Improving the Convergence of Schwarz Methods for Helmholtz Equation

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1 Introduction

Various domain decomposition methods have been proposed for the Helmholtz equation, with the Optimized Schwarz Method (OSM) being one of them (see e.g. [7] for a review of various domain decomposition methods, and [3] for the details of OSM). In this paper, we focus on OSM, which is based on the idea of using approximated half-space Dirichlet-to-Neumann (DtN) maps to improve the convergence of the Schwarz methods; current version of the OSM is based on polynomial approximation of the half-space DtN map. See [8] for a review of various approaches to approximating the half-space DtN map (more commonly referred to as Absorbing Boundary Conditions (ABCs)).

There are two approximations in the OSM that affect its convergence rate – the first being the approximation of the rest of the domain as unbounded and the second being the approximation of the half-space stiffness (square-root operator) as a polynomial. In contrast with the polynomial approximation used in OSM, we utilize the method of Perfectly Matched Discrete Layers (PMDL), which has close links to the well-known Perfectly Matched Layers (PML) (see [1]) and the rational approximation of the square-root operator. The resulting PMDL-Schwarz method is shown to converge faster than the second-order OSM. The rest of the paper contains a brief review of OSM and PMDL concepts, followed by an outline of the new PMDL-Schwarz method and illustration of its effectiveness with the help of convergence factor analysis and a numerical example.

Model Problem. We consider the governing equation,

$$-\frac{\partial^2 \hat{u}}{\partial x^2} - \frac{\partial^2 \hat{u}}{\partial y^2} - \omega^2 \hat{u} = \hat{f}, \quad (x,y) \in (-\infty,\infty) \times [0,L], \tag{1a}$$

$$\hat{u}(\cdot,0) = \hat{u}(\cdot,L) = 0. \tag{1b}$$

Applying Fourier Sine transform along the y direction, the above equation reduces to a 1-D form:

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$$-\frac{\partial^2 u}{\partial x^2} - k^2 u = f\,, \quad x \in (-\infty, \infty), \tag{2}$$

where $k = \sqrt{\omega^2 - k_y^2}$, k_y is the wavenumber along y and u, f are the Fourier symbols corresponding to $\hat{u}$, $\hat{f}$ respectively. For simplicity, we shall use the above 1-D equation to discuss the main ideas in this paper, but note that the proposed method is applicable to more complex equations and geometries. Also, since the focus of this paper is to improve the treatment of the transmission condition at an interface, it is sufficient to consider the case of two subdomains. Thus the domain is decomposed into two subdomains: $\Omega_1 \equiv (-\infty, 0)$ and $\Omega_2 \equiv (0, \infty)$, with the interface at $x = 0$.

2 Optimized Schwarz Methods

Optimized Schwarz Method is a domain decomposition method that is a variant of the Schwarz Alternating Method (see e.g. [7]). In the Schwarz Alternating Method, the displacement and traction continuity across the artificial interface are enforced by applying a mixed boundary condition of the form $\mathscr{B}(\cdot) \equiv \partial(\cdot)/\partial\mathbf{n} + \Lambda(\cdot)$ where $\mathbf{n}$ is the normal vector at the interface and the operator Λ is a parameter of the method. The Schwarz iteration scheme for solving (2) is given by:

$$-\frac{\partial^2 u_1^{j+1}}{\partial x^2} - k^2 u_1^{j+1} = f_1\,, \quad x \in \Omega_1\,, \qquad -\frac{\partial^2 u_2^{j+1}}{\partial x^2} - k^2 u_2^{j+1} = f_2\,, \quad x \in \Omega_2\,, \tag{3a}$$

$$\mathscr{B}_1 u_1^{j+1} = \mathscr{B}_1 u_2^{j}, \quad x = 0\,, \qquad \mathscr{B}_2 u_2^{j+1} = \mathscr{B}_2 u_1^{j+1}, \quad x = 0\,, \tag{3b}$$

$$\mathscr{B}_1(\cdot) \equiv \frac{\partial(\cdot)}{\partial \mathbf{n}_1} + \Lambda_1(\cdot)\,, \qquad \mathscr{B}_2(\cdot) \equiv \frac{\partial(\cdot)}{\partial \mathbf{n}_2} + \Lambda_2(\cdot)\,, \tag{3c}$$

where the operators $\Lambda_{1,2}$ are the parameters of the iteration that determine the convergence rate. The problem now reduces to choosing the parameters that lead to optimal convergence of the iteration scheme. The parameters are commonly chosen to be scalars but they can be operators that are optimized for convergence [3]. The dependence of the convergence on the choice of parameters is better understood by looking at the convergence factor ρ, which is defined as

$$\left|\hat{e}_i^{\,j+1}\right| = \rho \left|\hat{e}_i^{\,j}\right|\,, \tag{4}$$

where $\hat{e}_i^j = |u - u_i^j|$ is the error in the solution in subdomain i at iteration j. Thus, after one cycle of iteration, the error in solution reduces by ρ and the iterative scheme converges to a solution as long as $\rho < 1$.

For the Schwarz method in (3), the convergence factor can be shown to be (see for e.g. [3])

$$\rho = \left| \left(\frac{\Lambda_1 - \mathscr{K}_2}{\Lambda_1 + \mathscr{K}_1}\right) \left(\frac{\Lambda_2 - \mathscr{K}_1}{\Lambda_2 + \mathscr{K}_2}\right) \right|\,, \tag{5}$$

where $\mathscr{K}_1$ and $\mathscr{K}_2$ are the DtN maps of the subdomains Ω_1 and Ω_2 respectively. It is clear from (5) that the iterative scheme does not converge (because $\rho = 1$) for a pure Neumann ($\Lambda_i = 0$) or Dirichlet ($\Lambda_i = \infty$) interface condition. Also, if $\Lambda_1 = \mathscr{K}_2$ or $\Lambda_2 = \mathscr{K}_1$, then $\rho = 0$ and the Schwarz iterative scheme converges in two iterations, i.e., the parameters are optimal. However, DtN maps are known only for special cases and even then are usually non-local operators that are expensive to compute accurately. Thus we look for local approximations to these DtN maps that are accurate and computationally efficient.

Optimized Schwarz Methods [3] essentially approximate the DtN map of the subdomains by polynomial approximations of the DtN map of an unbounded domain, e.g. the second-order OSM makes the approximation

$$\mathscr{K}_1 = -\mathrm{i}\sqrt{\omega^2 - k_y^2} \approx p + q k_y^2 \,, \tag{6}$$

where p, q are parameters that are found by minimizing the convergence factor over the entire range of allowed vertical wavenumbers k_y. Note that there are other variants of OSM based on zeroth-order approximation; in this paper, we focus on the best available OSM, namely the second-order OSM.

3 A Schwarz Method with Improved Convergence

It appears to us that OSM uses polynomial approximation for reasons of implementability. A better approximation would be to use higher order rational approximations, which have been investigated extensively in the context of Absorbing Boundary Conditions (ABCs); it is now possible to implement these resulting ABCs and can also be used in the context of Schwarz methods. In this paper, we propose the use of a rational approximation in a recent ABC called Perfectly Matched Discrete Layers (formerly known as Continued Fraction ABCs – see [4]) instead of the polynomial approximation in (6).

The rational approximation corresponding to PMDL is given by:

$$\mathscr{K}_1 = -\mathrm{i}\sqrt{\omega^2 - k_y^2} \approx \mathscr{S}_n^{pmdl} \,, \tag{7}$$

where

$$\mathscr{S}_n^{pmdl} = p_n - \cfrac{q_n^2}{p_n + \left(p_{n-1} + \cfrac{q_{n-1}^2}{p_{n-1} + \left(p_{n-2} - \cfrac{q_{n-2}^2}{p_{n-2} + (\dots)} \right)} \right)} \,, \tag{8}$$

$$\left.\begin{aligned} p_i &= \frac{1}{4L_i}\left(4 - k^2 L_i^2\right) \\ q_i &= \frac{1}{4L_i}\left(-4 - k^2 L_i^2\right) \end{aligned}\right\} \quad i = 1 \dots n \,. \tag{9}$$

where L_i are the parameters that determine the accuracy of the approximation.

The error in the approximation (7) is typically analyzed through the so-called reflection coefficient, which has been shown to be (for details, see [4])

$$R = \prod_{i=1}^{n} \left| \frac{\mathscr{K}_1 - p_i}{\mathscr{K}_1 + p_i} \right|^2 . \tag{10}$$

If $R = 0$, then the approximation is exact, and the deviation from zero indicates magnitude of error in the approximation; smaller the value of R, better the approximation. So from (10) and (9), it is clear that the accuracy of proposed approximation hinges on the choice of L_i.

In general, L_i are chosen to be complex or imaginary to better approximate the DtN map for propagating wave modes and are chosen to be real when evanescent modes are important. While the parameters L_i can be optimized using the concepts discussed in [5], in this paper we choose L_i based on the OSM parameters (see Sect. 4).

Implementation of PMDL. While the rational form of the PMDL approximation in (8) is useful for analysis, the following matrix form proves to be useful for implementation:

$$\begin{bmatrix} \mathscr{S}_n^{pmdl} u_b \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & 0 & \cdots & 0 \\ q_1 & p_1 + p_2 & q_2 & & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & q_{n-1} & p_{n-1} + p_n & q_n \\ 0 & \cdots & 0 & q_n & p_n \end{bmatrix} \begin{bmatrix} u_b \\ u_{a,1} \\ u_{a,2} \\ \vdots \\ u_{a,n-1} \end{bmatrix}, \tag{11}$$

where p_i, q_i are given by (9) and $u_{a,i}$ are auxiliary variables that are introduced to facilitate the implementation and have no direct physical relevance to the problem. The equivalence between (8) and (11) can be easily seen by eliminating the auxiliary dof $u_{a,i}$ from (11) to recover (8). The matrix form of PMDL enables an easy implementation of the rational approximation as a simple tri-diagonal matrix.

PMDL, a link between Rational ABCs and Perfectly Matched Layers. While the matrix form of the PMDL approximation in (11) is based on the rational approximation in (8), it is intimately linked to impedance-preserving discretization of PML proposed in [4]. Unlike PML, the impedance is preserved even after discretization and thus the approximation is named perfectly matched discrete layers, PMDL. This link is substantial in that it provides a way to derive and easily implement PMDL approximations for more complex cases such as corners [4] and anisotropic elasticity [6].

The ease of implementation of PMDL is in fact the impetus behind proposed method. As implied by (10), the accuracy of approximation can be easily increased by adding auxiliary variables, which is equivalent to adding lines of nodes parallel to the interface. As will be shown in Sect. 4, addition of just one auxiliary variable, which has minimal increase in computational cost per iteration, significantly reduces the convergence factor and the number of iterations needed.

Implementation of the PMDL-Schwarz method. The proposed PMDL-Schwarz method is essentially the Schwarz Alternating method with the operator Λ_1 chosen to be the DtN map obtained using PMDL, i.e., $\Lambda_1 = \mathscr{S}_n^{pmdl}$ where $\mathscr{S}_n^{pmdl}$ is given by (11). Thus the interface condition in (3) for Ω_1 can be written as

$$\frac{\partial}{\partial \mathbf{n_1}}(u_1^{j+1} - u_2^j) + \mathscr{S}_n^{pmdl}(u_1^{j+1} - u_2^j) = 0. \tag{12}$$

Substituting (11) in (12), we get the PMDL-Schwarz formulation as

$$\begin{bmatrix} \frac{\partial u_1^{j+1}}{\partial \mathbf{n_1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} p_1 & q_1 & 0 & \cdots & 0 \\ q_1 & p_1+p_2 & q_2 & & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & q_{n-1} & p_{n-1}+p_n & q_n \\ 0 & \cdots & 0 & q_n & p_n \end{bmatrix} \begin{bmatrix} u_1^{j+1} \\ u_{a,1} \\ u_{a,2} \\ \vdots \\ u_{a,n-1} \end{bmatrix} = \begin{bmatrix} -\frac{\partial u_2^j}{\partial \mathbf{n_2}} + p_1 u_2^j \\ q_1 u_2^j \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{13}$$

Note that the formulation of the interface condition for Ω_2 can be derived in an identical manner and hence is not repeated here.

4 Comparison Between OSM and PMDL-Schwarz Methods

In this section, we compare the performance of OSM and PMDL-Schwarz method both theoretically (using convergence factors) and in a numerical simulation involving multiple domains and closed boundaries.

Convergence Factors: Consider the stiffness approximation of the second-order OSM (see [3]),

$$\mathscr{S}_{osm} = \frac{ab - \omega^2}{a+b} + \frac{1}{a+b}k_y^2. \tag{14}$$

Substituting $\Lambda_1 = \Lambda_2 = \mathscr{S}_{osm}$ in (5), we get the convergence factor of OSM to be

$$\rho_{osm} = \left| \frac{ab + k_y^2 - \omega^2 + \mathrm{i}\,(a+b)\sqrt{\omega^2 - k_y^2}}{ab + k_y^2 - \omega^2 - \mathrm{i}\,(a+b)\sqrt{\omega^2 - k_y^2}} \right|^2.$$

To compare, we use a two-layer PMDL-Schwarz method with $L_1 = 2/a$, and $L_2 = 2/b$, where a, b are the OSM parameters in (14). The stiffness approximation of the two-layer PMDL-Schwarz method is then given by

$$\mathscr{S}_n^{pmdl} = p_2 - \frac{q_2^2}{p_2 + p_1},$$
$$p_2 = \frac{1}{L_2} - \frac{(\omega^2 - k_y^2)L_2}{4}, \qquad q_2 = -\frac{1}{L_2} - \frac{(\omega^2 - k_y^2)L_2}{4},$$
$$p_1 = \frac{1}{L_1} - \frac{(\omega^2 - k_y^2)L_1}{4}.$$

Substituting $\Lambda_1 = \Lambda_2 = \mathscr{S}_n^{pmdl}$ in (5), we get the convergence factor of PMDL-Schwarz that can be simplified to

$$\rho_{pmdl} = \left(\left| \frac{ab + k_y^2 - \omega^2 + \mathrm{i}(a+b)\sqrt{\omega^2 - k_y^2}}{ab + k_y^2 - \omega^2 - \mathrm{i}(a+b)\sqrt{\omega^2 - k_y^2}} \right|^2 \right)^2.$$

Clearly $\rho_{pmdl} = \rho_{osm}^2$, and so the parameters of PMDL-Schwarz are chosen such that its convergence factor is the square of that of OSM and the method performs uniformly better over the entire range of wavenumbers k_y.

It is easy to numerically verify the above result for the model problem (1a), with the domain Ω decomposed into two semi-infinite layers. We take $a = 20.741\,\mathrm{i}$ and $b = 47.071$ to be the OSM parameters as these were shown in [3] to be optimal over the allowed wavenumber range $k_y \in [\pi, 60\pi]$. Figure 1a compares the convergence factors of OSM and PMDL-Schwarz method (with $L_1 = 2/a$ and $L_2 = 2/b$) and shows clearly that the proposed method performs better over the entire range of wavenumbers for a slightly increased computational cost (there is only one auxiliary variable introduced, which is similar to one line of nodes in 2-D).

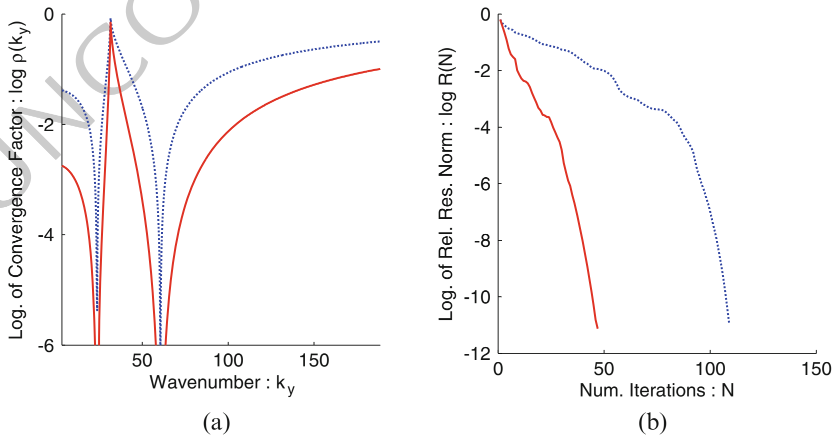

Fig. 1. Comparison between OSM (*dotted line*) and PMDL-Schwarz method (*solid line*) . **(a)** Convergence Factor. **(b)** Convergence of Solution

Numerical Example: In this example, Eq. (1a) is solved on a square domain ($\Omega \equiv [0,1] \times [0,1]$) with $\omega = 10\pi$ and a point source $f = 1/2$ is applied at $(0, 0.5)$. Homogeneous Neumann boundary condition is applied on the left ($x = 0$), Dirichlet condition at the top ($y = 1$) and bottom ($y = 0$), and an ABC on the right ($x = 1$). The computational domain is discretized using 60 bilinear finite elements along each direction. The domain is decomposed into nine subdomains with three subdomains along each dimension. The convergence plot is shown in Fig. 1b. As expected, the PMDL-Schwarz method converges twice as fast as the conventional OSM.

5 Discussion

We proposed a Schwarz method for Helmholtz equation based on the concepts of perfectly matched discrete layers (PMDL), a recently developed absorbing boundary condition that is related to the higher order rational approximations and the Perfectly Matched Layers. By examining the convergence factor and with the help of a numerical example, PMDL-Schwarz method is shown to converge faster than existing Optimized Schwarz Methods. Although not treated in this paper, it is important to mention that the PMDL is not just limited to the Helmholtz equation, but also to more complicated vector equations such as the elastic and electromagnetic wave equations. Thus, it is expected that the PMDL-Schwarz method would provide accelerated convergence in frequency domain computations in these contexts. Furthermore, as Waveform Relaxation Method in time domain share similar ideas with OSM (see e.g. [2]), PMDL ideas can also be used to improve the convergence of existing waveform relaxation methods. These extensions are subjects of ongoing research.

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