FETI-DP for Elasticity with Almost Incompressible Material Components

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1 Introduction

The purpose of this article is to present convergence bounds and some preliminary numerical results for a special category of problems of compressible and almost incompressible linear elasticity when using FETI-DP or BDDC domain decomposition methods.

We consider compressible and almost incompressible elasticity on the computational domain $\Omega \subset \mathbb{R}^3$ which is partitioned into a number of subdomains. We introduce nodes in the interior of the subdomains and on the interface. We distribute the material parameters such that in a neighborhood of the interface we have compressible and in the interior of a subdomain we have almost incompressible linear elasticity. Thus, each subdomain may contain an almost incompressible component in its interior surrounded by a hull of compressible material. We will also refer to this component as the incompressible inclusion.

By performing our analysis on the compressible hull, we can prove new condition number bounds. Such bounds will depend on the variation of the Poisson ratio ν in a neighborhood of the interface of the subdomains. More precisely, for compressible linear elasticity in a neighborhood of the interface and almost incompressible linear elasticity in the interior of the subdomains, we can prove a polylogarithmic condition number bound for the preconditioned FETI-DP system, which also depends on the thickness η of the compressible hull.

The condition number estimate presented in this contribution is based on the theory developed in [8] for compressible linear elasticity. It can be seen as an extension to certain configurations of incompressible components. For an algorithmic description of the FETI-DP method and the primal constraints applied in this paper, we refer to [5, 6]. The current work can also be seen as an extension of the work of [13–15]. There, the one-level FETI method for scalar elliptic problems is analyzed for special cases of coefficient jumps inside subdomains.

Coarse spaces for iterative substructuring methods that are robust either with respect to exact incompressibility constraints or with respect to almost incompressibility have been known for some time. For earlier work on Neumann-Neumann,

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FETI-DP, and BDDC methods for (almost) incompressible elasticity, see, e.g., [4, 9, 10, 12].

2 Almost Incompressible Linear Elasticity

Let $\Omega \subset \mathbb{R}^3$ be a polytope, which can be decomposed into smaller cubic subdomains. We can allow also for subdomains that are images of cubes under a reasonable mapping.

The domain is fixed on $\partial\Omega_D \subset \partial\Omega$, i.e., we impose Dirichlet boundary conditions, and the remaining part $\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$ is subject to a surface force g. Let $H_0^1(\Omega,\partial\Omega_D) := \left\{v \in (H^1(\Omega))^3 : \; v\,|_{\partial\Omega_D} = 0\right\}$ be the Sobolev space which is appropriate for the variational formulation. Furthermore, the linearized strain tensor $\varepsilon = (\varepsilon_{ij})_{ij}$ is defined as $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ with $u \in (H^1(\Omega))^3$.

Then, the linear elasticity problem is defined as follows.
Find the displacement $u \in H_0^1(\Omega,\partial\Omega_D)$, such that for all $v \in H_0^1(\Omega,\partial\Omega_D)$

$$\int_\Omega G\,\varepsilon(u) : \varepsilon(v)\,dx + \int_\Omega G\,\beta\,\mathrm{div}(u)\,\mathrm{div}(v)\,dx = < F, v >$$

with the material parameters G, β, and the right hand side

$$< F, v > = \int_\Omega f^T v\,dx + \int_{\partial\Omega_N} g^T v\,d\sigma.$$

The material parameters G and β can also be expressed using Young's modulus E and the Poisson ratio ν by $G = \frac{E}{1+\nu}$ and $\beta = \frac{\nu}{1-2\nu}$. We analyze linear elasticity problems with different material components. For the compressible part we use the standard displacement formulation, i.e., we discretize the displacement by piecewise quadratic tetrahedral finite elements.

For almost incompressible linear elasticity, i.e., when $\nu \to \frac{1}{2}$, the value of β tends to infinity, and the discretization of the standard displacement formulation of linear elasticity by low order finite elements leads to locking effects and slow convergence. As a remedy the displacement problem is replaced by a mixed formulation. Therefore, we introduce the pressure $p := G\,\beta\,\mathrm{div}(u) \in L_2(\Omega)$ as an auxiliary variable.

We consider the problem: Find $(u,p) \in H_0^1(\Omega,\partial\Omega_D) \times L_2(\Omega)$, such that

$$\begin{aligned}\int_\Omega G\,\varepsilon(u) : \varepsilon(v)\,dx + \int_\Omega \mathrm{div}(v)\,p\,dx &= \langle F, v\rangle \quad \forall v \in H_0^1(\Omega,\partial\Omega_D)\\ \int_\Omega \mathrm{div}(u)\,q\,dx - \int_\Omega \frac{1}{G\,\beta}\,p\,q\,dx &= 0 \quad \forall q \in L_2(\Omega).\end{aligned}$$

It is well-known that in the case of almost incompressible linear elasticity, the solution of this mixed formulation exists and is unique.

For the discretization of this mixed problem we can in principle use any inf-sup stable mixed finite element method. For simplicity we use $Q_2 - P_0$ mixed finite elements, i.e., we discretize the displacement with piecewise triquadratic hexahedral finite elements and the pressure with piecewise constant elements. This discretization is known to be inf-sup stable, which, in 3D, can be derived from the results in [11]. To obtain again a symmetric positive definite problem, the pressure is statically condensated element-by-element. We assume that a triangulation τ_h of Ω is given with shape regular finite elements, having a typical diameter h. Additionally, we assume that Ω can be represented exactly as a union of finite elements.

The domain Ω is now decomposed into N nonoverlapping subdomains Ω_i, $i = 1,\ldots,N$, with diameter H_i. The resulting interface is given by $\Gamma := \bigcup_{i\neq j}(\partial\Omega_i \cap \partial\Omega_j) \setminus \partial\Omega_D$. We assume matching finite element nodes on the neighboring subdomains across the interface Γ.

Then, for each subdomain we assemble the corresponding linear system

$$K^{(i)}u^{(i)} = f^{(i)}.$$

From the local linear systems, we obtain the FETI-DP saddle point problem, which is solved using a FETI-DP algorithm; see e.g., [1, 2, 5–8] for references on this algorithm. In this article we consider in particular the algorithm given in [5, 6, 8]; see the latter references for an algorithmic description of parallel FETI-DP methods using primal edge constraints and a transformation of basis. Here, in particular, we assume that all vertices are primal and all edge averages over all subdomain edges are the same across the interface Γ.

In our analysis, each of the N subdomains may contain an almost incompressible part, here also called an inclusion or a component, surrounded by a compressible hull. We will specify the definitions of a hull as follows.

Definition 1. *The hull of a subdomain Ω_i with width η is defined as*

$$\Omega_{i,\eta} := \{x \in \Omega_i : \ \mathrm{dist}(x, \partial\Omega_i) < \eta\}; \qquad \textit{see Fig. 1.}$$

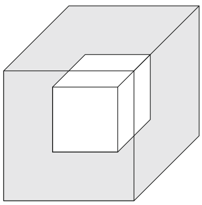

Fig. 1. $\Omega_{i,\eta}$: hull of Ω_i; see Definition 1

3 Convergence Analysis

In this section we provide a condition number estimate for the preconditioned FETI-DP matrix $M^{-1}F$, where F is the FETI-DP system matrix obtained from $K^{(i)}$ and

M^{-1} is the standard Dirichlet preconditioner; see [16]. We expand the convergence analysis, given in [8] for compressible linear elasticity, to the case where each subdomain can contain an almost incompressible inclusion surrounded by a compressible hull of thickness η. For the analysis, we make the following assumption; see [3] where the full details are provided.

Assumption 1 *For each subdomain, we have an inclusion which can be either almost incompressible or compressible, surrounded by a hull $\Omega_{i,\eta}$ of compressible material. The material coefficients $G(x)$ and $\beta(x)$ have a constant value in the interior inclusion and in the hull respectively, i.e.,*

$$G(x) = \begin{cases} G_{1,i} \; x \in \overline{\Omega}_{i,\eta} \\ G_{2,i} \; x \in \Omega_i \setminus \Omega_{i,\eta} \end{cases} \qquad \beta(x) = \begin{cases} \beta_{1,i} \; x \in \overline{\Omega}_{i,\eta} \\ \beta_{2,i} \; x \in \Omega_i \setminus \Omega_{i,\eta}. \end{cases}$$

Remark 1. Note that Assumption 1 allows that the Young modulus in the inclusion can be different from the one in the hull and that their quotient can be arbitrarily small or large.

The following assumption allows for the improved bound (2) in Theorem 1, which contains a linear factor H/η compared to the factor $(H/\eta)^4$ in (1).

Assumption 2 *For each subdomain Ω_i, $i = 1,\ldots,N$, we assume that $G_{1,i} \leq k_i \cdot G_{2,i}$, where $k_i > 0$ is a constant independent of $h, H, \eta, G_{1,i}$, and $G_{2,i}$.*

In the analysis provided in [3], for the edge term estimate, we need a further assumption.

Assumption 3 *For any pair of subdomains (Ω_i, Ω_k) which have an edge in common, we assume that there exists an acceptable path $(\Omega_i, \Omega_{j_1}, \ldots, \Omega_{j_n}, \Omega_k)$ from Ω_i to Ω_k, via a uniformly bounded number of other subdomains Ω_{i_q}, $q = 1,\ldots n$, such that the coefficients G_{1,j_q} of the Ω_{i_q} satisfy the condition*

$$TOL \cdot G_{1,j_q} \geq \min(G_{1,i}, G_{1,k}),\; q = 1,\ldots,n.$$

For a detailed description of the concept of acceptable paths, see [8, Sect. 5].

The following theorem is proven in [3].

Theorem 1. *Under the Assumptions 1 and 3, the condition number of the preconditioned FETI-DP system satisfies*

$$\kappa(M^{-1}F) \leq C \max(1, TOL) \left(1 + \log\left(\frac{H}{h}\right)\right) \left(1 + \log\left(\frac{\eta}{h}\right)\right) \left(\frac{H}{\eta}\right)^4, \qquad (1)$$

where $C > 0$ is independent of h, H, η, and the values of G_i and β_i, $i = 1,\ldots,N$ and hence also of E_i and ν_i.

If additionally Assumption 2 is satisfied, we have

$$\kappa(M^{-1}F) \leq C \max(1, TOL) \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \left(\frac{H}{\eta}\right), \qquad (2)$$

where $C > 0$ is independent of h, H, η, and the values of G_i and β_i, $i = 1,\ldots,N$ and hence also of E_i and ν_i.

4 Numerical Results

In this section, we present our numerical results for a linear elasticity problem in three dimensions. We consider almost incompressible inclusions in the interior of the subdomains. The inclusions are always surrounded by a compressible hull with $\nu = 0.3$. We use a FETI-DP algorithm with vertices and edge averages as primal constraints to control the rigid body modes. For the algorithmic concept, see for example [8]. The numerical results confirm our theoretical estimates.

Our tests are divided into different categories.

4.1 Variable Thickness of the Compressible Hull

Here, we present results for $3 \times 3 \times 3$ subdomains, a fixed $H/h = 11$, and a fixed Poisson ratio $\nu = 0.499999$ in each inclusion and $\nu = 0.3$ in each hull. For these computations we vary the thickness of the hull, i.e., $\eta = 0, h, \ldots, 5h$; see Table 1. For the case $\eta = 0$, we obtain a large condition number of $\kappa =$1,597.8. This is not surprising since we use a coarse space designed for compressible linear elasticity. In this case using a different, larger coarse space in 3D is the remedy; see, e.g., [10] or [12].

It is striking that already a hull with a thickness of one element, i.e., $\eta = h$, is sufficient to obtain a good condition number which is then not improved significantly by further increasing η. As a result, the number of iteration steps does not change for $\eta = h, \ldots, 5h$. In our theory, see Theorem 1, for this configuration of coefficients, our bound is linear in H/η. From the numerical results in Table 1 we cannot conclude that the bound is sharp. This might be due to the fact, that in 3D we cannot choose our mesh fine enough. However, for 2D problems using very fine meshes the linear dependence on H/η can be observed numerically; see Table 2.

Table 1. Growing η; $H/h = 11$; $1/H = 3$.

η	iterations	condition number
0	50	1597.8
$1h$	32	12.366
$2h$	32	12.250
$3h$	32	12.230
$4h$	32	12.231
$5h$	32	12.233

Growing η for $3 \times 3 \times 3$ subdomains, $E = 210$ on the whole domain, $\nu = 0.499999$ in each inclusion, and $\nu = 0.3$ in each hull. The results show only a weak dependence on η.

Table 2. Growing η; 2D; $H/h = 200$; $1/H = 3$

η	iterations	condition number
1/100	47	199.906
2/100	41	102.081
3/100	42	70.719
4/100	36	54.674

Linear elasticity in 2D with $\Omega = [0,1]^2$, discretized with $Q_1 - P_0$ stabilized finite elements; for a description of the discretization, see, e.g., [9]. The domain is decomposed into square subdomains with sidelength H, having square inclusions and a hull of thickness η. The Poisson ratio in each inclusion is chosen as $\nu = 0.4999999$ and in each hull as $\nu = 0.3$. The Young modulus is chosen as $E = 1$ on the whole domain. The results confirm the linear dependence on H/η.

4.2 Variable Incompressibility in the Inclusions

In Table 3, we vary the Poisson ratio in the inclusions from $\nu = 0.4$ up to $\nu = 0.499999$ while choosing a fixed number of elements in each subdomain, i.e., $H/h = 7$, and a thickness of the hull of $\eta = h$. We see that the condition number is indeed bounded independently of the almost incompressibility in the inclusions as expected from Theorem 1.

Table 3. Growing ν; $H/h = 7$; $1/H = 3$; $\eta = h$.

ν	iterations	condition number
0.4	27	9.4841
0.49	28	9.5038
0.499	28	9.5063
0.4999	28	9.5049
0.49999	28	9.5066
0.499999	29	9.5066

Growing ν for $3 \times 3 \times 3$ subdomains, $\eta = h$, $\nu = 0.3$ in the hulls, and $E = 210$ on the whole domain. A hull with a thickness of one element is clearly sufficient to obtain a good condition number.

4.3 Variable Young's Modulus in the Inclusions Combined with Variable Incompressibility in the Inclusions

In a last set of experiments, see Table 4, we consider subdomains with inclusions of a high and low Young modulus, i.e., $E = 1e+4$ and $E = 1e-4$, either combined with a Poisson ratio of $\nu = 0.4$ or $\nu = 0.499999$; see Fig. 2. The Young modulus of the hull is always $E = 1$ and its Poisson ratio is always $\nu = 0.3$. The four different parameter settings are determined by the number of the subdomain modulo four; see Fig. 2. In our theory, the condition number bound for such a configuration contains a factor $(H/\eta)^4$. However, the results in Table 4 are not worse than in the configurations where bound (1) of Theorem 1 applies, which contains only a linear H/η. The condition number is surprisingly low even if the thickness of the hull is only $\eta = h$. While this is a favorable result it also means that it is difficult to confirm numerically whether our theoretical bounds are sharp with respect to η.

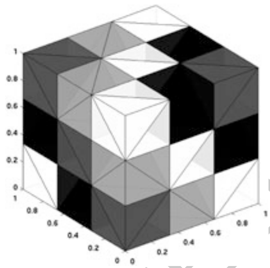

Fig. 2. Types of subdomains, see Table 4, identified by color

Table 4. Growing η; $H/h = 7$; $1/H = 3$.

distance η	iterations	condition number
0	>250	13426
$1h$	36	11.956
$2h$	29	9.2575
$3h$	29	9.4767
$4h$	27	9.4812

Growing η for $3 \times 3 \times 3$ subdomains. Four different kind of material parameter settings in the inclusions: $E = 1e+4$ and $\nu = 0.4$; $E = 1e-4$ and $\nu = 0.4$; $E = 1e+4$ and $\nu = 0.499999$; $E = 1e-4$ and $\nu = 0.499999$; for all hulls: $E = 1, \nu = 0.3$.

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