A Parallel Schwarz Method for Multiple Scattering Problems

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1 Introduction

Multiple scattering of waves is one of the important research topics in scientific and industrial fields. A number of numerical methods have been developed to compute waves scattered by several obstacles, e.g., acoustic waves scattered by schools of fish, water waves by ocean structures, and elastic waves by particles in composite materials.

In this paper, we focus on the acoustic scattering, which is described as solutions of boundary value problems of the Helmholtz equation in an unbounded domain. In order to compute such solutions numerically, the following method is well known [2, 6]: one introduces an artificial boundary and imposes an artificial boundary condition on it to reduce the original problem on the unbounded domain to a problem on a bounded domain enclosed by the artificial boundary. Recently, [4] have developed a new method by extending the method above to the multiple scattering problem. In their method, one introduces several disjoint artificial boundaries, each of which surrounds one of the obstacles, and imposes an exact non-reflecting boundary condition on such artificial boundaries. This boundary condition is called the multiple DtN (Dirichlet-to-Neumann) boundary condition.

In this paper, we parallelize their method by a parallel nonoverlapping Schwarz method due to [5]. The original unbounded domain is then decomposed into bounded subdomains, each of which is surrounded by one of the artificial boundaries, and the remaining unbounded subdomain. A particular feature of this method is including a problem in the unbounded subdomain, imposing Sommerfeld's radiation condition. This problem is reduced to a certain problem on the multiple artificial boundaries by the natural boundary reduction due to [2].

The remainder of this paper is organized as follows. In Sect. 2 we introduce the exterior Helmholtz problem, and present a parallel Schwarz algorithm and its convergence theorem, which is proved by the energy method due to [1] in Sect. 5. We introduce the multiple DtN operator associated with the problem on the unbounded subdomain in Sect. 3. We describe how to reduce the problem on the unbounded subdomain to the problem on the multiple artificial boundaries in Sect. 4.

2 Exterior Helmholtz Problem and Schwarz Method

We consider the following exterior Helmholtz problem:

$$\begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega_\infty, \\ u = 0 & \text{on } \bigcup_{j=1}^{J} \partial\mathcal{O}_j, \\ \lim_{r\longrightarrow+\infty} r^{\frac{1}{2}} \left(\dfrac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0, \end{cases} \tag{1}$$

where k is a positive constant, $\mathcal{O}_j$ $(1 \leq j \leq J)$ are bounded open sets of $\mathbb{R}^2$, $\Omega_\infty := \mathbb{R}^2 \setminus \left(\bigcup_{j=1}^{J} \overline{\mathcal{O}_j}\right)$, and f is a given datum. Assume that Ω_∞ is connected, f has a compact support, and $\partial\mathcal{O}_j$ $(1 \leq j \leq J)$ are of class C^∞. Problem (1) has a unique solution belonging to $H^2_{\text{loc}}(\overline{\Omega_\infty})$ for every compactly supported $f \in L^2(\Omega_\infty)$ (see [7]), where

$$H^m_{\text{loc}}(\overline{\Omega_\infty}) := \{u \mid u \in H^m(B) \ \text{ for all bounded open set } B \subset \Omega_\infty\} \quad (m \in \mathbb{N}).$$

2.1 Domain Decomposition

Suppose that for each $1 \leq j \leq J$, there exists a ball B_j with radius a_j and center c_j such that $\overline{\mathcal{O}_j} \subset B_j$, $\operatorname{supp} f \subset \bigcup_{j=1}^{J} B_j$, and $\overline{B_j} \cap \overline{B_l} = \emptyset$ if $j \neq l$. We introduce, for every $1 \leq j \leq J$, artificial boundaries: $\Gamma_j := \left\{x \in \mathbb{R}^2 \mid |x - c_j| = a_j\right\}$ and bounded domains: $\Omega_j := B_j \setminus \overline{\mathcal{O}_j}$. We further introduce the following unbounded domain: $\Omega_0 := \mathbb{R}^2 \setminus \left(\bigcup_{j=1}^{J} \overline{B_j}\right)$. Then we can decompose Ω_∞ into subdomains Ω_0, $\Omega_1, \ldots, \Omega_J$: $\overline{\Omega_\infty} = \bigcup_{j=0}^{J} \overline{\Omega_j}$, and we have $\Omega_j \cap \Omega_l = \emptyset$ if $j \neq l$.

2.2 A Parallel Schwarz Method

To solve problem (1), we consider the following parallel Schwarz method of Lions:

(1) Choose u_0^0 and u_j^0 $(1 \leq j \leq J)$.
(2) For $n = 1, 2, \ldots$, solve

$$\begin{cases} -\Delta u_0^n - k^2 u_0^n = 0 & \text{in } \Omega_0, \\ -\dfrac{\partial u_0^n}{\partial r_j} - \mathrm{i}ku_0^n = -\dfrac{\partial u_j^{n-1}}{\partial r_j} - \mathrm{i}ku_j^{n-1} & \text{on } \Gamma_j \quad (1 \leq j \leq J), \\ \lim_{r\longrightarrow+\infty} r^{1/2} \left(\dfrac{\partial u_0^n}{\partial r} - \mathrm{i}ku_0^n\right) = 0, \end{cases} \tag{2}$$

and for $1 \leq j \leq J$,

$$\begin{cases} -\Delta u_j^n - k^2 u_j^n = f & \text{in } \Omega_j, \\ u_j^n = 0 & \text{on } \partial\mathcal{O}_j, \\ \dfrac{\partial u_j^n}{\partial r_j} - \mathrm{i}ku_j^n = \dfrac{\partial u_0^{n-1}}{\partial r_j} - \mathrm{i}ku_0^{n-1} & \text{on } \Gamma_j. \end{cases} \tag{3}$$

Here, for each $1 \leq j \leq J$, (r_j, θ_j) are polar coordinates with origin c_j, and then the normal derivative on Γ_j is expressed by $\partial/\partial r_j$. As is well known, problems (2) and (3), respectively, have a unique solution.

Theorem 1. *Let u, u_0^n, and u_j^n $(1 \leq j \leq J)$ be the solutions of problems (1), (2), and (3), respectively. If*

$$\left. \frac{\partial u_0^0}{\partial r_j} - \mathrm{i}ku_0^0 \right|_{\Gamma_j} \quad \text{and} \quad \left. -\frac{\partial u_j^0}{\partial r_j} - \mathrm{i}ku_j^0 \right|_{\Gamma_j} \in H^{1/2}(\Gamma_j) \quad (1 \leq j \leq J), \tag{4}$$

then we have $u_0^n \longrightarrow u$ in $L^2(\Gamma)$ and $u_j^n \longrightarrow u$ in $H^1(\Omega_j)$ $(1 \leq j \leq J)$ as $n \longrightarrow +\infty$, where $\Gamma := \bigcup_{j=1}^{J} \Gamma_j$.

3 Multiple DtN Operator

We introduce the multiple DtN operator

$$S : H^{1/2}(\Gamma) \left(\cong \prod_{j=1}^{J} H^{1/2}(\Gamma_j) \right) \longrightarrow H^{-1/2}(\Gamma) \left(\cong \prod_{j=1}^{J} H^{-1/2}(\Gamma_j) \right)$$

defined by

$$S : p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_J \end{bmatrix} \longrightarrow \begin{bmatrix} \left. -\frac{\partial u}{\partial r_1} \right|_{\Gamma_1} \\ \left. -\frac{\partial u}{\partial r_2} \right|_{\Gamma_2} \\ \vdots \\ \left. -\frac{\partial u}{\partial r_J} \right|_{\Gamma_J} \end{bmatrix},$$

where $p_j := p|_{\Gamma_j}$ $(1 \leq j \leq J)$, and u is the solution of the following problem:

$$\begin{cases} -\Delta u - k^2 u = 0 & \text{in } \Omega_0, \\ u = p & \text{on } \Gamma, \\ \lim\limits_{r \longrightarrow +\infty} r^{1/2} \left(\dfrac{\partial u}{\partial r} - \mathrm{i}ku \right) = 0. \end{cases}$$

To represent S in an implicit form involving some operators which can be analytically represented, we introduce, for each $1 \leq j \leq J$, an operator

$$\mathcal{G}_j : H^{1/2}(\Gamma_j) \longrightarrow H^1_{\mathrm{loc}}(\overline{D_j})$$

defined by $\mathcal{G}_j[\varphi_j] := w_j$, where $w_j \in H^1_{\mathrm{loc}}(\overline{D_j})$ is the solution of the following problem:

$$\begin{cases} -\Delta w_j - k^2 w_j = 0 & \text{in } D_j, \\ w_j = \varphi_j & \text{on } \Gamma_j, \\ \lim\limits_{r_j \longrightarrow +\infty} r_j^{\frac{1}{2}} \left(\dfrac{\partial w_j}{\partial r_j} - \mathrm{i}kw_j \right) = 0, \end{cases} \tag{5}$$

where $D_j := \left\{ x \in \mathbb{R}^2 \mid |x - c_j| > a_j \right\}$.

Theorem 2. *If* $u \in H^1_{\mathrm{loc}}(\overline{\Omega_0})$ *satisfies*

$$\begin{cases} -\Delta u - k^2 u = 0 \quad \textit{in } \Omega_0, \\ \lim_{r \longrightarrow +\infty} r^{\frac{1}{2}} \left(\frac{\partial u}{\partial r} - \mathrm{i} k u \right) = 0, \end{cases}$$

then there uniquely exists a $\varphi \in H^{1/2}(\Gamma)$ *such that*

$$u = \sum_{j=1}^{J} \mathcal{G}_j[\varphi_j] \quad \textit{in } \Omega_0. \tag{6}$$

Proof. See [4]. □

Taking the traces and the normal derivatives $\partial/\partial r_j$ of both sides of (6) on Γ_j $(1 \le j \le J)$, we get

$$u = \varphi_j + \sum_{l \neq j} \mathcal{P}_{jl}[\varphi_l] \quad \text{on } \Gamma_j \tag{7}$$

and

$$\frac{\partial u}{\partial r_j} = -\mathcal{S}_j[\varphi_j] - \sum_{l \neq j} \mathcal{T}_{jl}[\varphi_l] \quad \text{on} \quad \Gamma_j, \tag{8}$$

respectively, where

$$\mathcal{P}_{jl}[\varphi_l] := \mathcal{G}_l[\varphi_l]\big|_{\Gamma_j}, \quad -\mathcal{T}_{jl}[\varphi_l] := \frac{\partial}{\partial r_j} \mathcal{G}_l[\varphi_l]\bigg|_{\Gamma_j} \quad (1 \le j \neq l \le J),$$

$$-\mathcal{S}_j[\varphi_j] := \frac{\partial}{\partial r_j} \mathcal{G}_j[\varphi_j]\bigg|_{\Gamma_j} \quad (1 \le j \le J).$$

We here remark that $\mathcal{S}_j$ is the single DtN operator associated with (5). Deleting φ_j from (7) and (8), we can see that the multiple DtN operator S can be represented as follows:

$$S = TC^{-1}, \tag{9}$$

where

$$C := \begin{bmatrix} I & \mathcal{P}_{12} & \cdots & \mathcal{P}_{1J} \\ \mathcal{P}_{21} & I & \cdots & \mathcal{P}_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{P}_{J1} & \mathcal{P}_{J2} & \cdots & I \end{bmatrix}, \qquad T := \begin{bmatrix} \mathcal{S}_1 & \mathcal{T}_{12} & \cdots & \mathcal{T}_{1J} \\ \mathcal{T}_{21} & \mathcal{S}_2 & \cdots & \mathcal{T}_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{T}_{J1} & \mathcal{T}_{J2} & \cdots & \mathcal{S}_J \end{bmatrix}.$$

As we all know, we can obtain an analytical representation of $\mathcal{G}_j$ by separation of variables, and hence, from this representation, we can derive analytical representations of the operators $\mathcal{S}_j$, $\mathcal{T}_{jl}$, and $\mathcal{P}_{jl}$.

Note that we have $C \in \mathrm{Isom}(H^{1/2}(\Gamma), H^{1/2}(\Gamma))$, the proof of which will appear elsewhere.

4 How to Solve Problem (2)

If u_0^n is the solution of problem (2), then we have

$$-\frac{\partial u_0^n}{\partial r_j} = S u_0^n|_{\Gamma_j} \quad (1 \le j \le J).$$

Hence we can reduce problem (2) to the following problem on Γ: find $p \in H^{1/2}(\Gamma)$ such that

$$Sp - \mathrm{i}kp = \lambda, \tag{10}$$

where $\lambda \in H^{-1/2}(\Gamma)$ and $\lambda = -\partial u_j^{n-1}/\partial r_j - \mathrm{i}k u_j^{n-1}$ on Γ_j. This process is often called the *natural boundary reduction* (cf. [2]). The solution p of this problem gives the trace on Γ of the solution u_0^n of problem (2).

By (9), we have

$$(S - \mathrm{i}kI)p = \lambda \Longleftrightarrow (T - \mathrm{i}kC)C^{-1}p = \lambda.$$

Hence, we can solve (10) by executing the following two processes:

- Solve $(T - \mathrm{i}kC)\varphi = \lambda$.
- Compute $p = C\varphi$.

Equation $(T - \mathrm{i}kC)\varphi = \lambda$ can be written in the following variational form: find $\varphi \in H^{1/2}(\Gamma)$ such that

$$\begin{aligned}
&\langle \mathcal{S}_j \varphi_j, \psi_j \rangle_j + \sum_{l \ne j} \int_{\Gamma_j} \mathcal{T}_{jl}[\varphi_l]\, \overline{\psi_j}\, d\Gamma_j \\
&\quad -\mathrm{i}k \left(\int_{\Gamma_j} \varphi_j \overline{\psi_j}\, d\Gamma_j + \sum_{l \ne j} \int_{\Gamma_j} \mathcal{P}_{jl}[\varphi_l]\, \overline{\psi_j}\, d\Gamma_j \right) \\
&= \langle \lambda_j, \psi_j \rangle_j \quad \forall \psi_j \in H^{1/2}(\Gamma_j), \quad 1 \le \forall j \le J,
\end{aligned} \tag{11}$$

where $\langle \cdot, \cdot \rangle_j$ is the duality pairing between $H^{-1/2}(\Gamma_j)$ and $H^{1/2}(\Gamma_j)$.

We can practically compute (11) by using the analytical representations of the operators $\mathcal{S}_j$, $\mathcal{T}_{jl}$, and $\mathcal{P}_{jl}$. Discretizing (11) by a FEM, we need to solve a linear system whose matrix is full and of order the number of nodes on Γ.

5 Proof of Theorem 1

We can prove Theorem 1 by the energy technique due to [1]. Let u, u_0^n, and u_j^n $(1 \le j \le J)$ be the solutions of problems (1), (2), and (3), respectively. Put $e_j^n := u - u_j^n$ $(0 \le j \le J)$. We should keep in mind that if (4) is satisfied, then $e_0^n \in H^2_{\mathrm{loc}}(\overline{\Omega_0})$ and $e_j^n \in H^2(\Omega_j)$ $(1 \le j \le J)$ for all $n \in \mathbb{N}$.

We now define a *pseudo energy* E^n by

$$E^n := \int_\Gamma |Se_0^n - \mathrm{i}ke_0^n|^2 \, d\Gamma + \sum_{j=1}^J \int_{\Gamma_j} \left| \frac{\partial e_j^n}{\partial r_j} - \mathrm{i}ke_j^n \right|^2 \, d\Gamma_j.$$

Lemma 1. $\{E^n\}_{n=1}^{\infty}$ *is a decreasing sequence.*

Proof. Because e_j^n satisfies the homogeneous equation in Ω_j and the homogeneous boundary condition on $\partial\mathcal{O}_j$, we have

$$\operatorname{Im}\left\{\int_{\Gamma_j} \frac{\partial e_j^n}{\partial r_j}\overline{e_j^n}\, d\Gamma_j\right\} = 0 \tag{12}$$

for every $n \in \mathbb{N}$ and for each $1 \leq j \leq J$. Hence, for every $n \in \mathbb{N}$, we have the following *energy equality*:

$$E^{n+1} = E^n + 4k \operatorname{Im}\left\{\int_{\Gamma} S e_0^n \overline{e_0^n}\, d\Gamma\right\}. \tag{13}$$

Since e_0^n also satisfies the homogeneous equation in Ω_0, we have

$$\operatorname{Im}\left\{\int_{\Gamma} S e_0^n \overline{e_0^n}\, d\Gamma\right\} = -kR \sum_{\mu=-\infty}^{\infty} \operatorname{Im}\left\{\frac{H_\mu^{(1)'}(kR)}{H_\mu^{(1)}(kR)}\right\} \left|e_{0,\mu}^n(R)\right|^2 \tag{14}$$

for an arbitrary positive number R satisfying $\Gamma \subset \{x \in \mathbb{R}^2 \mid |x| < R\}$, where $H_\mu^{(1)}$ is the first kind Hankel function of order μ, and $e_{0,\mu}^n(R)$ is the μth Fourier coefficient of e_0^n on $\Gamma^R := \{x \in \mathbb{R}^2 \mid |x| = R\}$. As we all know, we have

$$\operatorname{Im}\left\{\frac{H_\mu^{(1)'}(kR)}{H_\mu^{(1)}(kR)}\right\} > 0 \tag{15}$$

for all $\mu \in \mathbb{Z}$. Therefore, combining (13), (14) and (15) completes the proof of Lemma 1. □

Proposition 1. *We have* $S - \mathrm{i}kI \in \operatorname{Isom}(H^{1/2}(\Gamma),\, H^{-1/2}(\Gamma))$.

Proof. We can prove from the two facts that $S - \mathrm{i}kI$ is a bounded linear operator from $H^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$, and that the following problem:

$$\begin{cases} -\Delta u - k^2 u = 0 \;\text{ in } \Omega_0, \\ -\dfrac{\partial u}{\partial r_j} - \mathrm{i}ku = \lambda \;\text{ on } \Gamma_j \quad (1 \leq j \leq J), \\ \displaystyle\lim_{r \longrightarrow +\infty} r^{1/2}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0 \end{cases}$$

has a unique solution belonging to $H^1_{\mathrm{loc}}(\overline{\Omega_0})$ for every $\lambda \in H^{-1/2}(\Gamma)$. □

Proof of Theorem 1

Proposition 1 assures us that there exists a positive constant C such that

$$\|e_0^{n+1}\|_{H^{1/2}(\Gamma)} \leq C\|\lambda\|_{H^{-1/2}(\Gamma)}, \tag{16}$$

where $\lambda_j = -\partial e_j^n/\partial r_j - \mathrm{i}ke_j^n$. Using (12) and Lemma 1, we have

$$\|\lambda\|^2_{L^2(\Gamma)} = \sum_{j=1}^{J} \int_{\Gamma_j} \left| \frac{\partial e_j^n}{\partial r_j} - \mathrm{i}ke_j^n \right|^2 d\Gamma_j \leq E^1. \tag{17}$$

From (16) and (17), $\{e_0^n\}$ is a bounded sequence in $H^{1/2}(\Gamma)$, and hence $\{e_0^n\}$ has a subsequence $\{e_0^{n_l}\}$ such that $e_0^{n_l} \longrightarrow e_0$ in $H^{1/2}(\Gamma)$ weakly. This indicates that for all $q \in H^{1/2}(\Gamma)$,

$$\int_\Gamma Se_0^{n_l}\overline{q}\, d\Gamma = \langle e_0^{n_l}, S^*q\rangle \longrightarrow \langle e_0, S^*q\rangle = \langle Se_0, q\rangle, \tag{18}$$

where S^* is the DtN operator corresponding to the *incoming* radiation condition, and $\langle\cdot,\cdot\rangle$ is the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

On the other hand, from Lemma 1, (14), and (15), we have

$$\begin{aligned} E^1 &\geq \int_\Gamma |Se_0^n - \mathrm{i}ke_0^n|^2\, d\Gamma \\ &= \int_\Gamma \left\{|Se_0^n|^2 + k^2|e_0^n|^2\right\} d\Gamma - 2k\,\mathrm{Im}\left\{\int_\Gamma Se_0^n\overline{e_0^n}\, d\Gamma\right\} \\ &\geq \int_\Gamma |Se_0^n|^2\, d\Gamma. \end{aligned}$$

This implies that there exists a subsequence of $\{Se_0^{n_l}\}$, still denoted by $\{Se_0^{n_l}\}$, which converges in $L^2(\Gamma)$ weakly. Thus, we can see from (18) that $Se_0 \in L^2(\Gamma)$ and $Se_0^{n_l} \longrightarrow Se_0$ in $L^2(\Gamma)$ weakly. Further, by the compact imbedding of $H^{1/2}(\Gamma)$ in $L^2(\Gamma)$, $e_0^{n_l} \longrightarrow e_0$ in $L^2(\Gamma)$ strongly, and hence we have

$$\int_\Gamma Se_0^{n_l}\overline{e_0^{n_l}}\, d\Gamma \longrightarrow \int_\Gamma Se_0\overline{e_0}\, d\Gamma.$$

Now, from (13), we get

$$E^{n+1} = E^1 + 4k\sum_{m=1}^{n} \mathrm{Im}\left\{\int_\Gamma Se_0^m\overline{e_0^m}\, d\Gamma\right\},$$

and hence $\mathrm{Im}\left\{\int_\Gamma Se_0^m\overline{e_0^m}\, d\Gamma\right\} \longrightarrow 0$. Thereby we have $\mathrm{Im}\left\{\int_\Gamma Se_0\overline{e_0}\, d\Gamma\right\} = 0$. This implies $e_0 = 0$. Therefore, we can conclude that $u_0^n \longrightarrow u$ in $L^2(\Gamma)$.

We can show that $u_j^n \longrightarrow u$ in $H^1(\Omega_j)$ $(1 \leq j \leq J)$ in the same way as in the proof of Theorem 2.6 in [1]. □

6 Concluding Remarks

We demonstrated the convergence of the parallel Schwarz method of Lions for multiple scattering problems. Many techniques of acceleration of the convergence of the Schwarz method have been developed (see [3, 8]). The investigation of acceleration techniques is yet to be done for the Schwarz method presented in this paper.

References

1. B. Després. Domain decomposition method and the Helmholtz problem. In *Mathematical and Numerical Aspects of Wave Propagation Phenomena (Strasbourg, 1991)*, pp. 44–52. SIAM, Philadelphia, PA, 1991.
2. K. Feng. Finite element method and natural boundary reduction. In *Proceeding of the International Congress of Mathematicians*, Warsaw, Poland, 1983.
3. M.J. Gander, F. Magoulès, and F. Nataf. Optimized Schwarz methods without overlap for the Helmholtz equation. *SIAM J. Sci. Comput.*, 24(1):38–60, 2002.
4. M.J. Grote and C. Kirsch. Dirichlet-to-Neumann boundary conditions for multiple scattering problems. *J. Comput. Phys.*, 201(2):630–650, 2004.
5. P.-L. Lions. On the Schwarz alternating method. III: a variant for nonoverlapping subdomains. In T.F. Chan, R. Glowinski, J. Périaux, and O. Widlund, editors, *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations , held in Houston, Texas, March 20–22, 1989*, SIAM, Philadelphia, PA, 1990.
6. M. Masmoudi. Numerical solution for exterior problems. *Numer. Math.*, 51:87–101, 1987.
7. R.S. Phillips. On the exterior problem for the reduced wave equation. In *Partial Differential Equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, CA, 1971)*, pp. 153–160, AMS, Providence, RI, 1973.
8. D. Tromeur-Dervout. Aitken-Schwarz method: acceleration of the convergence of the Schwarz method. In F. Magoulès and T. Kako, editors, *Domain Decomposition Methods: Theory and Applications*, chapter 2, pp. 37–64, Gakkotosho, Tokyo, 2006.