# Monotone Multigrid Methods Based on Parametric Finite Elements

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**Summary.** In this paper, a particular technique for the application of elementary multilevel ideas to problems with warped boundaries is studied in the context of the numerical simulation of elastic contact problems. Combining a general multilevel setting with a different perspective, namely an advanced geometric modeling point of view, we present a (monotone) multigrid method based on a hierarchy of parametric finite element spaces. For the construction, a full-dimensional parameterization of high order is employed which accurately represents the computational domain.

The purpose of the volume parametric finite element discretization put forward here is two-fold. On the one hand, it allows for an elegant multilevel hierarchy to be used in preconditioners. On the other hand, it comes with particular advantages for the modeling of contact problems. After all, the long-term objective lies in an increased flexibility of *hp*-adaptive methods for contact problems.



## 1 Introduction

In the numerical simulation of elastic contact problems, the treatment of the non-penetration conditions at the potential contact boundary is of particular importance for both the quality of a finite element approximation and the overall efficiency of the algorithms. A vital challenge is to achieve an accurate description of geometric features, e.g., of warped surfaces, often incorporated in three-dimensional models from computer-aided design (CAD). Here, we investigate a new connection of different numerical methods, namely modern discretization techniques for partial differential equations on complex geometries on the one side and fast multilevel solvers for constrained minimization problems on the other side.

It is fair to say that the development of *hp*-adaptive methods for contact problems has not yet reached a mature state; see, e.g., [2] and the references therein. Partly, this is due to the difficulties concerning the geometric representation of the computational domain. A generally accepted paradigm is, though, that high order (finite element or boundary element) methods need high order meshes [11, 14]. This is especially difficult for three-dimensional multi-body contact problems. In this case, the application of non-conforming domain decomposition techniques [16] to realize

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an optimal information transfer across geometrically non-matching warped contact interfaces is a highly demanding task. For low order finite elements, this has been achieved, among others, by the authors; see [6].

The perspective we offer here is a parametric finite element method. For *hp*-adaptive methods, it is convenient to have a parameterization describing the geometry accurately ready to hand. This is because a change of the computational domain due to locally altered polynomial degree is not desirable. Therefore, it is reasonable to uncouple the representation of the geometry on the one hand and of a scale of approximation spaces for the discrete solution on the other hand. These two purposes are usually not separated properly. But of course, one can find curved elements of other than isoparametric structure in some form or another in the literature; see, e.g., [8, 17] or the monograph [3] and the references therein. Note that, for similar reasons, an "isogeometric" concept, which uses NURBS bases for both the description of the geometry and the discrete solution of the differential equation, has been introduced in [11].

For practical computations, the development of fast and robust solvers is equally important. As this issue has not yet been in the main focus of, e.g., the isogeometric analysis [11], we would like to contribute ideas from the field of multilevel methods for variational inequalities. More precisely, we show how to use a monotone multigrid method to efficiently solve the non-linear contact problem discretized with low order parametric finite elements. Note that the actual treatment of higher order elements is beyond the scope of the present discussion.

To obtain multilevel parametric finite element spaces in case $d = 3$, we use a full-dimensional parameterization, constructed by tetrahedral transfinite interpolation [15] of CAD data, to lift standard Lagrange elements to the computational domain. Note that, similarly, a surface parameterization has been used in a wavelet Galerkin scheme for boundary integral equations; see [10]. Such a procedure may serve as an essential prerequisite to tackle the problems mentioned above. In particular, many of the issues arising in the generation of *p*-version meshes for curved boundaries [14] can be avoided in a quite elegant way. In this sense, although rather expensive, the use of a high order parameterization permits maximal freedom in an *hp*-adaptive discretization scheme. We presume that the present concept can also be combined with the ideas in [6].

All in all, our results constitute real progress made in the development of an efficient *hp*-adaptive simulation environment for elastic contact problems in case of complex three-dimensional geometries.

## 2 Parametric Finite Elements

In this section, we introduce a parametric finite element discretization. On the one hand, this method uses much more geometric information from a CAD model than standard finite elements; on the other hand, we do not use the same functions for the discrete approximation of the displacement field as for the representation of the geometry, which is done in the so-called "isogeometric analysis" introduced in [11]. We use the associated space hierarchy in Sect. 3 to build a monotone multigrid method for low order elements.

In the following, the symbols $\varphi$ with some indices stand for certain full-dimensional parameterizations or finite element transformations. We denote the (closed) $d$-simplex by $\Delta^d$ and its faces by $\Delta_j^d$, $j \in \{1,\dots,d+1\}$. To describe the elastic body (here, $d = 3$) by a practicable parameterization, we consider a non-overlapping simplicial decomposition of the computational domain $\Omega \subset \mathbb{R}^d$ into a fixed number of $K \geq 1$ subdomains. Formally this reads as

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k = \bigcup_{k=1}^{K} \varphi_k(\Delta^d),$$

where the notation already indicates that the subdomains $(\Omega_k)_{k=1,\dots,K}$ appear as particular images of the simplex $\Delta^d$ under suitable parameterizations $(\varphi_k)_{k=1,\dots,K}$. This is illustrated in Fig. 1 (right).

Let us assume that the faces of the simplicial cells $\Omega_k$, namely the surfaces $\varphi_k(\Delta_j^d)$, $k \in \{1,\dots,K\}$, $j \in \{1,\dots,d+1\}$, are given as $B$-patches. This way to represent polynomial surfaces is analyzed in [4]. In this case, the author of [15] proposes to construct the full-dimensional mappings $\varphi_k : \Delta^d \to \mathbb{R}^d$, $k \in \{1,\dots,K\}$, as transfinite interpolations of the surface values from the CAD model using certain blending functions. Particularly, the single parameterizations are smooth and they match across these $B$-patch surfaces if the surfaces themselves match. This gives rise to a consistent global parameterization which we do not write down explicitly. We note that this global mapping is continuous but not necessarily differentiable across the interior interfaces. In addition, one can guarantee that each parameterization $\varphi_k$ satisfies the regularity assumption

$$\det(\nabla \varphi_k) > 0 \quad \text{in } \Delta^d. \tag{1}$$

In fact, this is one of the main results of [15].

In the following, we define the parametric finite element spaces in a rather straightforward way via a lift of standard Lagrange finite elements. For this purpose, let $(\mathscr{T}_\ell^k)_{\ell\in\mathbb{N}}$ be a family of nested simplicial meshes of $\Delta^d$ for each $k \in \{1,\dots,K\}$. To keep the global finite element spaces conforming, we assume that, at each level $\ell \in \mathbb{N}$, the meshes meeting at the faces of the simplicial subdomains $\Omega_k$ of $\Omega$ match. Let $\widehat{T}$ be the reference element; here, $\widehat{T} = \Delta^d$. Then, for each $T_\Delta \in \mathscr{T}_\ell^k$, there is an affine mapping $\varphi_{T_\Delta} : \widehat{T} \to \Delta^d$ such that $\varphi_{T_\Delta}(\widehat{T}) = T_\Delta$.

Now, we give a concise description of the parametric elements in $\Omega$ by employing the special finite element transformations

$$\varphi_T := \varphi_k \circ \varphi_{T_\Delta} : \widehat{T} \to \mathbb{R}^d, \tag{2}$$

which are diffeomorphisms between the reference element $\widehat{T}$ and the actual elements. That way, the parametric elements at level $\ell \in \mathbb{N}$ are identified as the images of the elements of the meshes $(\mathscr{T}_\ell^k)_{k=1,\dots,K}$; see Fig. 1. More precisely, a family of parametric meshes $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}}$ of $\Omega$ can be defined by

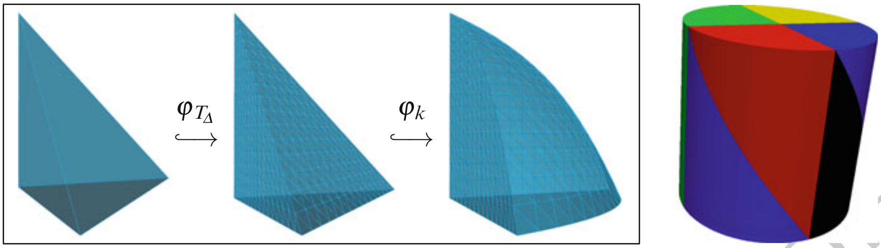



**Fig. 1.** From left to right: the reference element $\widehat{T} = \Delta^3$; a mesh of the simplex $\Delta^3$; a parametric mesh (here, $K = 1$) where each element is an image of an affine element; a tetrahedral decomposition of a cylinder with $K = 8$

$$\mathscr{T}_\ell := \left\{ T = \varphi_T(\widehat{T}) = \varphi_k(\varphi_{T_\Delta}(\widehat{T})) \mid 1 \le k \le K,\, T_\Delta \in \mathscr{T}_\ell^k \right\}, \quad \forall\, \ell \in \mathbb{N}.$$

Assume that this family of global meshes is shape regular and quasi-uniform. Note that assumption (1), combined with the continuous differentiability of the mappings $(\varphi_k)_{k=1,\dots,K}$ in the compactum $\Delta^d$, implies that it is sufficient to ensure these regularity conditions for each sequence $(\mathscr{T}_\ell^k)_{\ell\in\mathbb{N}}$ separately as far as we keep $K$ fixed.

Finally, let $\mathbb{P} := \mathbb{P}_r(\widehat{T})$ be the space of polynomials of degree $r$ in $\widehat{T}$. Then, for $\ell \in \mathbb{N}$, the parametric finite element space associated with the parametric mesh $\mathscr{T}_\ell$ is

$$\begin{aligned} X_\ell &:= \left\{ v \in \mathscr{C}^0(\Omega) \mid \forall\, T \in \mathscr{T}_\ell\ \exists\, w \in \mathbb{P} :\ v(\boldsymbol{x}) = w(\varphi_T^{-1}(\boldsymbol{x})),\ \forall\, \boldsymbol{x} \in T \right\} \\ &= \left\{ v \in \mathscr{C}^0(\Omega) \mid v \circ \varphi_T \in \mathbb{P},\ \forall\, T \in \mathscr{T}_\ell \right\}. \end{aligned} \tag{3}$$

Note that, in principle, the above definition makes sense for any reasonable set of finite element transformations $(\varphi_T)_{T\in\mathscr{T}_\ell}$. In case the mappings are constructed as in (2) via the high order parameterization from [15], this is a "superparametric" concept if the degree $r$ is small. This is in contrast to the subparametric or isoparametric finite elements which are usually considered in the literature; see [3].

From a practical point of view, virtually every kind of parameterization can be employed with the following qualification. For an efficient assembly of the stiffness matrix and the right hand side via sufficiently accurate (at best exact) numerical quadrature, the derivatives of the resulting finite element transformations (2) and the mappings themselves must be easy to evaluate; see, e.g., [1].

## Discretization of Signorini's Problem

Let us now apply the above concept to a contact problem in elasticity to find the deformation of a linear elastic body $\Omega$ in contact with a rigid obstacle. For this purpose, let the boundary be decomposed into pairwise disjoint parts: $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C$. Assume that the Dirichlet boundary $\Gamma_D$ is of positive Lebesgue measure in dimension $d-1$. Moreover, the condition $\overline{\Gamma}_C \cap \overline{\Gamma}_D = \emptyset$ may hold.

Let $\boldsymbol{n}$ be the outer normal vector field on $\partial\Omega \in \mathscr{C}^1$; the initial gap to the rigid obstacle in this direction is given as a function $g : \Gamma_C \to \mathbb{R}_{\ge 0}$. Then, for sufficiently smooth prescribed volume and surface force densities $\boldsymbol{f} = (f_i)$ and $\boldsymbol{p} = (p_i)$, the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ solves the boundary value problem

$$\begin{aligned} -\sigma_{ij}(\boldsymbol{u})_{,j} &= f_i \quad \text{in } \Omega, \\ \boldsymbol{u} &= \mathbf{0} \quad \text{on } \Gamma_D, \\ \sigma_{ij}(\boldsymbol{u}) n_j &= p_i \quad \text{on } \Gamma_N, \\ \boldsymbol{u} \cdot \boldsymbol{n} &\leq g \quad \text{on } \Gamma_C, \end{aligned} \tag{4}$$

where $\sigma_{ij}(\boldsymbol{u}) = A_{ijlm} u_{l,m}$ are the stresses and $\mathbf{A} = (A_{ijlm})$ is Hooke's tensor. The existence of a unique weak solution follows from Lions' and Stampacchia's lemma.

We use the vector-valued parametric finite element space $\boldsymbol{X}_\ell := (X_\ell)^d$ defined by (3) with $r = 1$ and denote the set of nodes by $\mathcal{N}_\ell$. As usual, the non-penetration conditions on the possible contact boundary $\Gamma_C$ are merely enforced at the potential contact nodes $\mathcal{N}_\ell^C = \mathcal{N}_\ell \cap \Gamma_C$; see below. Then, a discretization of Signorini's problem (4) with one-sided constraints is obtained by specifying a variational inequality

$$\text{find } \boldsymbol{u}_\ell \in \boldsymbol{K}_\ell \text{ such that } a(\boldsymbol{u}_\ell, \boldsymbol{v} - \boldsymbol{u}_\ell) \geq f(\boldsymbol{v} - \boldsymbol{u}_\ell), \ \forall \boldsymbol{v} \in \boldsymbol{K}_\ell, \tag{5}$$

on a suitable set of admissible displacements

$$\boldsymbol{K}_\ell := \left\{ \boldsymbol{v} \in \boldsymbol{X}_\ell \,|\, \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_D, \ (\boldsymbol{v} \cdot \boldsymbol{n})(p) \leq g(p), \ \forall p \in \mathcal{N}_\ell^C \right\}.$$

In the discrete variational inequality (5), the (bi-)linear forms $a$ and $f$ representing the elastic energy and the applied forces, respectively, are given by $a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega A_{ijlm} u_{l,m} v_{i,j} \, d\boldsymbol{x}$ and $f(\boldsymbol{v}) := \int_\Omega f_i v_i \, d\boldsymbol{x} + \int_{\Gamma_N} p_i v_i \, d\boldsymbol{a}$.

Although, from a modeling point of view, as much geometric information as possible should be used for an accurate description of contact phenomena, we remark that a strong pointwise non-penetration condition everywhere on $\Gamma_C$ is usually not suitable for the variational formulation on which the (parametric) finite element method relies. Besides, a decoupled set of constraints is preferable for a variety of reasons. The common remedy is to prescribe the contact constraints with respect to a suitable cone of Lagrange multipliers. This requires the introduction of appropriate sets of functionals in $(H^{\frac{1}{2}}(\Gamma_C))'$. To retain inequality constraints which can be enforced merely by looking at the nodes, one can employ discontinuous test spaces described, e.g., in [7].

The quality of a priori error estimates for the above discretization certainly depends on a number of aspects which have to be examined more closely. Beside regularity assumptions for the continuous solution, the balance of the primal degrees of freedom and the constraints by means of an inf-sup condition and certain properties of the parameterization, e.g., the regularity (1), influence the error analysis.

## 3 Monotone Multigrid Method for Parametric Elements

Similarly to some of the approaches reviewed in [5, Chap. 4], the scale of parametric finite element spaces constitutes an adjusted discretization technique which allows for an almost straightforward application of multilevel ideas. In this section, we examine the constructed space hierarchy, which we presume to possess the required approximation properties, and the corresponding natural transfer operators in a little more detail.

For the solution of the discrete variational inequality, we propose a monotone multigrid method [12]; see [13] for an overview of this and other solution strategies for contact problems and more references. Here, the non-penetration conditions at the potential contact nodes are treated by a non-linear block Gauß–Seidel smoother at the finest level $L$. Let $\tilde{\boldsymbol{u}} \in \boldsymbol{K}_L$ be a preliminary approximate solution (i.e., a current admissible iterate). Then, in the next step, a linear multilevel preconditioner depending on $\tilde{\boldsymbol{u}}$ is employed, which acts only on the space $\{\boldsymbol{v} \in \boldsymbol{X}_L \,|\, (\boldsymbol{v} \cdot \boldsymbol{n})(p) = 0,\ \forall\, p \in \mathcal{N}_L^C \text{ with } (\tilde{\boldsymbol{u}} \cdot \boldsymbol{n})(p) = g(p)\}$. The construction of the required coarse spaces from the spaces $(\boldsymbol{X}_\ell)_{\ell<L}$ involves local modifications of the coarse level matrices resulting from recursively truncated basis functions; see, e.g., [13].

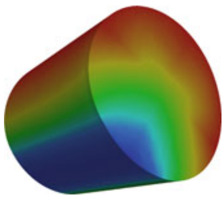
By construction, the spaces defined by (3) are nested. This is an immediate consequence of the fact that the parameterization is fixed and does not change with the index $\ell$. Still, let us formulate this statement in the following lemma and give an elementary proof of the assertion.

**Lemma 1.** *The parametric finite element spaces* $(X_\ell)_{\ell\in\mathbb{N}}$ *are nested.*

*Proof*. For $\ell \geq 1$, let $v \in X_{\ell-1}$ be arbitrary. Then, for $T \in \mathcal{T}_{\ell-1}$ there is a unique element $T_\Delta \in \mathcal{T}_{\ell-1}^k$ for some $k \in \{1,\ldots,K\}$ such that $\varphi_k(T_\Delta) = T$. Let $(T_\Delta^i)_{i=1,\ldots,N}$ be the children of $T_\Delta$ in $\mathcal{T}_\ell^k$. In general, $1 \leq N \leq 2^d$; in case of standard uniform refinement of the simplices, it is $N = 2^d$. We have the corresponding set of elements $(T^i)_{i=1,\ldots,N}$ in $\mathcal{T}_\ell$ with $T^i = \varphi_k(T_\Delta^i)$ for $i \in \{1,\ldots,N\}$. By assumption, $v \circ \varphi_T = v \circ \varphi_k \circ \varphi_{T_\Delta} \in \mathbb{P}$. Therefore, it is $v \circ \varphi_{T^i} = v \circ \varphi_k \circ \varphi_{T_\Delta^i} \in \mathbb{P}$ because $T_\Delta^i \subset T_\Delta$ and the finite element transformations are affine. As each element of $\mathcal{T}_\ell$ appears as the child of an element in $\mathcal{T}_{\ell-1}$ in the above fashion, we obtain $v \in X_\ell$. Consequently, $X_{\ell-1} \subset X_\ell$ for all $\ell \geq 1$.
□

Therefore, no advanced transfer concepts need to be studied here as the canonical inclusion $\mathcal{I}_{\ell-1}^\ell : X_{\ell-1} \to X_\ell$ is the most natural operator to be used as prolongation. Note that these operators only depend on the logical structure; as in the standard nested case, the representing matrices contain the entries 0, 0.5 and 1 and may be computed from the neighborhood relations in and between the simplicial meshes $(\mathcal{T}_{\ell-1}^k)_{k=1,\ldots,K}$ and $(\mathcal{T}_\ell^k)_{k=1,\ldots,K}$. This is because the respective multilevel basis is defined via a lift by proceeding as in (3). As a result, for a fixed finest level $L$, the computation of the matrices $\mathbf{I}_{\ell-1}^\ell \in \mathbb{R}^{|\mathcal{N}_\ell| \times |\mathcal{N}_{\ell-1}|}$ for $\ell \in \{1,\ldots,L\}$ between the nested spaces $(X_\ell)_{\ell=0,\ldots,L}$ does not need the parameterization. However, the computation of the outer normals $(\boldsymbol{n}(p))_{p\in\mathcal{N}_L^C}$ and also of the values $(g(p))_{p\in\mathcal{N}_L^C}$ for the prescription of the contact constraints may require access to the mappings $(\varphi_k)_{k=1,\ldots,K}$.

We anticipate that the constructed coarse spaces have the desired multilevel approximation properties. More precisely, under mild assumptions on the employed parameterization mappings $(\varphi_k)_{k=1,\ldots,K}$, the relevant Jackson- and Bernstein-type in-



| $L$ | #elements | #dof | #steps | $\tilde{\rho}$ | $\lvert\mathscr{A}_L\rvert$ |
|---|---|---|---|---|---|
| 0 | 96 | 107 | 8 (2) | 0.032 | 3 |
| 1 | 768 | 615 | 10 (3) | 0.031 | 15 |
| 2 | 6,144 | 3,915 | 11 (4) | 0.065 | 58 |
| 3 | 49,152 | 27,795 | 13 (6) | 0.091 | 199 |
| 4 | 393,216 | 209,187 | 14 (6) | 0.102 | 753 |
| 5 | 3,145,728 | 1,622,595 | 15 (8) | 0.114 | 2,984 |

**Fig. 2.** Contact problem of a parameterized cylinder with a rigid obstacle shaped like a broad channel. The *colors* indicate the displacement in $\boldsymbol{e}_3$-direction. Problem (5) is solved by a conjugate gradient method preconditioned by the monotone multigrid method ($\mathscr{V}(3,3)$-cycle)

equalities transfer from the standard finite element spaces to the parametric spaces; see also [9].

Finally, we point out that no modifications are necessary in the code of the solver provided that the local normal/tangential coordinate systems can be computed from the parameterization. Consequently, a monotone multigrid method can be employed for contact problems discretized with parametric finite elements in the quite straightforward way outlined above. Figure 2 shows a numerical example illustrating the performance of the method for $d = 3$. The number of active nodes where the constraints are binding is denoted by $|\mathscr{A}_L|$. We report on the asymptotic convergence rate $\tilde{\rho}$ of a conjugate gradient method preconditioned by the monotone multigrid method ($\mathscr{V}(3,3)$-cycle). Starting with the initial iterate zero at each refinement level (i.e., no nested iteration), we list the number of total steps needed to reduce the norm of the residual to less than $10^{-10}$. The count of included non-linear steps is given in brackets (e.g., for $L = 5$, the active set is found after 8 of the 15 cycles such that the remaining 7 steps are linear). Note that the pcg error reduction rate $\tilde{\rho}$ corresponds to this linear iteration phase where the active set has already been identified.

## 4 Conclusion

The results described in this paper certainly have preliminary character; the performance of the presented algorithms needs to be studied in more detail. This is work in progress. However, the experiments so far show that (monotone) multigrid methods based on parametric finite elements work as expected; see Fig. 2. Still, the effort of constructing a (high order) parameterization by the methodology developed in [15] especially pays if there is also a considerable gain on the modeling side. Here, the effect of this special resolution of the boundary on the discrete approximation of contact phenomena or general boundary effects needs to be investigated more closely.

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