u_h^*$ for u_h^* the solution of (3).

The main result of this paper is the following theorem:

Theorem 1. *For any $u \in V^h$, it holds that*

$$c\, a_h(u,u) \le a_h(Tu,u) \le C \left(1 + \log\left(\frac{H}{\underline{h}}\right)\right)^2 a_h(u,u),$$

where $H = \max_k H_k$ with $H_k = \mathrm{diam}(\Omega_k)$, $\underline{h} = \min_k h_k$, and c, C are positive constants independent of all mesh parameters h_k, H_k and the coefficients ρ_k.

Sketch of the Proof

We present here only a sketch of the proof which is based on the abstract ASM scheme, cf. e.g. [12].

We have to check three key assumptions, cf. [12]. For our method the assumption II (Strengthened Cauchy-Schwarz Inequalities), is satisfied with a constant independent of the number of subdomains by a coloring argument.

Note that T_0 is the orthogonal projection onto V_0 (in terms of the bilinear form $a_h(\cdot,\cdot)$) which is scaled by $(1 + \log(H/\underline{h}))^{-1}$, i.e., we have

$$a_h(u,u) = (1 + \log(H/\underline{h}))\, b_0(u,u) \qquad \forall u \in V_0.$$

It can also be shown following the lines of proof of [11] that

$$a_h(E_k u, E_k u) \le C_1\, (1 + \log(H/\underline{h}))^2 b_k(u,u) \qquad \forall u \in V_k,$$

where C_1 is a constant independent of mesh parameters and subdomain coefficients. Thus these two estimates yields that the constant ω in the assumption III (Local Stability), is bounded by $C_1\, (1 + \log(H/\underline{h})^2$.

It remains to prove assumption I (Stable Decomposition), i.e., we have to prove that there exists a positive constant C_0^2 such that for any $u \in V_h$ there are $w_0 \in V_0$ and $w_k \in V_k$, $k = 1, \ldots, N$ such that $u = w_0 + \sum_{k=1}^{N} E_k w_k$ and

$$b_0(w_0, w_0) + \sum_{k=1}^{N} b_k(w_k, w_k) \le C_0^2 a_h(u,u). \tag{10}$$

We first define decomposition for $u = (u_1, \ldots, u_N) \in V^h$. Let $w_0 = I_0 u$ and $w_k = u_k - I_{h,k} u_k \in V_c^h(\Omega_k)$. Note that

$$w_0 + \sum_{k=1}^{N} E_k w_k = I_0 u + \sum_{k=1}^{N} E_k (u_k - I_{h,k} u_k) = \sum_{k=1}^{N} E_k u_k = u.$$

Next, we see that

$$\sum_{k=1}^{N} b_k(w_k, w_k) = \sum_{k=1}^{N} \rho_k |u_k - I_{H,k} u_k|^2_{H^2(\Omega_k)} \tag{11}$$
$$= \sum_{k=1}^{N} \rho_k |u_k|^2_{H^2(\Omega_k)} = a_h(u,u).$$

Again following the lines of proof of [11], we can show that

$$a_h(I_0 u, I_0 u) \leq C_0^2 (1 + \log(H/\underline{h})) a_h(u,u),$$

where C_0^2 is a constant independent of mesh parameters and subdomain coefficients, thus

$$b_0(w_0, w_0) = (1 + \log(H/\underline{h}))^{-1} a_h(I_0 u, I_0 u) \leq C_0^2\, a_h(u,u).$$

The last estimate and (11) yield us the bound in (10) and this concludes the sketch of the proof.

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References

[1] Bernardi, C., Maday, Y., Patera, A.T.: A new nonconforming approach to domain decomposition: the mortar element method. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, vol. XI (Paris, 1989–1991)*, vol. 299 of *Pitman Res. Notes Math. Ser.*, pages 13–51. Longman Sci. Tech., Harlow, 1994.
[2] Bjørstad, P.E., Dryja, M., Rahman, T.: Additive Schwarz methods for elliptic mortar finite element problems. *Numer. Math.*, 95(3):427–457, 2003.
[3] Braess, D., Dahmen, W., Wieners, C.: A multigrid algorithm for the mortar finite element method. *SIAM J. Numer. Anal.*, 37(1):48–69, 1999.
[4] Brenner, S.C., Scott, L.R.: *The mathematical theory of finite element methods*, vol. 15 of *Texts in Applied Mathematics*. Springer, New York, 2nd ed., 2002.
[5] Brenner, S.C., Sung, L.-Y.: Balancing domain decomposition for nonconforming plate elements. *Numer. Math.*, 83(1):25–52, 1999.
[6] Dryja, M.: A Neumann-Neumann algorithm for a mortar discetization of elliptic problems with discontinuous coefficients. *Numer. Math.*, 99:645–656, 2005.
[7] Kim, H.H., Widlund, O.B.: Two-level Schwarz algorithms with overlapping subregions for mortar finite elements. *SIAM J. Numer. Anal.*, 44(4):1514–1534, 2006.

[8] Marcinkowski, L.: Domain decomposition methods for mortar finite element discretizations of plate problems. *SIAM J. Numer. Anal.*, 39(4):1097–1114, 2001.
[9] Marcinkowski, L.: A mortar element method for some discretizations of a plate problem. *Numer. Math.*, 93(2):361–386, 2002.
[10] Marcinkowski, L.: An Additive Schwarz Method for mortar Morley finite element discretizations of 4th order elliptic problem in 2d. *Electron. Trans. Numer. Anal.*, 26:34–54, 2007.
[11] Marcinkowski, L.: A Neumann-Neumann algorithm for a mortar finite element discretization of 4th order elliptic problems in 2d. Tech. Report 173, Institute of Applied Mathematics and Mechanics, Warsaw University, June 2007.
[12] Toselli, A., Widlund, O. *Domain decomposition methods—algorithms and theory*, vol. 34 of *Springer Series in Computational Mathematics*. Springer, Berlin, 2005.
[13] Wohlmuth, B.I.: *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, vol. 17 of *Lectures Notes in Computational Science and Engineering*. Springer, Berlin, 2001.
[14] Xu, X., Li, L., Chen, W.: A multigrid method for the mortar-type Morley element approximation of a plate bending problem. *SIAM J. Numer. Anal.*, 39(5): 1712–1731, 2001/02.