Sharp Condition Number Estimates for the Symmetric 2-Lagrange Multiplier Method

Stephen W. Drury[1] and Sébastien Loisel[2]

[1] Dept. of Mathematics, McGill University, 805 Sherbrooke Street West, Montreal, Quebec, Canada, H3A 2K6, drury@math.mcgill.ca
[2] Dept. of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom, S.Loisel@hw.ac.uk

Summary. Domain decomposition methods are used to find the numerical solution of large boundary value problems in parallel. In optimized domain decomposition methods, one solves a Robin subproblem on each subdomain, where the Robin parameter a must be tuned (or optimized) for good performance. We show that the 2-Lagrange multiplier method can be analyzed using matrix analytical techniques and we produce sharp condition number estimates.

1 Introduction

Consider the model problem

$$-\Delta u = f \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is the domain, f is a given forcing and $u \in H_0^1(\Omega)$ is the unknown solution. In the present paper, we describe a symmetric 2-Lagrange multiplier (S2LM) domain decomposition method to solve elliptic problems such as (1). When we discretize (1) using e.g. piecewise linear finite elements, we obtain a linear system of the form

$$A\mathbf{u} = \mathbf{f}, \tag{2}$$

where $\mathbf{u} \in \mathbb{R}^n$ is the finite element coefficient vector of the approximation to the solution u of (1).

We now consider the domain decomposition [9] $\Omega = \Gamma \cup \Omega_1 \cup \ldots \cup \Omega_p$, where $\Omega_1, \ldots, \Omega_p$ are the (open, disjoint) "subdomains" and $\Gamma = \Omega \cap \bigcup_{k=1}^{p} \partial\Omega_k$ is the "artificial interface". We introduce the "local problems"

$$\begin{cases} -\Delta u_k = f & \text{in } \Omega_k, \quad \text{(PDE)} \\ u_k = 0 & \text{on } \partial\Omega_k \cap \partial\Omega, \quad \text{(natural b.c.)} \\ (a + D_\nu)u_k = \lambda_k & \text{on } \partial\Omega_k \cap \Gamma, \quad \text{(artificial b.c.)} \end{cases} \tag{3}$$

where $a > 0$ is the Robin tuning parameter and $k = 1, \ldots, p$ and D_ν denotes the directional derivative in the outwards pointing normal ν of $\partial\Omega_k$. The interface Γ is

R. Bank et al. (eds.), *Domain Decomposition Methods in Science and Engineering XX*, Lecture Notes in Computational Science and Engineering 91,
DOI 10.1007/978-3-642-35275-1__29, © Springer-Verlag Berlin Heidelberg 2013

artificial in that it is not a natural part of the "physical problem" (1) but instead is introduced purely for the purpose of calculation.

We again discretize the systems (3) using a finite element method. The Robin b.c. in (3) gives rise to a mass matrix on the interface $\Gamma \cap \partial\Omega_k$, which we lump. If the grid is uniform, this mass matrix is aI (we absorb any h factors into the a coefficient) – we make this simplification for the remainder of the present paper.

$$\begin{bmatrix} A_{IIk} & A_{I\Gamma k} \\ A_{\Gamma Ik} & A_{\Gamma\Gamma k}+aI \end{bmatrix} \overbrace{\begin{bmatrix} \mathbf{u}_{Ik} \\ \mathbf{u}_{\Gamma k} \end{bmatrix}}^{\mathbf{u}_k} = \overbrace{\begin{bmatrix} \mathbf{f}_{Ik} \\ \mathbf{f}_{\Gamma k} \end{bmatrix}}^{\mathbf{f}_k} + \begin{bmatrix} 0 \\ \boldsymbol{\lambda}_k \end{bmatrix}. \tag{4}$$

Here, we have used the suggestive subscripts I for interior nodes and Γ for the artificial interface nodes.

The FETI-2LM algorithm was introduced in [4] for cases without cross-points, while the general case including cross points was introduced and analyzed in [7]. The method consists of finding the value of $\boldsymbol{\lambda} = [\boldsymbol{\lambda}_1^T,\ldots,\boldsymbol{\lambda}_p^T]^T$ which yields solutions $\mathbf{u}_1,\ldots,\mathbf{u}_p$ to (4) in such a way that $\mathbf{u}_1,\ldots,\mathbf{u}_p$ meet continuously across Γ and glue together into the unique solution $\mathbf{u}$ of (2).

The main result of the present paper is a new estimate of the condition number of FETI-2LM algorithms using matrix analytical techniques. This new idea produces sharp condition number estimates with much more straightforward proof techniques than the techniques used in [7] (where the estimates are not sharp). As a result, the present paper is a logical follow-up to [7].

The present paper focuses on 1-level algorithms which are known not to scale. Scalable algorithms are considered in [8] and [3].

Our paper is organized as follows. In Sect. 2, we give the symmetric 2-Lagrange multiplier method for general domains with cross points. In Sect. 3, we give spectral estimates including our main result, Theorem 1, on the condition number of the symmetric 2-Lagrange multiplier system. In Sect. 4, we verify this Theorem with some numerical experiments.

2 The Symmetric 2-Lagrange Multiplier Method

We now describe the 2-Lagrange multiplier method that we analyze in the present paper. Consider the local problems (4) and eliminate the interior degrees of freedom to obtain the relation

$$a\overbrace{\begin{bmatrix} \mathbf{u}_{\Gamma 1} \\ \vdots \\ \mathbf{u}_{\Gamma p} \end{bmatrix}}^{\mathbf{u}_G} = \overbrace{\begin{bmatrix} a(S_1+aI)^{-1} & & \\ & \ddots & \\ & & a(S_p+aI)^{-1} \end{bmatrix}}^{Q} \left(\overbrace{\begin{bmatrix} \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_p \end{bmatrix}}^{\mathbf{g}} + \overbrace{\begin{bmatrix} \boldsymbol{\lambda}_1 \\ \vdots \\ \boldsymbol{\lambda}_p \end{bmatrix}}^{\boldsymbol{\lambda}} \right), \tag{5}$$

where

$$S_k = A_{\Gamma\Gamma k} - A_{\Gamma Ik}A_{IIk}^{-1}A_{I\Gamma k} \quad \text{and} \quad \mathbf{g}_k = \mathbf{f}_{\Gamma k} - A_{\Gamma Ik}A_{IIk}^{-1}\mathbf{f}_{Ik}$$

are the "Dirichlet-to-Neumann maps" and "accumulated right-hand-sides" and where $\mathbf{u}_{\Gamma j}$ denotes those degrees of freedom of the local solution $\mathbf{u}_j$ associated with the artificial interface Γ.

The matrices S_k are symmetric and semidefinite. Since $Q = a(S+aI)^{-1}$, we find that the spectrum $\sigma(Q)$ is contained in the set $[\varepsilon, 1-\varepsilon] \cup \{1\}$ for some $\varepsilon > 0$. The eigenvalue 1 of Q comes from the kernel of S and hence the kernel of $Q-I$ is spanned by the indicating functions of the subdomains that "float".

2.1 Relations Between (4) and (2) and Continuity

We define the boolean restriction matrix R_k by selecting rows of the $n \times n$ identity matrix corresponding to those vertices of Ω that are in $\bar{\Omega}_k \cap \Omega$. As a result, from a finite element coefficient vector $\mathbf{v}$ corresponding to a finite element function $v \in H_0^1(\Omega)$, we can define a finite element coefficient vector $\mathbf{v}_k = R_k \mathbf{v}$, which corresponds to a finite element function $v \in H^1(\Omega_k) \cap H_0^1(\Omega)$, which is obtained by restricting v to Ω_k.

The identity $\int_\Omega = \sum_{k=1}^p \int_{\Omega_k}$ induces the following relations between (4) and (2):

$$A = \sum_{k=1}^{p} R_k^T \overbrace{\begin{bmatrix} A_{IIk} & A_{I\Gamma k} \\ A_{\Gamma Ik} & A_{\Gamma\Gamma k} \end{bmatrix}}^{A_{Nk}} R_k \quad \text{and} \quad \mathbf{f} = \sum_{k=1}^{p} R_k^T \mathbf{f}_k. \tag{6}$$

Each interface vertex $\mathbf{x}_i \in \Gamma$ is adjacent to $m_i \geq 2$ subdomains. As a result, the "many-sided trace" $\mathbf{u}_G$ defined by (5) contains m_i entries corresponding to $\mathbf{x}_i$, one per subdomain adjacent to $\mathbf{x}_i$. We define the orthogonal projection matrix K which averages function values for each interface vertex $\mathbf{x}_i$. A many-sided trace $\mathbf{u}_G$ corresponds to local functions $\mathbf{u}_1, \ldots, \mathbf{u}_p$ that meet continuously across Γ if and only if

$$K\mathbf{u}_G = \mathbf{u}_G. \tag{7}$$

2.2 A Problem in $\boldsymbol{\lambda}$

The **symmetric 2-Lagrange multiplier** (S2LM) system is given by

$$(Q-K)\boldsymbol{\lambda} = -Q\mathbf{g}. \tag{8}$$

We further let E be the orthogonal projection onto the kernel of $Q-I$.

Lemma 1. *Assume that $\|EK\| < 1$. The problem (2) is equivalent to (8).*

Proof. In order to solve (2) using local problems (4), one should find Robin boundary values $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_p$ which result in local solutions $\mathbf{u}_1, \ldots, \mathbf{u}_p$ that meet continuously across Γ. As a result, we impose the condition (7), which we multiply by $a > 0$ and convert to an expression in $\boldsymbol{\lambda}$ using (5) to obtain $Ka(S+aI)^{-1}(\boldsymbol{\lambda}+\mathbf{g}) = a(S+aI)^{-1}(\boldsymbol{\lambda}+\mathbf{g})$ or

$$(I-K)Q\boldsymbol{\lambda} = (K-I)Q\mathbf{g} \tag{9}$$

With this continuity condition, there is clearly a unique $\mathbf{u}$ which restricts to the $\mathbf{u}_j$:

$$\mathbf{u}_j = R_j\mathbf{u}, \quad j = 1,\ldots,p. \tag{10}$$

Imposing continuity is not sufficient, we must also ensure that the "fluxes" match. Indeed, if we impose on the solution $\mathbf{u}$ of (10) that the Eq. (2) should hold, one obtains

$$\mathbf{f} = A\mathbf{u} \stackrel{(6)}{=} \sum_{j=1}^{p} R_j^T A_{Nj} R_j \mathbf{u} \stackrel{(10)}{=} \sum_{j=1}^{p} R_j^T A_{Nj} \mathbf{u}_j \tag{11}$$

$$\stackrel{(4),(6)}{=} \mathbf{f} + \sum_{j=1}^{p} R_j^T \begin{pmatrix} 0 \\ \boldsymbol{\lambda}_j - a\mathbf{u}_{\Gamma j} \end{pmatrix} \tag{12}$$

Canceling the $\mathbf{f}$ terms on each side and multiplying by K, we obtain $K\boldsymbol{\lambda} - Ka\mathbf{u}_G = 0$. Using (5), we obtain

$$K(Q-I)\boldsymbol{\lambda} = -KQ\mathbf{g}. \tag{13}$$

We add (9) and (13) to obtain (8).

To see that the solution of (8) is unique, observe that the ranges of E and K intersect trivially by the hypothesis that $\|EK\| < 1$. As a result, the eigenspace of Q of eigenvalue 1 intersects trivially with the range of K and $Q-K$ is nonsingular. □

We will further discuss the choice of the parameter a in Sect. 3.1.

3 Spectral Estimates

If we use GMRES or MINRES on the symmetric indefinite system (8), the residual norm can be estimated as a function of the condition number of $Q-K$, cf. [2]. In order to estimate the condition number of $Q-K$, we begin by giving a canonical form for the pair of projections E and K.

Lemma 2. *Let E and K be orthogonal projections. There is a choice of orthonormal basis that block diagonalizes E and K simultaneously and such that the blocks E_k and K_k of E and K satisfy*

$$E_k \in \left\{0, 1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\} \quad \textit{and} \quad K_k \in \left\{0, 1, \begin{bmatrix} c_k^2 & c_k s_k \\ c_k s_k & s_k^2 \end{bmatrix}\right\}, \tag{14}$$

where $c_k = \cos\theta_k > 0$, $s_k = \sin\theta_k > 0$ and $\theta_k \in (0, \pi/2)$ is a "principal angle" relating E and K.

The canonical form (14) can be obtained from the CS decomposition [1] by starting from $E = \operatorname{diag}(I, 0)$ and picking orthonormal bases for the range and kernel of K. Due to space constraints, we omit this argument.

We also give a technical lemma which describes the spectrum of a sum of certain symmetric matrices.

Lemma 3. *Let X, Y be symmetric matrices of dimensions $m \times m$. Let $0 < y_{\min} < y_{\max}$ and assume that $|\sigma(Y)| \subset [y_{\min}, y_{\max}]$. Denote by $\rho(X)$ the spectral radius of X and assume that $\rho(X) < y_{\min}$. Then,*

$$|\sigma(X+Y)| \subset [y_{\min} - \rho(X), y_{\max} + \rho(X)]. \tag{15}$$

Proof. This follows from a Theorem of Weyl [5, Theorem 4.3.1, pp. 181–182]. □

3.1 Condition Number of $Q - K$

We now come to our main result.

Theorem 1. *Let $\varepsilon > 0$. Assume that $\sigma(Q) \subset [\varepsilon, 1-\varepsilon] \cup \{1\}$. Let E, K be orthogonal projections and assume that $\|EK\| < 1$. Then we have the sharp estimates*

$$|\sigma(Q-K)| \subset \left[\frac{\varepsilon + \sqrt{(1+\varepsilon)^2 - 4\|EK\|^2\varepsilon} - 1}{2}, 1\right], \quad \textit{and} \tag{16}$$

$$\kappa(Q-K) \leq \frac{2}{\varepsilon + \sqrt{(1+\varepsilon)^2 - 4\|EK\|^2\varepsilon} - 1} = O((1-\|EK\|)^{-1}\varepsilon^{-1}). \tag{17}$$

Proof. Let $X = Q - \frac{1}{2}I - \varepsilon E$ and $Y = \frac{1}{2}I + \varepsilon E - K$. Then, $Q - K = X + Y$ and we are in a position to use Lemma 3. We now estimate the spectral properties of X and Y.

Spectral properties of X: Recall that E projects onto the eigenspace of Q with eigenvalue 1. As a result, after some orthonormal change of basis, we find that $Q = \operatorname{diag}(Q_0, I)$ and $E = \operatorname{diag}(0, I)$ and hence

$$\rho(X) \leq \frac{1}{2} - \varepsilon. \tag{18}$$

Spectral properties of Y: Lemma 2 shows that E and K block diagonalize simultaneously and Y is also block diagonal in the same basis. Using (14), we find that the kth block Y_k of Y is given by

$$Y_k = \begin{cases} \frac{1}{2} & \text{if } E_k = K_k = 0, \\ -\frac{1}{2} & \text{if } E_k = 0,\ K_k = 1, \\ \frac{1}{2} + \varepsilon & \text{if } E_k = 1,\ K_k = 0, \\ \begin{bmatrix} \frac{1}{2} + \varepsilon - c_k^2 & -c_k s_k \\ -c_k s_k & \frac{1}{2} - s_k^2 \end{bmatrix} & \text{otherwise;} \end{cases} \tag{19}$$

where the case $E_k = K_k = 1$ is excluded by the hypothesis that $\|EK\| < 1$. As a result, the eigenvalues of Y_k are in the set $\{\pm\frac{1}{2}, \frac{1}{2} + \varepsilon, \lambda_\pm(c_k^2)\}$, where

$$\lambda_\pm(c_k^2) = \frac{\varepsilon \pm \sqrt{(1+\varepsilon)^2 - 4c_k^2\varepsilon}}{2}. \tag{20}$$

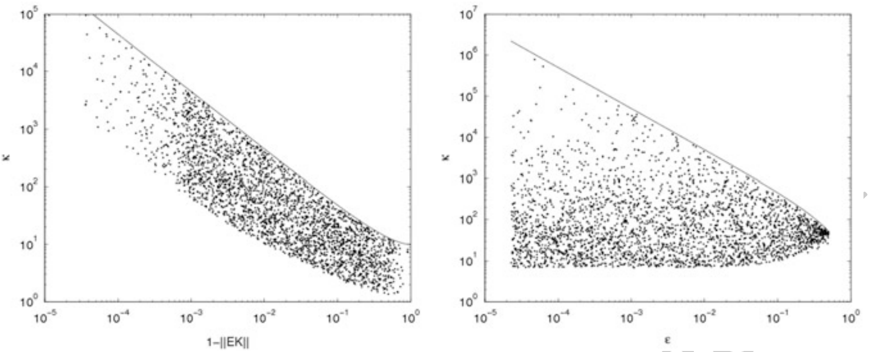

Fig. 1. Comparing random $Q-K$ (points) versus the estimate (17) (solid). *Left*: $\varepsilon = 0.1$, varying $\|EK\|$, 3,000 repetitions. *Right*: $\|EK\| = 0.99$, varying ε, 3,000 repetitions

Note that $\|EK\| = \sqrt{\rho(EKE)} = \max_k c_k$ and that the functions $\lambda_\pm(c_k^2)$ are monotonic in c_k^2. Hence, we find the following bounds for the modulus of an eigenvalue of Y:

$$|\sigma(Y)| \subset \left[\overbrace{\frac{\sqrt{(1+\varepsilon)^2 - 4\|EK\|^2\varepsilon} - \varepsilon}{2}}^{y_{\min}}, \overbrace{\frac{1}{2}+\varepsilon}^{y_{\max}} \right]. \tag{21}$$

Combining (15), (18), and (21) gives (16).

The examples $Q = \operatorname{diag}(1, 1-\varepsilon)$ and $K = \begin{bmatrix} c^2 & c\sqrt{1-c^2} \\ c\sqrt{1-c^2} & 1-c^2 \end{bmatrix}$ for $c = 0$ and $c = \|EK\|$ give the extreme eigenvalues of (21) and hence our estimates are sharp. $\square$

In view of Theorem 1, the Robin parameter a should be chosen so as to make ε as large as possible. This occurs precisely when a is the geometric mean of the extremal positive eigenvalues of S. More details can be found in [7].

4 Numerical Verification

We verify numerically the validity of Theorem 1 by generating random 5×5 matrices Q and E as follows. We set $Q = \operatorname{diag}(\varepsilon, q, 1-\varepsilon, 1, 1)$ where q is chosen randomly between ε and $1-\varepsilon$. We generate randomly a 2-dimensional space and set K to be the orthogonal projection onto that space. We compare the resulting condition number $\kappa = \kappa(Q-K)$ against (17), cf. Fig. 1.

We observe that our estimates are correct and sharp for such "generic" random matrices, although some "lucky" random matrices produce much milder condition numbers than our estimates.

5 Conclusions

We have analyzed a domain decomposition method with optimized Robin boundary conditions. Our estimates rely on new matrix analytical techniques and are sharp. By further estimating the quantities $\|EK\|$ and ε (cf. [7]) our estimates are consistent with and generalize the estimates calculated using Fourier transforms in the optimized Schwarz literature (e.g. [6]). An upcoming paper [8] will further analyze the weak scaling property of a 2-level algorithm and large-scale implementations are being developed. There are also several remaining open problems, such as the analysis of FETI-2LM for nonsymmetric and/or nonlinear problems and the analysis of substructuring preconditioners.

Bibliography

[1] Chandler Davis and W. M. Kahan. Some new bounds on perturbation of subspaces. *Bulletin of the American Mathematical Society*, pages 863–868, 1969.

[2] Tobin A. Driscoll, Kim-Chuan Toh, and Lloyd N. Trefethen. From potential theory to matrix iterations in six steps. *SIAM Review*, pages 547–578, 1998.

[3] Stephen W. Drury and Sébastien Loisel. The performance of optimized Schwarz and 2-Lagrange multiplier preconditioners for GMRES. *Manuscript*, 2011.

[4] Charbel Farhat, Antonini Macedo, Michel Lesoinne, Francois-Xavier Roux, Frédéric Magoulès, and Armel de La Bourdonnaie. Two-level domain decomposition methods with lagrange multipliers for the fast iterative solution of acoustic scattering problems. *Computer Methods in Applied Mechanics and Engineering*, 184:213–239, 2000.

[5] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, 1990.

[6] S. Loisel, J. Côté, M. J. Gander, L. Laayouni, and A. Qaddouri. Optimized domain decomposition methods for the spherical Laplacian. *SIAM Journal on Numerical Analysis*, 48:524–551, 2010.

[7] Sébastien Loisel. Condition number estimates for the nonoverlapping optimized Schwarz method and the 2-Lagrange multiplier method for general domains and cross points. *Submitted to SIAM Journal on Numerical Analysis*, 2011.

[8] Sébastien Loisel. Condition number estimates and weak scaling for 2-level 2-Lagrange multiplier methods for general domains and cross points. *Submitted*, 2011.

[9] Andrea Toselli and Olof B. Widlund. *Domain Decomposition Methods – Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer Berlin Heidelberg, 2005.