A Domain Decomposition Solver for the Discontinuous Enrichment Method for the Helmholtz Equation

Charbel Farhat[1], Radek Tezaur[1], and Jari Toivanen[1]

Department of Aeronautics & Astronautics, Stanford University, Mail Code 3035, Stanford, CA 94305, U.S.A. cfarhat@stanford.edu, rtezaur@stanford.edu, toivanen@stanford.edu

1 Introduction

The discontinuous enrichment method (DEM) [4] for the Helmholtz equation approximates the solution as a sum of a piecewise polynomial continuous function and element-wise supported plane waves [5]. A weak continuity of the plane wave part is enforced using Lagrange multipliers. The plane wave enrichment improves the accuracy of solutions considerably. In the mid-frequency range, severalfold savings in terms of degrees of freedom over comparable higher order polynomial discretizations have been observed, which translates into even larger savings in compute time [6, 9]. The partition of unity method [8] and the ultra weak variational formulation [1] also employ plane waves in the construction of discretizations. It was shown recently in [10] that DEM without the polynomial field is computationally more efficient than these methods.

So far only direct solution methods have been used with DEM. This paper describes an iterative domain decomposition method which will enable to solve much larger problems with DEM. The method is a generalization of the FETI-H version [3] of the FETI method [2] and the domain decomposition method for DEM without the polynomial part described in [7]. It is based on a non-overlapping decomposition of the domain into subdomains. On the subdomain interfaces Lagrange multipliers are introduced to enforce the continuity of the polynomial part strongly and the continuity of the enrichment weakly. An efficient iterative solution procedure with a two-level preconditioner resembling that of the FETI-H method is constructed for the Lagrange multipliers on the interfaces between the subdomains.

2 Problem Formulation and Discretization

The solution $u \in H^1(\Omega)$ of a Helmholtz problem modeling acoustic scattering from a rigid obstacle, for example, satisfies the equations

R. Bank et al. (eds.), *Domain Decomposition Methods in Science and Engineering XX*, Lecture Notes in Computational Science and Engineering 91,
DOI 10.1007/978-3-642-35275-1__23, © Springer-Verlag Berlin Heidelberg 2013

$$
\begin{aligned}
-\Delta u - k^2 u &= f && \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= g_1 && \text{on } \Sigma_1 \\
\frac{\partial u}{\partial \nu} &= iku + g_2 && \text{on } \Sigma_2,
\end{aligned}
\tag{1}
$$

where k is the wavenumber, Σ_1 is the boundary of a sound-hard scatterer, Σ_2 is the far-field boundary, and ν denotes the unit outward normal.

Let the domain Ω be split into n_e elements, $\Omega = \cup_{e=1}^{n_e} \Omega_e$. In DEM, the solution is sought in the form $u = u^P + u^E$, where u^P is a standard continuous piecewise polynomial finite element function, and u^E is an enrichment function discontinuous across element interfaces. A weak inter-element continuity of the solution is enforced by Lagrange multipliers λ^E. The following hybrid variational formulation is used: Find $u \in \mathscr{V}$ and $\lambda^E \in \mathscr{W}^E$ such that

$$
\begin{aligned}
a(u,v) + b(\lambda^E, v) &= r(v) && \forall v \in \mathscr{V} \\
b(\mu^E, u) &= 0 && \forall \mu^E \in \mathscr{W}^E.
\end{aligned}
$$

The forms a, b, and r are defined by

$$
\begin{aligned}
a(u,v) &= \int_\Omega (\nabla u \cdot \nabla v - k^2 uv)\, d\Omega - \int_{\Sigma_2} ikuv\, d\Gamma, \\
b(\lambda^E, v) &= \sum_{e=1}^{n_e} \sum_{e'=1}^{e-1} \int_{\Gamma_{e,e'}} \lambda^E \left(v_{\Omega'_e} - v|_{\Omega_e} \right) d\Gamma, \quad \text{and} \\
r(v) &= \int_\Omega f v\, d\Omega + \int_{\Sigma_1} g_1 v\, d\Gamma + \int_{\Sigma_2} g_2 v\, d\Gamma,
\end{aligned}
$$

where $\Gamma_{e,e'} = \partial\Omega_e \cap \partial\Omega_{e'}$. For the considered discretization, the space $\mathscr{V}$ consists of functions of the form $u = u^P + u^E$, where u^E is a superposition of n_θ planar waves, i.e.

$$
u^E(\mathbf{x}) = \sum_{p=1}^{n_\theta} e^{ik\theta_p \cdot \mathbf{x}} u^E_{e,p}, \qquad \mathbf{x} \in \Omega_e.
$$

In two dimensions, $\theta_p = (\cos\vartheta_p, \sin\vartheta_p)^T, \vartheta_p = 2\pi(p-1)/n_\theta, p = 1,\ldots,n_\theta$. The Lagrange multipliers space $\mathscr{W}^E$ is then chosen using functions of the form

$$
\lambda^E(\mathbf{x}) = \sum_{p=1}^{n_\lambda} e^{ik\eta_p \tau_{e,e'} \cdot \mathbf{x}} \lambda_{e,e',p}, \qquad \mathbf{x} \in \Gamma_{e,e'},
$$

where $\tau_{e,e'}$ is a unit tangent vector and η_p is a scalar. This choice yields a family of quadrilateral elements, denoted by Q-n_θ-n_λ. In particular, the elements Q-8-2 and Q-16-4 used in the numerical experiments in this paper use $\eta_1 = -\eta_2 = 0.5$ and $\{\eta_p\}_{p=1}^4 = \{\pm 0.2, \pm 0.75\}$, respectively. For details on stability, implementation, and accuracy, the reader is referred to [5, 6].

3 Domain Decomposition Formulation

The elements are divided into n^d disjoint subsets E^j defining subdomains Ω^j such that $\bar{\Omega}^j = \cup_{e \in E^j} \bar{\Omega}_e$. Subdomain problems are given by regularized bilinear forms

$$\widetilde{a}^j(u^j, v^j) = \int_{\Omega^j} (\nabla u^j \cdot \nabla v^j - k^2 u^j v^j) d\Omega - \int_{\Sigma_2 \cap \partial\Omega^j} iku^j v^j \, d\Gamma \\ - \gamma \sum_{\substack{j'=1 \\ j' \neq j}}^{n^d} \int_{\Gamma^{j,j'}} s^{j,j'} iku^j v^j \, d\Gamma,$$

where $\Gamma^{j,j'} = \partial\Omega^j \cap \partial\Omega^{j'}$. The functions u^j and v^j belong to the restriction of $\mathscr{V}$ into Ω^j and the last term ensures the subdomain problems cannot be singular; for details see [7]. The coefficients $s^{j,j'}$ are chosen so that the regularization terms cancel out for a continuous function. The continuity of the polynomial part of the solution $\tilde{u}^P = \sum_{j=1}^{n^d} u^{P,j}$ across the subdomain interfaces is enforced using a Lagrange multiplier λ^P. For this purpose, a bilinear form

$$c(\lambda^P, \tilde{v}) = \sum_{j=1}^{n^d} \sum_{j'=1}^{j-1} \sum_{l} \lambda^P_{j,j',l} \left(\tilde{v}^P|_{\Omega^{j'}} - \tilde{v}^P|_{\Omega^j} \right) (\mathbf{x}_{j,j',l})$$

is defined, where $\mathbf{x}_{j,j',l}$ is the location of the lth mesh node on $\Gamma^{j,j'}$. The mesh nodes are given by the Lagrange interpolation points of the piecewise polynomial functions. The domain decomposition formulation then reads:

Find $\tilde{u} \in \widetilde{\mathscr{V}}$, λ^E, and λ^P such that

$$\begin{aligned} \widetilde{a}(\tilde{u}, \tilde{v}) + b(\lambda^E, \tilde{v}) + c(\lambda^P, \tilde{v}) &= \widetilde{r}(\tilde{v}) && \forall \tilde{v} \in \widetilde{\mathscr{V}} \\ b(\mu^E, \tilde{u}) &= 0 && \forall \mu^E \in \mathscr{W}^E \\ c(\mu^P, \tilde{u}) &= 0 && \forall \mu^P \in \mathscr{W}^P, \end{aligned} \tag{2}$$

where $\widetilde{\mathscr{V}}$ is spanned by $\sum_{j=1}^{n^d} v_j$, $\widetilde{a}(\tilde{u}, \tilde{v}) = \sum_{j=1}^{n^d} a^j(u^j, v^j)$, and $\widetilde{r}$ is the sum of subdomain contributions of r.

4 Linear Systems and Condensations

The formulation (2) leads to the saddle point system of linear equations

$$\begin{pmatrix} \mathbf{r}A^{PP} & \mathbf{r}A^{PE} & 0 & \mathbf{C}^{PL} \\ \mathbf{r}A^{EP} & \mathbf{r}A^{EE} & \mathbf{B}^{EL} & 0 \\ 0 & \mathbf{B}^{LE} & 0 & 0 \\ \mathbf{C}^{LP} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}^P \\ \boldsymbol{u}^E \\ \lambda^E \\ \lambda^P \end{pmatrix} = \begin{pmatrix} \mathbf{r}^P \\ \mathbf{r}^E \\ 0 \\ 0 \end{pmatrix}, \tag{3}$$

where the superscripts P, E, and L refer to the polynomial part, the enrichment part, and the Lagrange multiplier, respectively, and $\boldsymbol{u}^P, \boldsymbol{u}^E, \lambda^E, \lambda^P$ are vectors of the subdomain-by-subdomain polynomial degrees of freedom (depicted by black dots in Fig. 1), the element-by-element enrichment degrees of freedom (magenta arrows), the enrichment element-to-element continuity Lagrange multipliers (red arrows), and the polynomial subdomain-to-subdomain continuity Lagrange multipliers (black arrows), respectively. The enrichment unknowns $\boldsymbol{u}^E$ can be condensed out on the element level (Fig. 1 top and left) to obtain

$$\begin{pmatrix} \bar{\mathbf{r}A} & \bar{\mathbf{B}}^T & \bar{\mathbf{C}}^T \\ \bar{\mathbf{B}} & \bar{\mathbf{D}} & 0 \\ \bar{\mathbf{C}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}^P \\ \lambda^E \\ \lambda^P \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{r}} \\ \bar{\mu} \\ 0 \end{pmatrix}, \tag{4}$$

where

$$\begin{aligned} \bar{\mathbf{r}A} &= \mathbf{r}A^{PP} - \mathbf{r}A^{PE}\left(\mathbf{r}A^{EE}\right)^{-1}\mathbf{r}A^{EP}, & \bar{\mathbf{B}} &= -\mathbf{B}^{LE}\left(\mathbf{r}A^{EE}\right)^{-1}\mathbf{r}A^{EP}, \\ \bar{\mathbf{C}} &= \mathbf{C}^{LP}, & \bar{\mathbf{D}} &= -\mathbf{B}^{LE}\left(\mathbf{r}A^{EE}\right)^{-1}\mathbf{B}^{EL}, \\ \bar{\mathbf{r}} &= \mathbf{r}^P - \mathbf{r}A^{PE}\left(\mathbf{r}A^{EE}\right)^{-1}\mathbf{r}^E, & \bar{\mu} &= -\mathbf{B}^{LE}\left(\mathbf{r}A^{EE}\right)^{-1}\mathbf{r}^E. \end{aligned}$$

The enrichment Lagrange multipliers λ^E can be divided into two parts—those on the boundaries between the subdomains and those inside the subdomains, denoted by the subscript B and I, respectively. The system (4) can then be written in the block form

$$\begin{pmatrix} \bar{\mathbf{r}A} & \bar{\mathbf{B}}_{II}{}^T & \bar{\mathbf{B}}_{BB}{}^T & \bar{\mathbf{C}}^T \\ \bar{\mathbf{B}}_{II} & \bar{\mathbf{D}}_{II} & \bar{\mathbf{D}}_{IB} & 0 \\ \bar{\mathbf{B}}_{BB} & \bar{\mathbf{D}}_{BI} & \bar{\mathbf{D}}_{BB} & 0 \\ \bar{\mathbf{C}} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}^P \\ \lambda_I^E \\ \lambda_B^E \\ \lambda^P \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{r}} \\ \bar{\mu}_I \\ \bar{\mu}_B \\ 0 \end{pmatrix}.$$

Finally, the elimination on the subdomain level of the unknowns $\boldsymbol{u}^P$ and the interior

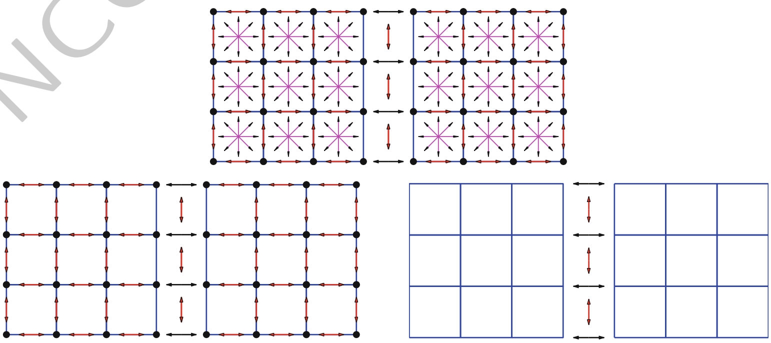

Fig. 1. 2×1 domain decomposition of a DEM discretization with bilinear polynomials and Q-8-2 elements resulting in the system (3) (*top*); variables left after condensation of enrichment dofs (4) (*left*); and elimination of the subdomain interior dofs (5) (*right*)

enrichment Lagrange multipliers λ_I^E gives the Schur complement system (cf. Fig. 1 right)

$$\mathbf{F}\begin{pmatrix}\lambda_B^E \\ \lambda^P\end{pmatrix} = \mathbf{b}. \tag{5}$$

It is noted that the matrix $\mathbf{F}$ is a sum of subdomain matrices. Once the Lagrange multipliers λ_B^E and λ^P have been solved from (5), the rest of the unknowns is recovered by post-processing, first to obtain $\boldsymbol{u}^P$ and λ_I^E, then to obtain $\boldsymbol{u}^E$.

5 Preconditioning

The system (5) is solved efficiently using a Krylov iterative method with a two-level preconditioner which is a generalization of those described in [3, 7].

Here, the subdomain preconditioners are based on the bilinear forms

$$\hat{a}^j(u^j,v^j) = \int_{\Omega^j} (\nabla u^j \cdot \nabla v^j - k^2 u^j v^j) d\Omega - \int_{\partial\Omega^j \setminus \Sigma_1} iku^j v^j \, d\Gamma,$$
$$\hat{b}^j(\lambda^E,v^j) = \sum_{e\in E^j} \sum_{e'=e+1}^{n_e} \int_{\Gamma_{e,e'}} \lambda^E v|_{\Omega_e} d\Gamma - \sum_{e\in E^j} \sum_{e'=1}^{e-1} \int_{\Gamma_{e,e'}} \lambda^E v|_{\Omega_e} d\Gamma, \quad \text{and}$$
$$\hat{c}^j(\lambda^P,v^j) = \sum_{j'=j+1}^{n^d} \sum_l \lambda^P_{j,j',l} v^P|_{\Omega^j}(\mathbf{x}_{j,j',l}) - \sum_{j'=1}^{j-1} \sum_l \lambda^P_{j,j',l} v^P|_{\Omega^j}(\mathbf{x}_{j,j',l}).$$

Repeating the same steps described above for obtaining $\mathbf{F}$ in (5) but with matrices based on $\hat{a}^j$, and restricting the resulting matrix to the unknowns corresponding to the interfaces of the subdomain Ω^j, a matrix denoted by $\mathbf{F}^j$ is obtained (cf. [7]). An additive subdomain-by-subdomain preconditioner is then defined by

$$\mathbf{K} = \sum_{j=1}^{n^d} \left(\mathbf{R}^j\right)^T \left(\mathbf{F}^j\right)^{-1} \mathbf{R}^j,$$

where $\mathbf{R}^j$ is the restriction on the interfaces associated with Ω^j. Linear systems with $\mathbf{F}^j$ can be solved efficiently using an **LU** decomposition.

The system (5) is solved iteratively on the orthogonal complement of a coarse space spanned by the columns of a matrix $\mathbf{Q}$ (cf. [3, 7]). A projector to the orthogonal complement of the coarse space is given by

$$\mathbf{P} = \mathbf{I} - \mathbf{Q}(\mathbf{Q}^T\mathbf{F}\mathbf{Q})^{-1}\mathbf{Q}^T\mathbf{F}.$$

The solution $\lambda = [\lambda_B^E, \lambda^P]^T$ of (5) can be decomposed into two parts $\lambda = \lambda^0 + \mathbf{P}\lambda^1$, where $\lambda^0 = \mathbf{Q}(\mathbf{Q}^T\mathbf{F}\mathbf{Q})^{-1}\mathbf{Q}^T\mathbf{b}$ and λ^1 satisfies

$$\mathbf{P}^T\mathbf{F}\lambda^1 = \mathbf{P}^T\mathbf{b}.$$

Including the preconditioner $\mathbf{K}$ leads to the following equation

$$\mathbf{PKP}^T\mathbf{F}\lambda^1 = \mathbf{PKF}\lambda^1 = \mathbf{PKP}^T\mathbf{b},$$

which is solved by GMRES.

The coarse space is based on plane waves propagating in n_q uniformly distributed directions. Each set of n_q plane waves are supported by one subdomain interface $\Gamma^{j,j'}$ and their normal derivatives on the interface are approximated using an L^2-projection into the space of Lagrange multipliers giving rise to n_q columns of $\mathbf{Q}$. Currently, the coarse space acts only on the interface enrichment Lagrange multipliers λ_B^E. The maximum dimension of the coarse space is $n_q n_i$, where n_i is the number of nonzero measure interfaces $\Gamma^{j,j'}$. A **QR** factorization is used to remove nearly linearly dependent vectors. More details are given in Sect. 3.4 of [7].

6 Numerical Results

The model problem considered here is given by (1) with the computational domain $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : 1 < \|x\| < 2\}$, and the boundaries $\Gamma_1 = \{\mathbf{x} \in \mathbb{R}^2 : \|x\| = 1\}$ and $\Gamma_2 = \{\mathbf{x} \in \mathbb{R}^2 : \|x\| = 2\}$. The right-hand side function and the boundary functions are chosen as

$$\begin{aligned}
f(\mathbf{x}) &= (-\Delta - k^2)(x_1^2 + x_2^2) = -4 - k^2(x_1^2 - x_2^2),\\
g_1(\mathbf{x}) &= -\frac{\partial e^{-ikx_1}}{\partial \nu} + \frac{\partial (x_1^2 + x_2^2)}{\partial \nu} = -ikx_1 e^{ikx_1} - 2(x_1^2 + x_2^2), \quad \text{and}\\
g_2(\mathbf{x}) &= \frac{\partial (x_1^2 + x_2^2)}{\partial \nu} - ik(x_1^2 + x_2^2) = (1 - ik)(x_1^2 + x_2^2).
\end{aligned}$$

The solution is a sum of that given by the scattering of the plane wave e^{-ikx_1} by a sound-hard disk inside Γ_1 and the polynomial $x_1^2 + x_2^2$. Two wavenumbers, $k = 8\pi$ and 16π are considered, in which case the diameter of the scatterer is 8 and 16 wavelengths, respectively. The solution at $k = 16\pi$ is shown in Fig. 2. Meshes of 96×8 ($k = 8\pi$) and 192×16 ($k = 16\pi$) elements result in two elements per wavelength in the radial direction.

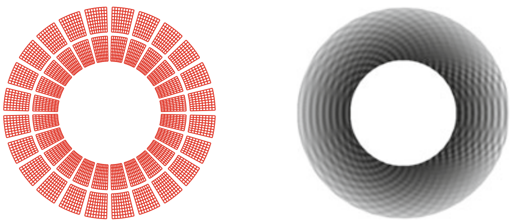

Fig. 2. The 24×2 domain decomposition for the 192×16 mesh (*left*) and the real part of the solution at $k = 16\pi$ (*right*)

Table 1. Results for the 96×8 mesh with the wavenumber $k = 8\pi$.

		12 x 1 subdomains			24 x 2 subdomains			
			$n_q = 0$	$n_q = 8$		$n_q = 0$	$n_q = 8$	
poly	enrich	N	iter.	iter.	N	iter.	iter.	error
Q_1	none	108	49		336	213		0.683405
Q_2	none	204	33		624	195		0.141341
none	Q-8-2	192	35	31	576	163	7	0.438341
Q_1	Q-8-2	300	34	31	912	184	28	0.004677
Q_2	Q-8-2	396	34	31	1200	206	48	0.004472
none	Q-16-4	384	35	30	1152	151	39	0.019767
Q_1	Q-16-4	492	36	31	1488	160	54	0.000024
Q_2	Q-16-4	588	36	31	1776	176	73	0.000013

Table 2. Results for the 192×16 mesh with the wavenumber $k = 16\pi$.

		12 x 1 subdomains			24 x 2 subdomains			
			$n_q = 0$	$n_q = 16$		$n_q = 0$	$n_q = 16$	
poly	enrich	N	iter.	iter.	N	iter.	iter.	error
Q_1	none	204	79		624	350		0.568750
Q_2	none	396	40		1200	368		0.174451
none	Q-8-2	384	44	34	1152	264	16	0.478914
Q_1	Q-8-2	588	42	34	1776	281	31	0.007441
Q_2	Q-8-2	780	42	34	2352	295	56	0.007826
none	Q-16-4	768	42	33	2304	233	42	0.021694
Q_1	Q-16-4	972	42	35	2928	238	52	0.000011
Q_2	Q-16-4	1164	42	33	3504	253	123	0.000010

Bilinear (Q_1) and biquadratic (Q_2) bases are used for the polynomial part $\boldsymbol{u}^P$. Q-8-2 and Q-16-4 elements are used for the enrichment $\boldsymbol{u}^E$ and its Lagrange multipliers λ^E. The domain is decomposed into 12×1 and 24×2 subdomains (Fig. 2). The GMRES iterations are terminated once the norm of the residual is reduced by 10^{-8}. Tables 1 and 2 summarize the performance results obtained for various element types. In these tables, N is the size of the system (5), i.e. the number of Lagrange multipliers enforcing continuity between subdomains. The error is the relative l_2 error of the averaged nodal values with respect to the analytical solution of the problem.

The errors in the last column of Tables 1 and 2 clearly show the benefit of discretizations with both polynomial and enrichment fields for this problem. The combined discretizations increase the accuracy by at least two orders of magnitude. The iteration counts without a coarse space ($n_q = 0$) are roughly the same for all discretizations and not quite satisfactory for the 24×2 decomposition. However, these are reduced substantially when the coarse space is added.

Bibliography

[1] Olivier Cessenat and Bruno Despres. Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz problem. *SIAM J. Numer. Anal.*, 35(1):255–299, 1998.

[2] Charbel Farhat and Francoise-Xavier Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Internat. J. Numer. Meths. Engrg.*, 32(6):1205–1227, 1991.

[3] Charbel Farhat, Antonini Macedo, and Michel Lesoinne. A two-level domain decomposition method for the iterative solution of high frequency exterior Helmholtz problems. *Numer. Math.*, 85(2):283–308, 2000.

[4] Charbel Farhat, Isaac Harari, and Leopoldo P. Franca. The discontinuous enrichment method. *Comput. Methods Appl. Mech. Engrg.*, 190(48):6455–6479, 2001.

[5] Charbel Farhat, Isaac Harari, and Ulrich Hetmaniuk. A discontinuous Galerkin method with Lagrange multipliers for the solution of Helmholtz problems in the mid-frequency regime. *Comput. Methods Appl. Mech. Engrg.*, 192(11–12):1389–1419, 2003.

[6] Charbel Farhat, Radek Tezaur, and Paul Weidemann-Goiran. Higher-order extensions of a discontinuous Galerkin method for mid-frequency Helmholtz problems. *Internat. J. Numer. Methods Engrg.*, 61(11):1938–1956, 2004.

[7] Charbel Farhat, Radek Tezaur, and Jari Toivanen. A domain decomposition method for discontinuous Galerkin discretizations of Helmholtz problems with plane waves and Lagrange multipliers. *Internat. J. Numer. Methods Engrg.*, 78 (13):1513–1531, 2009.

[8] Jens M. Melenk and Ivo Babuška. The partition of unity finite element method: basic theory and applications. *Comput. Methods Appl. Mech. Engrg.*, 139(1–4):289–314, 1996.

[9] Radek Tezaur and Charbel Farhat. Three-dimensional discontinuous Galerkin elements with plane waves and Lagrange multipliers for the solution of mid-frequency Helmholtz problems. *Internat. J. Numer. Methods Engrg.*, 66(5):796–815, 2006.

[10] Dalei Wang, Radek Tezaur, Jari Toivanen, and Charbel Farhat. Overview of the discontinuous enrichment method, the ultra-weak variational formulation, and the partition of unity method for acoustic scattering in the medium frequency regime and performance comparisons. *Internat. J. Numer. Methods Engrg.*, 89 (4):403–417, 2012.