A FETI-2LM Method for Non-Matching Grids

François-Xavier Roux

High Performance Computing Unit, ONERA, 29 avenue de la Division Leclerc, 92320, Châtillon, France roux@onera.fr

1 Introduction

In this paper, a new solution methodology based on the FETI-2LM method for non conforming grids is introduced. Thanks to the regularizing properties of the Robin interface matching conditions of the FETI-2LM method, each non conforming condition can be localized inside one subdomain, in such a way that the FETI-2LM method applies exactly in the same way as in the conforming case.

The paper is organized as follows: section 2 recalls the principle of FETI-2LM method, section 3 briefly describes the mortar method for non conforming domains, the new methodology for localizing the multi-point constraints on the non conforming interface derived from the mortar method is introduced in section 4 and section 5 generalizes the methodology in the case of multi-level splitting of a mesh including a non conforming interface.

2 FETI-2LM method

2.1 Discrete Approach

Consider the linear problem $Kx = b$ arising from a finite element discretization of a PDE. The mesh of the entire domain is split in two meshes like in Fig.1, the two subdomains are denoted by Ω_1 and Ω_2, and their interface by Γ_3. Then, the global stiffness matrix and right hand sides have the following block structure:

$$K = \begin{bmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{1}$$

and the subdomain stiffness matrices and right hand sides are:

$$K_1 = \begin{bmatrix} K_{11} & K_{13} \\ K_{31} & K_{33}^{(1)} \end{bmatrix}, \; y_1 = \begin{bmatrix} b_1 \\ b_3^{(1)} \end{bmatrix} \quad K_2 = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33}^{(2)} \end{bmatrix}, \; y_2 = \begin{bmatrix} b_2 \\ b_3^{(2)} \end{bmatrix} \tag{2}$$

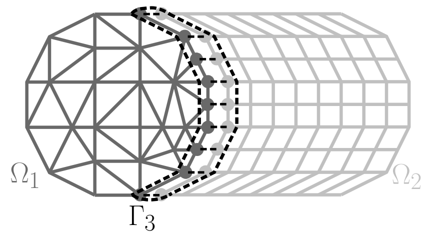

Fig. 1. Two meshes with interface.

with $K_{33}^{(1)} + K_{33}^{(2)} = K_{33}$ and $b_3^{(1)} + b_3^{(2)} = b_3$.

The FETI-2LM method [3] is based on introducing independent generalized Robin boundary conditions on interface Γ_3. Discretized local Robin problem takes the following form:

$$\begin{bmatrix} K_{ii} & K_{i3} \\ K_{3i} & K_{33}^{(i)} + A_{33}^{(i)} \end{bmatrix} \begin{bmatrix} x_i \\ x_3^{(i)} \end{bmatrix} = \begin{bmatrix} b_i \\ b_3^{(i)} + \lambda_3^{(i)} \end{bmatrix} \tag{3}$$

To be the restrictions in the subdomains of the solution of the global problem, the solutions of the local problems must first satisfy the discrete continuity condition:

$$x_3^{(1)} - x_3^{(2)} = 0 \tag{4}$$

The second interface matching condition, that is a discrete condition of equilibrium, is nothing else than the last row of the global discrete system:

$$K_{31}x_1 + K_{32}x_2 + K_{33}x_3 = b_3 \tag{5}$$

Note that these two conditions can be derived by simple algebraic manipulation from any linear system of equations whose matrix has the block form of (1).

Given the splitting of block matrix K_{33} and of vector b_3, the discrete equilibrium equation (5) can be rewritten as:

$$K_{31}x_1 + K_{32}x_2 + K_{33}^{(1)}x_3^{(1)} + K_{33}^{(2)}x_3^{(2)} = b_3^{(1)} + b_3^{(2)} \tag{6}$$

Since the last row of the discrete Robin problem (3) gives:

$$K_{3i}x_i + K_{33}^{(i)}x_3^{(i)} + A_{33}^{(i)}x_3^{(i)} = b_3^{(i)} + \lambda_3^{(i)} \tag{7}$$

equation (6) can be alternatively written as:

$$A_{33}^{(1)}x_3^{(1)} + A_{33}^{(2)}x_3^{(2)} = \lambda_3^{(1)} + \lambda_3^{(2)} \tag{8}$$

Finally, the two discrete interface conditions (4) and (8) can be combined to give the equivalent mixed equations:

$$\begin{aligned} A_{33}^{(1)} x_3^{(2)} + A_{33}^{(2)} x_3^{(2)} &= \lambda_3^{(1)} + \lambda_3^{(2)} \\ A_{33}^{(1)} x_3^{(1)} + A_{33}^{(2)} x_3^{(1)} &= \lambda_3^{(1)} + \lambda_3^{(2)} \end{aligned} \tag{9}$$

By eliminating inner unknowns in the discrete Robin problem (3), the relation between the trace of the solution on interface $x_3^{(i)}$ and the discrete augmented flux $\lambda_3^{(i)}$ can be explicitly computed:

$$(K_{33}^{(i)} - K_{3i}K_{ii}^{-1}K_{i3} + A_{33}^{(i)})x_3^{(i)} = \lambda_3^{(i)} + b_3^{(i)} - K_{3i}K_{ii}^{-1}b_i \tag{10}$$

Denote by $S^{(i)} = (K_{33}^{(i)} - K_{3i}K_{ii}^{-1}K_{i3})$, the Schur complement matrix and by $c_3^{(i)} = b_3^{(i)} - K_{3i}K_{ii}^{-1}b_i$, the condensed right-hand-side.

Replacing $x_3^{(1)}$ and $x_3^{(2)}$ by their values as function of $\lambda_3^{(1)}$ and $\lambda_3^{(2)}$ derived from equation (10) in the mixed interface equations (9), leads to the condensed interface problem associated to the FETI-2LM method:

$$\begin{bmatrix} I & I - (A_{33}^{(1)} + A_{33}^{(2)})(S^{(2)} + A_{33}^{(2)})^{-1} \\ I - (A_{33}^{(1)} + A_{33}^{(2)})(S^{(1)} + A_{33}^{(1)})^{-1} & I \end{bmatrix} \begin{bmatrix} \lambda_3^{(1)} \\ \lambda_3^{(2)} \end{bmatrix} = \begin{bmatrix} (S^{(2)} + A_{33}^{(2)})^{-1} c_3^{(2)} \\ (S^{(1)} + A_{33}^{(1)})^{-1} c_3^{(1)} \end{bmatrix} \tag{11}$$

2.2 Optimal Interface Operator

FETI-2LM method consists in solving the condensed interface problem (11) via a Krylov space method. Of course, the gradient is not computed using explicit formula (11) but using the implicit one (9) where $x_3^{(1)}$ and $x_3^{(2)}$ are computed by solving the local Robin problems (3).

The main ingredient for the method to be effective is the choice of the operator $A_{33}^{(i)}$ associated with the generalized Robin condition. Analysis of the condensed interface problem (11) clearly shows that the optimal choice consists in taking in each subdomain the Schur complement of the rest of the domain:

$$A_{33}^{(1)} = S^{(2)} \qquad A_{33}^{(2)} = S^{(1)} \tag{12}$$

With such a choice, the matrix of the condensed interface problem (11) in the 2-subdomain case is simply the identity matrix and the method is then a direct solver.

In practice the Schur complement is of course too expensive to compute and, since it is a dense matrix, using it would also give a very large bandwidth to the stiffness matrices of the local generalized Robin problems. Sparse approximation of the Schur complement must be used. A purely algebraic methodology has been developed in [5]. It consist in building the approximate Schur complement by assembling exact Schur complements computed on small patches along the interface.

3 Mortar Method

Lagrange multiplier based domain decomposition methods have been extended to the case on non-conforming meshes, especially with the mortar method [1]. At the continuous level, the principle of the method consists in introducing a weak formulation of the interface continuity condition:

$$\int_{\Gamma_3} (u_1 - u_2)\mu = 0 \quad \forall\, \mu \in W \tag{13}$$

where u_i is the solution in subdomain Ω_i of the continuous PDE and W is a suitable set of Lagrange multipliers.

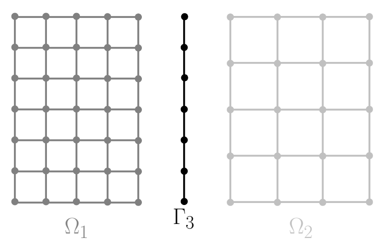

Fig. 2. Non-conforming interfaces.

For the discrete non-conforming case, optimal approximation results have been proved for elliptic second order PDEs when W is the mortar space of one of the two neighboring subdomains.

For instance, for a 2-D problem and linear finite element space V, the mortar space is a subset of the set of the traces on interface Γ of functions belonging to V, consisting of functions that are piecewise constant on the last segments of Γ, like in Fig. 3.

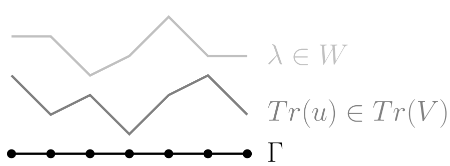

Fig. 3. Mortar space in 2-D for linear elements.

Suppose that, like in Fig. 2, the mortar side is Ω_1, then the global mixed problem associated with the weak formulation (13) of the continuity constraint takes the following discrete form:

$$\begin{bmatrix} K_1 & 0 & B_1^t \\ 0 & K_2 & B_2^t \\ B_1 & B_2 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} \quad \text{with } \xi_1 = \begin{bmatrix} x_1 \\ x_3^{(1)} \end{bmatrix},\ \xi_2 = \begin{bmatrix} x_2 \\ x_3^{(2)} \end{bmatrix}, \tag{14}$$

$B_1 = M_{31}R_1$ and $B_2 = -M_{32}R_2$. The R_i matrix is the restriction from subdomain Ω_i to its interface $\partial\Omega_i$, and the M_{3i} matrix is the mass matrix obtained by integration of products of mortar basis functions (living on Γ_3) and traces on $\partial\Omega_i$ of the basis functions of V_i associated with the degrees of freedom of $\partial\Omega_i$.

The FETI method can be applied for solving the global problem (14) [4]. The only difference with the conforming FETI method lies in the fact that, since the B_i matrices are not signed boolean matrices any more, the preconditioning phase must include a scaling taking into account the inhomogeneity induced by the B_i matrices [2].

4 A FETI-2LM Method for Non-conforming Interfaces

In order to avoid dealing with non-conforming interfaces, the multi-point constraints associated with the discrete weak continuity condition:

$$B_1\xi_1 + B_2\xi_2 = 0 \tag{15}$$

may be included inside one subdomain. This means that the targeted subdomain must annex the interface degrees of freedom of the neighboring subdomain. In the case of Fig. 4, it is subdomain Ω_2. The opposite choice could be made as well.

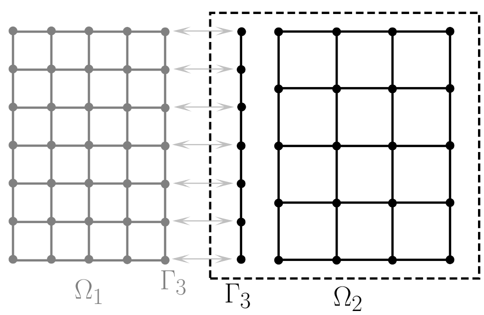

Fig. 4. Inclusion of non-conforming interface in one subdomain.

Then the actual interface between subdomain Ω_1 and extended subdomain Ω_2 is conforming and the weak condition (15) gives multi-point constraints for inner degrees of freedom of extended subdomain Ω_2. This means that the new local stiffness matrices are:

$$\begin{bmatrix} K_1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & M_{31}^t \\ 0 & K_2 & B_2^t \\ M_{31} & B_2 & 0 \end{bmatrix} \tag{16}$$

The restriction matrix R_1 is not present in the mixed matrix of the extended subdomain Ω_2 since only the interface degrees of freedom of Ω_1 have been annexed.

The main apparent issue with this approach is the fact that the matrix of the extended subdomain Ω_2 is highly singular since the annexed degrees of freedom have no stiffness. But, thanks to the generalized Robin conditions on the interface with the FETI-2LM method, each local stiffness matrix is augmented by an approximation of the Schur complement on the interface of the stiffness matrix of the neighboring subdomain:

$$S^{(1)} = K_{33}^{(1)} - K_{31}K_{11}^{-1}K_{13} \qquad S^{(2)} = \begin{bmatrix} 0 \; M_{31}^t \end{bmatrix} \begin{bmatrix} K_2 & B_2^t \\ B_2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ M_{31} \end{bmatrix} \tag{17}$$

If $A_{33}^{(1)} \approx S^{(2)}$ and $A_{33}^{(2)} \approx S^{(1)}$ are approximations of the Schur complements defined in equation (17), the augmented stiffness matrices of the generalized Robin problems of the FETI-2LM method are:

$$\begin{bmatrix} K_1 + R_1^t A_{33}^{(1)} R_1 \end{bmatrix} \qquad \begin{bmatrix} A_{33}^{(2)} & 0 & M_{31}^t \\ 0 & K_2 & B_2^t \\ M_{31} & B_2 & 0 \end{bmatrix} \tag{18}$$

None of these matrices is singular any more, in the case where $A_{33}^{(1)} = S^{(2)}$ and $A_{33}^{(2)} = S^{(1)}$, since they are obtained by eliminating unknowns of Ω_2 or inner unknowns of Ω_2, in the well posed global problem (14). The same property holds in general, provided that $A_{33}^{(1)}$ and $A_{33}^{(2)}$ are consistent enough approximations of $S^{(2)}$ and $S^{(1)}$.

The procedure developed in [5] for computing algebraic approximation of the Schur complement applies without any modification to the case where the local matrix has a mixed form, like the matrix of the extended subdomain Ω_2 in (17).

5 Localization of Non-conforming Interface Matching Conditions

In most cases, non-conforming interfaces exist only for engineering or geometrical reason in a limited area of the computational domain. There can be only one non-conforming interface which splits the entire domain into two unbalanced subdomains. In order to get enough subdomains for the domain decomposition solver to be efficient, each initial non-conforming domain must be split into smaller subdomains in such a way that the total number of subdomains is large enough and that the subdomains are balanced.

Therefore, the initial non-conforming interface may be split into several interfaces. Since each non-conforming interface matching conditions couples several degrees of freedom on each side of the interface, it frequently happens that a multi-point constraint associated with a mortar Lagrange multiplier involves degrees of freedom located in more than one subdomain on each side of the non-conforming interface. This leads to a very serious implementation issue since the FETI methods require that each interface connects one subdomain only on each side.

A solution consists in localizing all degrees of freedom associated with a mortar Lagrange multiplier on each side of the non conforming interface to a single subdomain. This means that some degrees of freedom must be annexed by one neighboring subdomain located on the same side of the non-conforming interface, like in Fig. 5, in which Ω_{ij} denotes j^{th} subdomain arising from the splitting of initial domain Ω_i.

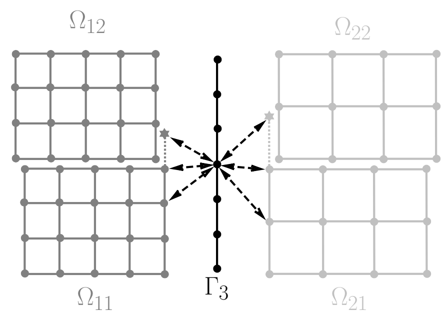

Fig. 5. Localization of non-conforming interface degrees of freedom.

Once again, this procedure adds degrees of freedom with no stiffness, e.g., subdomain Ω_{11}. But the generalized Robin boundary condition on the conforming interface between Ω_{11} and Ω_{12} adds the necessary regularizing terms. And since the FETI-2LM method ensures exact continuity of the solutions along a conforming interface, the value of the solution in Ω_{11} for the annexed degrees of freedom of Ω_{12} is exactly the same as in Ω_{12}.

Thanks to this technique, the initial single non-conforming interface can be split into several non-conforming interfaces, each of them coupling only one subdomain on each side. Therefore, the methodology introduced in section 4 can be applied on each of them.

6 Conclusion

The methodology presented in this paper allows the localization of each multi-point constraint associated with non-conforming interface matching conditions in one subdomain. This localization is to be made in a pre-processing phase. It allows any multi-level splitting of a mesh including a non-conforming interface without any modification in the formulation of the non conforming interface matching conditions.

Thanks to this localization, the FETI-2LM method with the automatic algebraic computation of approximate optimal generalized Robin interface conditions can be implemented without any change from the standard conforming case.

This methodology has been successfully implemented for test problems. Comparison must be made now, in term of convergence speed, between this new non-conforming FETI-2LM method and the classical non-conforming FETI-1LM.

References

[1] Bernardi, C., Maday, Y., Patera, A.T.: Domain decomposition by the mortar element method. In H.G. Kaper ans M. Garbey, editor, *Asymptotic and Numerical Methods for Partial Differential Equations with Critical Parameters*, pages 269–286. N.A.T.O. ASI, Kluwer Academic, 1993.

[2] Farhat, C., Lacour, C., Rixen, D.: Incorporation of linear multipoint constraints in substructure based iterative solvers. part 1 : A numerically scalable algorithm. *Internat. J. Numer. Methods Engrg.*, 43:997–1016, 1998.

[3] Farhat, C., Macedo,A. Lesoinne, M., Roux, F.-X., Magoulès, F., de La Bourdonnaye, A.: Two-level domain decomposition methods with lagrange multipliers for the fast iterative solution of acoustic scattering problems. *Comput. Methods Appl. Mech. Engrg.*, 184:213–240, 2000.

[4] Lacour, C.: Non-conforming domain decomposition method for plate and shell problems. In J. Mandel, C. Farhat, and X.-C. Cai, eds., *Tenth International Conference on Domain Decomposition Methods*, pages 304–310. AMS, Contemporary Mathematics 218, 1998.

[5] Magoulès, F., Roux, F.-X., Series, L.: Algebraic approximation of dirichlet-to-neumann maps for the equations of linear elasticity. *Comput. Methods Appl. Mech. Engrg.*, 195:3742–3759, 2006.