Domain Decomposition Method for Stokes Problem with Tresca Friction

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1 Introduction

Development of numerical methods for the solution of Stokes system with slip boundary conditions (Tresca friction conditions) is a challenging task whose difficulty lies in the nonlinear conditions. Such boundary conditions have to be taken into account in many situations arising in practice, in flow of polymers (see [10] and references therein).

The paper is devoted to domain decomposition methods (DDM in short) for the Stokes problem with the slip boundary conditions. The original domain is cut into two sub-domains and the augmented Lagrangian formulation for separate resulting Poisson problems in both domains is used for computations. To relate solutions of these two sub-problems to the original solution, one has to introduce additional constraints "gluing" them together. The domain decomposition formulation is based on the Uzawa block relaxation method for the augmented Lagrangian involving three supplementary conditions. The paper is concluded by preliminary several numerical examples.

2 Setting Stokes Problem with Nonlinear Boundary Conditions

Let us consider a domain $\Omega \subset \mathbb{R}^2$ with the Lipschitz boundary $\partial\Omega$ which is split into two non-empty and non-overlapping parts Γ_0 and Γ. We denote by n the outward unit normal to $\partial\Omega$ and u_n, respectively u_t, the normal, respectively the tangential, component of u. We also make use of σ_t for the tangential component of the stress vector $\sigma(u)n$. The problem consists in finding the velocity field u and the pressure p for the following Stokes problem with nonlinear boundary condition of Tresca friction type:

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$$\left\{\begin{array}{rrl} & -div(\nu\varepsilon(u)) + \nabla p = \mathrm{f} & \text{in } \Omega \\ & div(u) = 0 & \text{in } \Omega \\ & u = 0 & \text{on } \Gamma_0 \\ & u_n = 0 & \text{on } \Gamma \\ & |\sigma_t| \leq g & \text{on } \partial\Omega \\ |\sigma_t| < g \Rightarrow & u_t = 0 & \text{on } \Gamma \\ |\sigma_t| = g \Rightarrow \exists k > 0 \text{ a constant such that} & u_t = -k\sigma_t & \text{on } \Gamma \end{array}\right. \tag{1}$$

where f is in $\mathbf{L}^2(\Omega)$, $g \in L^2(\Gamma)$, $g > 0$ is the given slip bound on Γ and $|\cdot|$ is the euclidean norm.

One can derive the variational formulation of (1):

$$\left\{\begin{array}{l} \text{Find } u \in \mathbf{V}_{div}(\Omega) \text{ such that} : \forall v \in \mathbf{V}_{div}(\Omega) \\ a(u, v-u) + j(v) - j(u) \geq L(v-u), \end{array}\right. \tag{2}$$

with

$$\mathbf{V}(\Omega) = \left\{ v \in \mathbf{H}^1(\Omega),\ v_{|\Gamma_0} = 0, v_n = 0 \text{ on } \Gamma \right\},$$

$$\mathbf{V}_{div}(\Omega) = \left\{ v \in \mathbf{V}(\Omega)\ ,\ div(v) = 0 \text{ in } \Omega \right\},$$

$$a(u,v) = \int_\Omega \nu\varepsilon(u) : \varepsilon(v)\, d\Omega, \quad L(v) = \int_\Omega \mathrm{f}\, v\, d\Omega, \quad j(v) = \int_\Gamma g|v_t|\, d\Gamma.$$

Problem (2) is an elliptic variational inequality of the second kind which has a unique solution [3]. Moreover, since the bilinear form $a(\cdot,\cdot)$ is symmetric (2) is equivalent to the following constrained non-differentiable minimization problem:

$$\text{Find } u \in \mathbf{V}_{div}(\Omega) \text{ such that}: \quad \mathscr{J}(u) \leq \mathscr{J}(v) \quad \forall v \in \mathbf{V}_{div}(\Omega), \tag{3}$$

where $\mathscr{J}(v) = \dfrac{1}{2} a(v,v) + j(v) - L(v)$ is the total potential energy functional.

3 Uzawa DDM for Stokes Problem with Tresca Friction

We now study the domain decomposition of (3). We first rewrite (3) in the following more useful form. Suppose that $\varphi = v_t$, then the minimization problem (3) becomes:

$$\left\{\begin{array}{l} \text{Find } (u, \Phi) \in \Pi \text{ such that:} \\ \Sigma(u,\Phi) \leq \Sigma(v,\varphi) \forall (v,\varphi) \in \Pi, \end{array}\right. \tag{4}$$

where

$$\Pi = \{(v,\varphi) \in \mathbf{V}_{div}(\Omega) \times H^{\frac{1}{2}}(\Gamma) \text{ such that } \varphi = v_t\},$$

and Σ is the Lagrangian defined on Π by:

$$\forall(\varphi,v)\in\Pi \qquad \Sigma(v,\varphi)=\frac{1}{2}a(v,v)-L(v)+j(\varphi). \tag{5}$$

Let $\{\Omega_1,\Omega_2\}$ be a partition of Ω, as shown in Fig. 1, and let

$$\begin{aligned}
&\Gamma_{12}=\Gamma_{21}=\partial\Omega_1\cap\partial\Omega_2, \quad \Gamma_i=\Gamma\cup\partial\Omega_i, \quad \Gamma_i^0=\Gamma_0\cup\partial\Omega_i,\\
&v_i=v|_{\Omega_i}, \quad p_i=p|_{\Omega_i},\\
&\mathbf{V}(\Omega_i)=\Big\{v_i\in\mathbf{H}^1(\Omega_i),\ v_{i|\Gamma_i^0}=0,\ v_i.n_{i|\Gamma_i}=0\Big\},\\
&\mathbf{V}_{div}(\Omega_i)=\Big\{v_i\in\mathbf{V}(\Omega_i)\ ,\ div(v_i)=0 \text{ in } \Omega_i\Big\}.
\end{aligned}$$

Restrictions of the functionals a and Σ over Ω_i are defined by a_i and Σ_i respectively. Inner products over a given part S of $\partial\Omega_i$, $i=1,2$, and Ω_i are defined by

$$(u,v)_S=\int_S uv\mathrm{d}\Gamma \quad \text{and} \quad (u,v)_{\Omega_i}=\int_{\Omega_i} uv\mathrm{d}x.$$

We treat the pressure as a Lagrange multiplier associated with the constraint

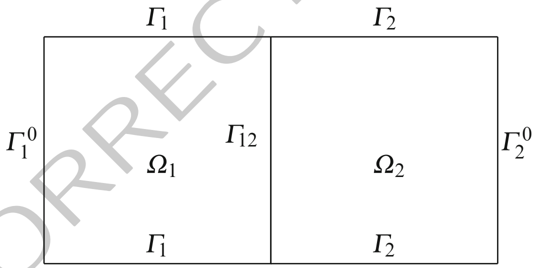

Fig. 1. Decomposition of Ω into two subdomains

$div(u)=0$. Using the decomposition of Fig. 1, the functional (5) becomes

$$\Sigma(v,\varphi)=\Sigma_1(v_1,\varphi_1)+\Sigma_2(v_2,\varphi_2). \tag{6}$$

It is clear that problem (3) is equivalent to the following constrained minimization problem:

$$\begin{aligned}
&\forall(v_i,\varphi_i)\in\mathbf{V}(\Omega_i)\times H^{\frac{1}{2}}(\Gamma_i), i=1,2\\
&\Sigma(u_1,\Phi_1)+\Sigma(u_2,\Phi_2)\le\Sigma_1(v_1,\varphi_1)+\Sigma_2(v_2,\varphi_2)\\
&div(u_i)=0 \quad \text{in } \Omega_i,\\
&u_{it}-\Phi_i=0 \quad \text{in } \Gamma_i,\\
&u_i-\Psi=0 \quad \text{in } \Gamma_{12}.
\end{aligned} \tag{7}$$

The auxiliary interface unknown Ψ is added to the continuity constraint to avoid coupling between u_1 and u_2 in the penalty term. This so-called *three-field formulation* has been used in domain decomposition of elliptic problems [9]. To ensure the uniqueness of the pressure, the following constraint can be added

$$\int_{\Omega_1} p_1 \, d\Omega_1 + \int_{\Omega_2} p_2 \, d\Omega_1 = 0. \tag{8}$$

Then, we introduce the set

$$\mathfrak{P} = \left\{ (q_1, q_2) \in L^2(\Omega_1) \times L^2(\Omega_2) \text{ such that } \int_{\Omega_1} q_1 \, d\Omega_1 + \int_{\Omega_2} q_2 \, d\Omega_1 = 0 \right\}$$

We can associate to (7) the augmented Lagrangian functional $\mathscr{L}_r$ defined by

$$\begin{aligned} \mathscr{L}_r \;\; (u, \Phi, \Psi, p, \mu, \lambda) &= \Sigma(u_1, \Phi_1) + \Sigma(u_2, \Phi_2) \\ &+ \sum_{i=1}^{2} \left[(\mu_i, \Phi_i - u_{it})_{\Gamma_i} - (p_i, div(u_i))_{\Omega_i} + (\lambda_i, u_i - \Psi)_{\Gamma_{12}} \right] \\ &+ \sum_{i=1}^{2} \left[\frac{r_1}{2} ||div(u_i)||^2_{L^2(\Omega_i)} + \frac{r_2}{2} ||\Phi_i - u_{it}||^2_{L^2(\Gamma_i)} + \frac{r_3}{2} ||u_i - \Psi||^2_{L^2(\Gamma_{12})} \right]. \end{aligned} \tag{9}$$

where r_1, r_2 and r_3 are the penalty parameters which are strictly positive.

Remark 1. The standard L^2 scalar product (not equivalent to the $H^{1/2}$ scalar product) on the interface Γ_{12} and Γ_i is used in the definition of (9). This approach is easy to implement but it has some negative effects on the convergence of our algorithm.

Then, problem (7) is equivalent to the following saddle-point problem:

$$\begin{cases} \text{Find } (u, \Phi, \Psi, p, \mu, \lambda) \in \mathscr{H} \quad \text{ such that: } \forall (v, \Phi, \Psi, q, \tilde{\mu}, \tilde{\lambda}) \in \mathscr{H} \\ \\ \mathscr{L}_r(u, \Phi, \Psi, q, \tilde{\mu}, \tilde{\lambda}) \le \mathscr{L}_r(u, \Phi, \Psi, p, \mu, \lambda) \le \mathscr{L}_r(v, \Phi, \Psi, p, \mu, \lambda). \end{cases} \tag{10}$$

where $u=(u_1,u_2) \in \mathbf{V}(\Omega_1) \times \mathbf{V}(\Omega_2)$, $\Phi=(\Phi_1, \Phi_2) \in L^2(\Gamma_1) \times L^2(\Gamma_2)$, $\Psi \in (L^2(\Gamma_{12}))^2$, $p = (p_1, p_2) \in \mathfrak{P}$, $\mu = (\mu_1, \mu_2) \in L^2(\Gamma_1) \times L^2(\Gamma_2)$ and $\lambda \in (L^2(\Gamma_{12}))^2$. $\mathscr{H}$ is the Cartesian product of all these spaces.

3.1 Uzawa Block Relaxation Method: UBR2

In order to solve (10) we use Uzawa block relaxation algorithm based on ALG2, see [4]. This leads to the following iterations:

Initialization: Φ^{-1}, Ψ^{-1}, p^0, λ^0, μ^0 and $r_i > 0$ fixed.
Repeat until convergence:

1. Find $u^k \in \mathbf{V}(\Omega_1) \times \mathbf{V}(\Omega_2)$ such that: $\quad \forall v \in \mathbf{V}(\Omega_1) \times \mathbf{V}(\Omega_2)$

$$\mathscr{L}_r(u^k, \Phi^{k-1}, \Psi^{k-1}, p^k, \mu^k, \lambda^k) \le \mathscr{L}_r(v, \Phi^{k-1}, \Psi^{k-1}, p^k, \mu^k, \lambda^k). \tag{11}$$

2. `Find` $\Phi^k \in L^2(\Gamma_1) \times L^2(\Gamma_2)$ `such that:` $\quad \forall \Phi \in L^2(\Gamma_1) \times L^2(\Gamma_2)$

$$\mathscr{L}_r(u^k, \Phi^k, \Psi^{k-1}, p^k, \mu^k, \lambda^k) \leq \mathscr{L}_r(u^k, \Phi, \Psi^{k-1}, p^k, \mu^k, \lambda^k). \tag{12}$$

3. `Find` $\Psi^k \in (L^2(\Gamma_{12}))^2$ `such that:` $\quad \forall \Psi \in (L^2(\Gamma_{12}))^2$.

$$\mathscr{L}_r(u^k, \Phi^k, \Psi^k, p^k, \mu^k, \lambda^k) \leq \mathscr{L}_r(u^k, \Phi^k, \Psi, p^k, \mu^k, \lambda^k). \tag{13}$$

4. `Lagrange multipliers update`

$$p_i^{k+1} = p_i^k - r_1 div(u_i^k), \tag{14}$$
$$\lambda_i^{k+1} = \lambda_i^k + r_2(u_{i|\Gamma_{12}}^k - \Psi^k), \tag{15}$$
$$\mu_i^{k+1} = \mu_i^k + r_3(u_{it}^k - \Phi_i^k). \tag{16}$$

Subproblem (11) is equivalent to solving, in each subdomain, the following problem:

$$\begin{aligned} &\text{Find} \quad u_i^k \in \mathbf{V}(\Omega_i) \text{ such that} \\ &a(u_i^k, v) + r_1(\nabla.u_i^k, \nabla.v_i)_{\Omega_i} + r_2(u_i, v_i)_{\Gamma_{12}} + r_3(u_t^k, v_t)_{\Gamma} = (\mathrm{f}_i, v_i) + (p_i, \nabla.v_i)_{\Omega_i} \\ &\qquad + (r_2 \Psi^k - \lambda^k, v_i)_{\Gamma_{12}} + (r_3 \Phi_i^{k-1} - \mu_i^k, v_{it})_{\Gamma_i} \quad \forall v_i \in \mathbf{V}(\Omega_i). \end{aligned} \tag{17}$$

The subproblems of steps 2 and 3 are uncoupled and consists in the following calculations:

$$\Phi_i^k = \begin{cases} \dfrac{||\mu_i^k + r_3 u_{it}^k||_{0,\Gamma_i} - g}{r_3 ||\mu_i^k + r_3 u_{it}^k||_{0,\Gamma_i}} (\mu_i^k + r_3 u_{it}^k) & \text{if} \quad ||\mu_i^k + r_3 u_{it}^k||_{0,\Gamma_i} \geq g \\ & 0 \text{ unless} \end{cases} \tag{18}$$

and

$$\Psi^k = \frac{1}{2r_2}(\lambda_1^k + \lambda_2^k) + \frac{1}{2}(u_1^k + u_2^k)|_{\Gamma_{12}}. \tag{19}$$

Remark 2. For sake of simplicity the given slip bound g is assumed to be non-negative constant in (18).

Remark 3. After update (14), p^{k+1} must be projected onto $\mathfrak{P}$ to ensure the uniqueness of the pressure.

Remark 4. The main advantage of this formulation is that (17) reduces to 2D uncoupled elliptic problems which can be solved in parallel. Moreover, the matrices derived from discret problems are symmetric and positive definite.

4 Numerical Experiments

The domain decomposition algorithm **UBR2**, with $r_1 = r_2 = r_3$, presented in the previous section was implemented in Matlab V7.9 on a Core2 Duo-1.8 Ghz processor PC. For discrete velocity-pressure-Lagrange multipliers spaces, we use the P^1-iso-P^2/P^1 finite element. These spaces are well known to satisfy the discrete Babuska-Brezzi inf-sup condition [1].
For all the numerical experiments presented, the domain Ω is the square $[0,0.1]^2$, while $\Omega_1 = [0,0.05]\times[0,0.1]$ and $\Omega_2 = [0.05,0.1]\times[0,0.1]$. The fluid can slip on $\Gamma_1 \cup \Gamma_2 = [0,0.1]\times\{0.1\}\cup[0,0.1]\times\{0\}$, We set $g = 0.015$ which is consistent with experimental values, see [5]. The viscosity is taken equal to 0.1 and the stopping tolerance ε is 10^{-6}. In addition we enforce parabolic profile on both $\Gamma_1^0 = \{0\}\times[0,0.1]$ and $\Gamma_2^0 = \{0.1\}\times[0,0.1]$:

$$u|_{\Gamma_1^0} = u|_{\Gamma_2^0} = \begin{bmatrix} y(1-y) \\ -y(1-y) \end{bmatrix}$$

Remark 5. We choose this profile to enforce shear stress near the solid wall to reach the threshold without considering a complicated domain geometry.

In Fig. 2 we report the velocity field for the solution of Stokes problem with Tresca friction (1) in Ω and in $\Omega_1 \cup \Omega_2$. We can see that we have the same velocity profile. In Table 1 we report the discrete mesh size h, the corresponding number of degree

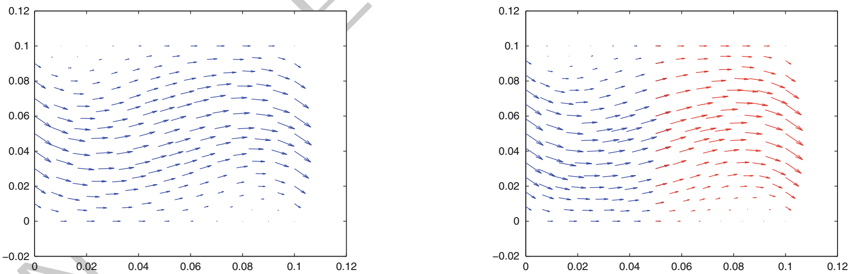

Fig. 2. Fluid flow with Tresca BC for one (*left*) and two domains (*right*)

of freedom (d.o.f) and number of elements on each subdomain in the follows experiments. Table 2 shows the number of iterations IT, the sequential CPU (in seconds) times and the parallel CPU* times (when subproblems (17) for $i = 1,2$ are solved in parallel). For several mesh size and for N_{SD} (Number of Sub-Domains) equal to 1 or 2. We notice that the **UBR2** algorithm is a h-dependent algorithm and the domain decomposition method to be preferable when dealing with parallel computing using parallel solver.

Table 3 show how the number of iterations and the optimal value of the relaxation parameter r_{opt} depend on h. We remark that the speed of convergence is very sensitive to r; this explains the strong increase in the number of iterations for a finer mesh.

N_{SD}	$h = 0.02$	$h = 0.01$	$h = 0.0067$	$h = 0.005$	$h = 0,004$
	n/n_Δ	n/n_Δ	n/n_Δ	n/n_Δ	n/n_Δ
1	189/336	665/1284	1577/3032	2829/5496	4393/8548
2	112/188	370/676	806/1516	1396/2668	2220/4284

Table 1. h: mesh size; n: number of d.o.f. by domain n_Δ: number of elements by domain.

N_{SD}	$h = 0.02$	$h = 0.01$	$h = 0.0067$	$h = 0.005$	$h = 0,004$
	IT/CPU/CPU*	IT/CPU/CPU*	IT/CPU/CPU*	IT/CPU/CPU*	IT/ CPU/CPU*
1	199/0.41/-	349/2.8/-	453/10.8/-	509/30.36/-	595/67.3/-
2	486/1/0.81	769/4.8/3.27	993/15.3/7.96	1294/41.14/21.98	1599/99.34/51.59

Table 2. Standard speed-up for h: mesh size; IT: number of iterations; CPU & CPU*: CPU times.

N_{SD}	$h = 0.02$	$h = 0.01$	$h = 0.0067$	$h = 0.005$	$h = 0,004$
	r_{opt}/IT	r_{opt}/IT	r_{opt}/IT	r_{opt}/IT	r_{opt}/IT
1	335/199	590/349	740/453	840/509	1010/595
2	116/486	124/769	175/993	230/1294	290/1599

Table 3. Convergence rate with respect r_{opt}.

5 Conclusion

The augmented Lagrangian formulation (9) of domain decomposed Stokes problem with Tresca friction leads to a numerical strategy which solves a classical Poisson problem (17) (in each subdomain Ω_i) and the contribution of Tresca friction (18) in a decoupled way. Nevertheless, this algorithm has a mesh dependent convergence and its practical implementation still facing the issue of the optimal choice of the penalties, $r_i, i = 1,2,3$. To improve this algorithm, different preconditioners will be investigated, especially the Steklov-Poincaré operator on the interface (see e.g. [6–8]) and the Cahouet-Chabard preconditioner [2] for the pressure multiplier.

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