A Hybrid Discontinuous Galerkin Method for Darcy-Stokes Problems

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Summary. We propose and analyze a hybrid discontinuous Galerkin method for the solution of incompressible flow problems, which allows to deal with pure Stokes, pure Darcy, and coupled Darcy-Stokes flow in a unified manner. The flexibility of the method is demonstrated in numerical examples.

1 Model Problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain in $d = 2$ or 3 dimensions. Given data $\mathbf{f} \in [L^2(\Omega)]^d$ and $g \in L^2(\Omega)$, we consider the generalized Stokes problem

$$\sigma \mathbf{u} - 2\mu \operatorname{div} \varepsilon(\mathbf{u}) + \nabla p = \mathbf{f} \qquad \text{and} \qquad \operatorname{div} \mathbf{u} = g \quad \text{in } \Omega. \tag{1}$$

As usual, $\mathbf{u}$ denotes the velocity, p the pressure, and $\varepsilon(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the symmetric part of the velocity gradient tensor. We require that

$$\sigma \geq 0, \quad \mu \geq 0, \quad \text{and} \quad M \geq \sigma + \mu \geq m > 0 \qquad \text{in } \Omega.$$

For convenience, we assume that σ, the reciprocal of the permeability, and the viscosity μ are constant, and consider homogeneous boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0 \quad \text{if} \quad \mu > 0 \qquad \text{or} \qquad \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0 \quad \text{if} \quad \mu = 0. \tag{2}$$

The unique solvability of the boundary value problem (1)–(2) is guaranteed, if the pressure p and the data g have zero average. For the case $\mu > 0$, we then have $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$, where $\mathbf{H}_0^1(\Omega) := \{\mathbf{v} \in [H^1(\Omega)]^d : \mathbf{v}|_{\partial\Omega} = 0\}$ and $L_0^2 := \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\}$. In the Darcy limit $\mu = 0$, we only have $\mathbf{u} \in \mathbf{H}_0(\operatorname{div};\Omega) := \{\mathbf{v} \in [L^2(\Omega)]^d : \operatorname{div} \mathbf{v} \in L^2(\Omega),\ \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\}$.

For the approximation of problem (1)–(2), we consider a hybrid discontinuous Galerkin method, which is capable of treating incompressible flow in the Stokes

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and Darcy regimes, as well as coupled problems in a unified manner. Our analysis extends the results of [7] for Stokes flow. Related work on stabilized non-conforming and discontinuous Galerkin methods for Darcy-Stokes flow can be found in [4, 8] and the references given there. We refer to [1, 5] for a unified treatment of discontinuous Galerkin methods for elliptic problems and their hybridization.

2 Notation and Preliminaries

Let $\mathscr{T}_h = \{T\}$ be a shape-regular quasi-uniform partition of Ω into affine families of triangles and/or quadrilaterals (tetrahedra and/or hexahedra) of size $\mathtt{h}$. By $\partial\mathscr{T}_h := \{\partial T : T \in \mathscr{T}_h\}$, we denote the set of element boundaries, and by $\mathscr{E}_h := \{E_{ij} = \partial T_i \cap \partial T_j : i > j\} \cup \{E_{i,0} = \partial T_i \cap \partial\Omega\}$ the set of edges (faces) between elements or on the boundary; $\mathscr{E} = \bigcup_{E\in\mathscr{E}_h} E$ is called the *skeleton*.

For $s \geq 0$, let $H^s(\mathscr{T}_h) := \{v \in L^2(\Omega) : v|_T \in H^s(T) \text{ for all } T \in \mathscr{T}_h\}$ denote the broken Sobolev space with inner product $(u,v)_{s,\mathscr{T}_h} := \sum_{T\in\mathscr{T}_h}(u,v)_{s,T}$ and norm $\|u\|_{s,\mathscr{T}_h}$; the subindex is omitted for $s = 0$. Piecewise defined derivatives are denoted with the standard symbols. The traces of functions in $H^1(\mathscr{T}_h)$ lie in $L^2(\partial\mathscr{T}_h)$, which is equipped with the scalar product $\langle u,v\rangle_{\partial\mathscr{T}_h} := \sum_{T\in\mathscr{T}_h}\langle u,v\rangle_{\partial T}$ and norm $|v|_{\partial\mathscr{T}_h}$. Any function in $L^2(\mathscr{E})$ can be identified with a function in $L^2(\partial\mathscr{T}_h)$ by doubling its values on the element interfaces. Bold symbols are used for vector valued functions.

Let $\mathscr{P}_{\mathtt{p}}(T)$ denote the polynomials of degree $\leq \mathtt{p}$ over T, and recall that

$$|v_{\mathtt{p}}|^2_{\partial T} \leq c_T\, \mathtt{p}^2 \mathtt{h}^{-1} \|v_{\mathtt{p}}\|^2_T \qquad \text{for all } v_{\mathtt{p}} \in \mathscr{P}_{\mathtt{p}}(T). \tag{3}$$

Explicit bounds for the constant c_T in the discrete trace inequaliy (3) are known for all elements under consideration. The parameter c_T can be replaced by the shape regularity parameter $\gamma := \max\{c_T : T \in \mathscr{T}_h\}$, which is assumed to be independent of $\mathtt{h}$. We then choose a stabilization parameter α such that

$$4\gamma \mathtt{p}^2 \mathtt{h}^{-1} \leq \alpha \leq 4\gamma' \mathtt{p}^2 \mathtt{h}^{-1}, \tag{4}$$

with γ' independent of $\mathtt{p}$ and $\mathtt{h}$, and we define two norms on $L^2(\partial\mathscr{T}_h)$ by

$$|v|_{\pm 1/2,\partial\mathscr{T}_h} := \Big(\sum\nolimits_{T\in\mathscr{T}_h} |v|^2_{\pm 1/2,\partial T}\Big)^{1/2} \quad \text{with} \quad |v|_{\pm 1/2,\partial T} := \alpha^{\pm 1/2}|v|_{\partial T}.$$

Similar norms are frequently used for the analysis of mixed, non-conforming and discontinuous Galerkin methods; see e.g. [1].

3 The Hybrid DG Method

Let us fix $\mathtt{p} \geq 1$, and choose $\mathtt{q} = \mathtt{p} - 1$ or $\mathtt{q} = \mathtt{p}$. For the approximation of velocity and pressure in (1)–(2), we will utilize the finite element spaces

$$\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{L}^2(\mathscr{T}_h) : \mathbf{v}_h|_T \in [\mathscr{P}_{\mathrm{p}}(T)]^d \quad \text{for all } T \in \mathscr{T}_h\},$$
$$Q_h := \{q_h \in L^2_0(\Omega) : q_h|_T \in \mathscr{P}_{\mathrm{q}}(T) \quad \text{for all } T \in \mathscr{T}_h\}.$$

We further choose $\hat{\mathrm{p}} = \mathrm{p}$ or $\hat{\mathrm{p}} = \mathrm{q}$, and define a space

$$\widehat{\mathbf{V}}_h := \{\widehat{\mathbf{v}}_h \in \mathbf{L}^2(\mathscr{E}) : \widehat{\mathbf{v}}_h|_E \in [\mathscr{P}_{\hat{\mathrm{p}}}(E)]^d \quad \text{for all } E \in \mathscr{E}_h,\ \ \widehat{\mathbf{v}}_h = 0 \text{ on } \partial\Omega\},$$

of piecewise polynomials for representing velocities on the skeleton. The conditions $\mathrm{p}-1 \le \mathrm{q} \le \mathrm{p}$ and $\mathrm{q} \le \hat{\mathrm{p}}$ are explicitly used in the analysis of a Fortin operator; see Proposition 5. In view of Lemma 1, we also require that $\hat{\mathrm{p}} \ge 1$. Note that the Dirichlet boundary condition has been included explicitly in the definition of the hybrid space $\widehat{\mathbf{V}}_h$. We further denote by $\pi^{\mathrm{p}} : \mathbf{H}^1(\mathscr{T}_h) \to \mathbf{V}_h$ and $\widehat{\pi}^{\hat{\mathrm{p}}} : \mathbf{L}^2(\mathscr{E}) \to \widehat{\mathbf{V}}_h$, the L^2 orthogonal projections onto the discrete spaces. The boundary value problem (1)–(2) is then approximated by the following finite element scheme.

Method 1. Find $\mathbf{u}_h \in \mathbf{V}_h$, $\widehat{\mathbf{u}}_h \in \widehat{\mathbf{V}}_h$, and $p_h \in Q_h$, such that

$$\begin{cases} \mathbf{a}_h(\mathbf{u}_h,\widehat{\mathbf{u}}_h;\mathbf{v}_h,\widehat{\mathbf{v}}_h) + \mathbf{b}_h(\mathbf{v}_h,\widehat{\mathbf{v}}_h;p_h) = (\mathbf{f},\mathbf{v}_h)_{\mathscr{T}_h}, \\ \mathbf{b}_h(\mathbf{u}_h,\widehat{\mathbf{u}}_h;q_h) \qquad\qquad\qquad\qquad\ \ = (g,q_h)_{\mathscr{T}_h}, \end{cases}$$

for all $\mathbf{v}_h \in \mathbf{V}_h$, $\widehat{\mathbf{v}}_h \in \widehat{\mathbf{V}}_h$, and $q_h \in Q_h$. The bilinear forms are defined as

$$\mathbf{a}_h(\mathbf{u},\widehat{\mathbf{u}};\mathbf{v},\widehat{\mathbf{v}}) := \sigma\mathbf{d}_h(\mathbf{u},\widehat{\mathbf{u}};\mathbf{v},\widehat{\mathbf{v}}) + 2\mu\mathbf{s}_h(\mathbf{u},\widehat{\mathbf{u}};\mathbf{v},\widehat{\mathbf{v}}),$$
$$\mathbf{b}_h(\mathbf{v},\widehat{\mathbf{v}};q) := -(\operatorname{div}\mathbf{v},q)_{\mathscr{T}_h} + \langle \mathbf{v}-\widehat{\mathbf{v}}, q\mathbf{n}\rangle_{\partial\mathscr{T}_h},$$

and the bilinear forms $\mathbf{d}_h$ and $\mathbf{s}_h$ are given by

$$\mathbf{d}_h(\mathbf{u},\widehat{\mathbf{u}};\mathbf{v},\widehat{\mathbf{v}}) := (\mathbf{u},\mathbf{v})_{\mathscr{T}_h} + \alpha\langle(\widehat{\pi}^{\hat{\mathrm{p}}}\mathbf{u}-\widehat{\mathbf{u}})\cdot\mathbf{n}, (\widehat{\pi}^{\hat{\mathrm{p}}}\mathbf{v}-\widehat{\mathbf{v}})\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h},$$
$$\mathbf{s}_h(\mathbf{u},\widehat{\mathbf{u}};\mathbf{v},\widehat{\mathbf{v}}) := (\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathscr{T}_h} - \langle\varepsilon(\mathbf{u})\cdot\mathbf{n}, \mathbf{v}-\widehat{\mathbf{v}}\rangle_{\partial\mathscr{T}_h}$$
$$\qquad - \langle\mathbf{u}-\widehat{\mathbf{u}}, \varepsilon(\mathbf{v})\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h} + \alpha\langle\widehat{\pi}^{\hat{\mathrm{p}}}\mathbf{u}-\widehat{\mathbf{u}}, \widehat{\pi}^{\hat{\mathrm{p}}}\mathbf{v}-\widehat{\mathbf{v}}\rangle_{\partial\mathscr{T}_h}.$$

One easily verifies that any regular solution of (1)–(2) also satisfies the discrete variational principle above.

Proposition 1 (Consistency). *Let* $(\mathbf{u},p)$ *denote a solution of* (1)–(2)*, and assume additionally that* $\mathbf{u} \in \mathbf{H}^2(\mathscr{T}_h)$ *and* $p \in H^1(\mathscr{T}_h)$*. Then*

$$\mathbf{a}_h(\mathbf{u},\mathbf{u};\mathbf{v}_h,\widehat{\mathbf{v}}_h) + \mathbf{b}_h(\mathbf{v}_h,\widehat{\mathbf{v}}_h;p) = (\mathbf{f},\mathbf{v}_h)_{\mathscr{T}_h} \quad \text{and} \quad \mathbf{b}_h(\mathbf{u},\mathbf{u};q_h) = (g,q_h)_{\mathscr{T}_h}$$

for all $\mathbf{v}_h \in \mathbf{V}_h$*,* $\widehat{\mathbf{v}}_h \in \widehat{\mathbf{V}}_h$*, and* $q_h \in Q_h$*; thus, Method 1 is consistent.*

In the Darcy limit $\mu = 0$, it suffices to require $\mathbf{u} \in \mathbf{H}^1(\mathscr{T}_h)$.

4 Stability and Error Analysis

An important ingredient for our analysis will be the following result.

Lemma 1 (Discrete Korn inequality). *Let* $\hat{\mathtt{p}} \geq 1$. *Then there is a* $\kappa > 0$ *independent of* $\mathtt{h}$, *such that for all* $\mathbf{v} \in \mathbf{H}^1(\mathscr{T}_h)$ *and* $\widehat{\mathbf{v}} \in \mathbf{L}^2(\mathscr{E})$, *there holds*

$$\|\varepsilon(\mathbf{v})\|^2_{\mathscr{T}_h} + |\widehat{\pi}^{\hat{\mathtt{p}}}(\mathbf{v} - \widehat{\mathbf{v}})|^2_{1/2,\partial\mathscr{T}_h} \geq \kappa \|\nabla \mathbf{v}\|^2_{\mathscr{T}_h}. \tag{5}$$

Proof. The statement follows via the triangle inequality from Korn's inequality for piecewise H^1 functions [3, Eq. (1.12)] established by Brenner. □

Proposition 2. *For any* $(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h \times \widehat{\mathbf{V}}_h$ *there holds*

$$\mathbf{s}_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; \mathbf{v}_h, \widehat{\mathbf{v}}_h) \geq \min\{\tfrac{5}{12}, \tfrac{\kappa}{4}\}\big(\|\nabla \mathbf{u}\|^2_{\mathscr{T}_h} + |\widehat{\pi}^{\hat{\mathtt{p}}}(\mathbf{u} - \widehat{\mathbf{u}})|^2_{1/2,\partial\mathscr{T}_h}\big).$$

Proof. By Young's inequality, Eq. (3) and (4), we obtain

$$-2\langle \varepsilon(\mathbf{v}_h) \cdot \mathbf{n}, \mathbf{v}_h - \widehat{\mathbf{v}}_h \rangle_{\partial T} \geq -\tfrac{3}{4}\|\varepsilon(\mathbf{v}_h)\|^2_T - \tfrac{1}{3}|\widehat{\pi}^{\hat{\mathtt{p}}}(\mathbf{v}_h - \widehat{\mathbf{v}}_h)|^2_{1/2,\partial T}.$$

The result then follows by Lemma 1, and the definition of $\mathbf{s}_h$. □

For appropriately characterizing the coercivity of the bilinear form $\mathbf{d}_h$, let us introduce the discrete kernel space for the bilinear form $\mathbf{b}_h$, namely

$$\mathbf{K}_h := \{(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h \times \widehat{\mathbf{V}}_h : \mathbf{b}_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; q_h) = 0 \ \forall q_h \in Q_h\}.$$

Proposition 3. *For any pair of functions* $(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{K}_h$ *there holds*

$$\mathbf{d}_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; \mathbf{v}_h, \widehat{\mathbf{v}}_h) \geq \|\mathbf{v}_h\|^2_{\mathscr{T}_h} + \|\operatorname{div} \mathbf{v}_h\|^2_{\mathscr{T}_h} + \tfrac{3}{4}|\widehat{\pi}^{\hat{\mathtt{p}}}(\mathbf{v}_h - \widehat{\mathbf{v}}_h) \cdot \mathbf{n}|^2_{1/2,\partial\mathscr{T}_h}.$$

Proof. Note that for every $T \in \mathscr{T}_h$ we have $\operatorname{div} \mathbf{v}_h|_T \in \mathscr{P}_{\mathtt{q}}(T)$. Testing with $q_h = \operatorname{div} \mathbf{v}_h$ and using (3) yields

$$\|\operatorname{div} \mathbf{v}_h\|^2_T = \langle (\mathbf{v}_h - \widehat{\mathbf{v}}_h) \cdot \mathbf{n}, \operatorname{div} \mathbf{v}_h \rangle_{\partial T} \leq \tfrac{1}{2}|(\widehat{\pi}^{\hat{\mathtt{p}}}\mathbf{v}_h - \widehat{\mathbf{v}}_h) \cdot \mathbf{n}|_{1/2,\partial T} \|\operatorname{div} \mathbf{v}_h\|_T,$$

and hence $\|\operatorname{div} \mathbf{v}_h\|_{\mathscr{T}_h} \leq \frac{1}{2}|(\widehat{\pi}^{\hat{\mathtt{p}}}\mathbf{v}_h - \widehat{\mathbf{v}}_h) \cdot \mathbf{n}|_{1/2,\partial\mathscr{T}_h}$. The result then follows by adding and subtracting $\|\operatorname{div} \mathbf{v}_h\|^2_{\partial\mathscr{T}_h}$ from the bilinear form $\mathbf{d}_h$. □

The two coercivity estimates suggest to utilize the following energy norms for the stability analysis of Method 1, namely, $\|q\|_{0,\mathscr{T}_h}$ and

$$\begin{aligned}\|(\mathbf{v}, \widehat{\mathbf{v}})\|^2_{1,\mathscr{T}_h} := \sigma\big(&\|\mathbf{v}\|^2_{\mathscr{T}_h} + \|\operatorname{div} \mathbf{v}\|^2_{\mathscr{T}_h} + |\widehat{\pi}^{\hat{\mathtt{p}}}(\mathbf{v} - \widehat{\mathbf{v}}) \cdot \mathbf{n}|^2_{1/2,\partial\mathscr{T}_h}\big) \\ &+ \mu\big(\|\nabla \mathbf{v}\|^2_{\mathscr{T}_h} + |\widehat{\pi}^{\hat{\mathtt{p}}}(\mathbf{v} - \widehat{\mathbf{v}})|^2_{1/2,\partial\mathscr{T}_h}\big).\end{aligned}$$

Remark 1. If $\mu = 0$, then $\|(\cdot,\cdot)\|_{1,\mathscr{T}_h}$ is only a semi-norm on $\mathbf{V}_h \times \widehat{\mathbf{V}}_h$. This deficiency can be overcome by eliminating the tangential velocities in the definition of the hybrid space, or by penalizing also the jump of the tangential velocities in the bilinear form $\mathbf{d}_h$. Both remedies do not affect our analysis.

A combination of Propositions 2 and 3 now yields the kernel ellipticity for $\mathbf{a}_h$.

Proposition 4 (Coercivity). *For any element* $(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{K}_h$ *there holds*

$$\mathbf{a}_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; \mathbf{v}_h, \widehat{\mathbf{v}}_h) \geq \min\{\tfrac{3}{4}, \tfrac{\kappa}{2}\} \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|^2_{1,\mathscr{T}_h}.$$

The constants C_i appearing in the following results depend on the bounds m and M, but are else independent of the parameters μ, σ, and of $\mathtt{h}$ and $\mathtt{p}$. Let us next consider the operator $(\pi^{\mathrm{p}}, \widehat{\pi}^{\hat{\mathrm{p}}}) : \mathbf{H}^1_0(\Omega) \to \mathbf{V}_h \times \widehat{\mathbf{V}}_h$.

Proposition 5 (Fortin operator). *For any* $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ *there holds*

$$b_h(\pi^{\mathrm{p}}\mathbf{v}, \widehat{\pi}^{\hat{\mathrm{p}}}\mathbf{v}; q_h) = b(\mathbf{v}, q_h) \quad \forall q_h \in Q_h, \tag{6}$$

and

$$\|(\pi^{\mathrm{p}}\mathbf{v}, \widehat{\pi}^{\hat{\mathrm{p}}}\mathbf{v})\|_{1,\mathscr{T}_h} \leq C_\pi \, \mathtt{p}^{1/2} \|\mathbf{v}\|_{1,\Omega}. \tag{7}$$

Proof. Equation (6) follows from the properties of the projections, and (7) results from stability estimates for the L^2 projections; cf. [7] for details. □

The inf-sup stability of $\mathbf{b}_h$ now follows directly from the previous result.

Proposition 6 (Inf-sup condition). *For any* $q_h \in Q_h$ *there holds*

$$\sup_{(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h \times \widehat{\mathbf{V}}_h} \frac{\mathbf{b}_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; q_h)}{\|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_{1,\mathscr{T}_h}} \geq C_\beta \, \mathtt{p}^{-1/2} \|q_h\|_{0,\mathscr{T}_h}. \tag{8}$$

As a consequence of Propositions 4 and 6, we obtain by Brezzi's theorem that Method 1 has a unique solution and thus is well-defined. Next, we show the boundedness of the bilinear forms with respect to a pair of stronger norms defined by $|||q_h|||^2_{0,\mathscr{T}_h} := \|q_h\|^2_{\mathscr{T}_h} + |q_h \cdot \mathbf{n}|^2_{-1/2,\partial\mathscr{T}_h}$ and

$$|||(\mathbf{v}_h, \widehat{\mathbf{v}}_h)|||^2_{1,\mathscr{T}_h} := \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|^2_{1,\mathscr{T}_h} + \mu |\partial_{\mathbf{n}} \mathbf{v}_h|^2_{-1/2,\partial\mathscr{T}_h},$$

The norms $\|\cdot\|_{0,\mathscr{T}_h}$, $\|(\cdot,\cdot)\|_{1,\mathscr{T}_h}$ and $|||\cdot|||_{0,\mathscr{T}_h}$, $|||(\cdot,\cdot)|||_{1,\mathscr{T}_h}$ are equivalent on the finite element spaces with equivalence constants less than two. This yields coercivity and inf-sup stability of $\mathbf{a}_h$ and $\mathbf{b}_h$ also with respect to the stronger norms. The following bounds then follow from the Cauchy-Schwarz inequality.

Proposition 7 (Boundedness). *For any* $\widehat{\mathbf{u}}$, $\widehat{\mathbf{v}} \in \widehat{\mathbf{V}}_h \oplus \mathbf{L}^2(\mathscr{E})$ *and every function* $\mathbf{u}$, $\mathbf{v} \in \mathbf{V}_h \oplus (\mathbf{H}^1_0(\Omega) \cap \mathbf{H}^2(\mathscr{T}_h))$, *there holds*

$$\mathbf{a}_h(\mathbf{u}, \widehat{\mathbf{u}}; \mathbf{v}, \widehat{\mathbf{v}}) \leq C_a |||(\mathbf{u}, \widehat{\mathbf{u}})|||_{1,\mathscr{T}_h} |||(\mathbf{v}, \widehat{\mathbf{v}})|||_{1,\mathscr{T}_h},$$

and for all $p \in Q_h \oplus (L^2_0(\Omega) \cap H^1(\mathscr{T}_h))$, *there holds additionally*

$$\mathbf{b}_h(\mathbf{u}, \widehat{\mathbf{u}}; p) \leq C_b |||(\mathbf{u}, \widehat{\mathbf{u}})|||_{1,\mathscr{T}_h} |||p|||_{0,\mathscr{T}_h}.$$

Standard polynomial approximation results [2] imply the following properties.

Proposition 8 (Approximation). *Assume $s \geq 1$. Then for any function $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{s+1}(\mathscr{T}_h)$ there exist elements $\mathbf{v}_h \in \mathbf{V}_h$ and $\widehat{\mathbf{v}}_h \in \widehat{\mathbf{V}}_h$ such that*

$$|||(\mathbf{u} - \mathbf{v}_h, \mathbf{u} - \widehat{\mathbf{v}}_h)|||_{1,\mathscr{T}_h} \leq C_{ap}\, \mathtt{p}^{1/2-s} h^{\min\{\mathtt{p},s\}} \|\mathbf{u}\|_{s+1,\mathscr{T}_h},$$

and for any $p \in L_0^2(\Omega) \cap H^s(\mathscr{T}_h)$ there exists a $q_h \in Q_h$ such that

$$|||p - q_h|||_{0,\mathscr{T}_h} \leq C_{ap}\, \mathtt{p}^{-s} h^{\min\{s,\mathtt{q}+1\}} \|p\|_{s,\mathscr{T}_h}.$$

The a-priori estimates now follow by combination of the previous results.

Proposition 9 (Error estimate). *Let $(\mathbf{u}, p)$ be the solution of (1)–(2), and let $(\mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h)$ denote the approximation defined by Method 1. Then*

$$\begin{aligned} |||(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \widehat{\mathbf{u}}_h)|||_{1,\mathscr{T}_h} &+ \mathtt{p}^{-1/2} |||p - p_h|||_{0,\mathscr{T}_h} \\ &\leq C_{err}\, \mathtt{p}^{1/2} h^{\min\{\mathtt{p},s\}} \left(\mathtt{p}^{1/2-s} \|\mathbf{u}\|_{s+1,\mathscr{T}_h} + \mathtt{p}^{-s} \|p\|_{s,\mathscr{T}_h}\right). \end{aligned}$$

Proof. The result follows with the usual arguments from the consistency, discrete stability, and boundedness of the bilinear forms, and the approximation properties of the finite element spaces; for details, see [7] or [9].

5 Remarks

The analysis of Sect. 4 applies almost verbatim to spatially varying material parameters μ and σ. In particular, a coupling of Darcy and Stokes equations in different parts of the domain is possible and treated automatically. A numerical example for such a case is presented in the next section.

Our results can be extended to shape regular meshes and varying polynomial degree. Also meshes with a bounded number of hanging nodes on each edge or face, and even more general non-conforming mortar meshes can be treated. We refer to [6, 7] for a detailed discussion of conditions on the mesh and polynomial degree distribution.

The coercivity and boundedness estimates also hold for more general finite element spaces, but we explicitly utilized the complete discontinuity of the spaces in the proof of the inf-sup condition. Other constructions of a Fortin-operator, cf. e.g. [9], would allow to relax this assumption.

Our analysis also covers equal order approximations $\mathtt{q} = \mathtt{p}$, which are stabilized sufficiently by the jump penalty terms.

All degrees of freedom except the piecewise constant pressure and the hybrid velocities can be eliminated by static condensation on the element level. This leads to small global systems, which for $\hat{\mathtt{p}} = 0$ exhibit the same sparsity pattern as a non-conforming $P_1 - P_0$ discretization. For $\hat{\mathtt{p}} = 0$, the discrete Korn inequality (5) is not valid, so this choice had to be excluded here. If $\varepsilon(\mathbf{u})$ in (1) is replaced by $\frac{1}{2}\nabla\mathbf{u}$, we however obtain a stable scheme.

6 Numerical Results

Let us now illustrate the capability of the proposed method to deal with incompressible flow in various regimes. Our numerical results were obtained with an implementation of Method 1 in NGSolve.[3]

As a first example, we consider the generalized Stokes equation (1) on the unit square $\Omega = (-1,1)^2$ with a known analytic solution given by

$$\mathbf{u} = \left(20xy^3, 5x^4 - 5y^4\right), \quad p = 60x^2y - 20y^3, .$$

The data $\mathbf{f}$ and g can be obtained from Eq. (1). For the numerical solution, we employed Method 1 with $\mathtt{p} = \hat{\mathtt{p}} = 2$ and $\mathtt{q} = 1$ on a sequence of uniformly refined meshes for different values of μ and σ. The analytic solution was used to compute the errors listed in Table 1. As predicted by the theory, we can observe the optimal quadratic convergence.

Table 1. Energy errors obtained by simulation on a sequence of uniformly refined meshes for $(\sigma,\mu) \in \{(1,0),(\frac{1}{2},\frac{1}{2}),(0,1)\}$, resembling Darcy, Brinkman, and Stokes flow.

level	Darcy	rate	Brinkman	rate	Stokes	rate
0	4.3996	–	3.4058	–	3.8578	–
1	1.1261	1.96	0.8628	1.98	0.9764	1.98
2	0.2799	2.00	0.2146	2.00	0.2428	2.00
3	0.0678	2.04	0.0533	2.00	0.0603	2.00

As a second test case, we consider a coupled Darcy-Stokes flow on a domain consisting of two subdomains Ω_D and Ω_S, as depicted in Fig. 1. The flow in the subdomains is governed by

$$\sigma_i \mathbf{u}_i - 2\mu_i \operatorname{div} \varepsilon(\mathbf{u}_i) + \nabla p_i = 0 \quad \text{and} \quad \operatorname{div} \mathbf{u}_i = 0 \qquad \text{in } \Omega_i,$$

with $\mu_D = 0$ in the Darcy domain Ω_D, and $\sigma_S = 0$ in the Stokes domain Ω_S, and the subproblems are coupled across the interface $\partial\Omega_D \cap \partial\Omega_S$ through

$$\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n}, \;\; p_S - 2\mu(\varepsilon(\mathbf{u}_S)\cdot\mathbf{n})\cdot\mathbf{n} = p_D, \;\; \mathbf{u}_S \cdot \tau + 2\gamma(\varepsilon(\mathbf{u}_S)\cdot\mathbf{n})\cdot\tau = 0.$$

For $\gamma = 0$, these conditions arise naturally when considering the generalized Stokes problem (1) with discontinuous coefficients. In the case $\gamma \neq 0$ the third *Beaver-Joseph-Saffman* condition, which restricts the tangential components of the normal stresses, gives rise to an additional interface term that has to be included in the definition of the bilinear form $\mathbf{a}_h$; for details see [8] and the references given there.

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[3] visit: http://sourceforge.net/apps/mediawiki/ngsolve

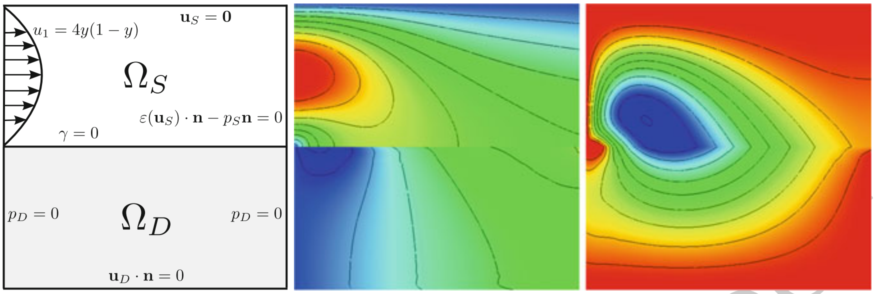

Fig. 1. From *left* to *right*: problem setup, and isolines of x- and y-components of the velocity for parameters $\mu_S = 1$, $\sigma_S = 0$ and $\mu_D = 0$, σ_D=1; $\gamma = 0$. A part of the flow soaks through the porous medium. The normal component of the velocity is (almost) continuous across the interface, while no continuity is obtained for the tangential component

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