A Subspace Correction Method for Nearly Singular Linear Elasticity Problems

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1 Introduction

The focus of this work is on constructing a robust (uniform in the problem parameters) iterative solution method for the system of linear algebraic equations arising from a nonconforming finite element discretization based on reduced integration. We introduce a specific space decomposition into two overlapping subspaces that serves as a basis for devising a uniformly convergent subspace correction algorithm. We consider the equations of linear elasticity in primal variables. For nearly incompressible materials, i.e., when the Poisson ratio ν approaches $1/2$, this problem becomes ill-posed and the resulting discrete problem is nearly singular.

Subspace correction methods for nearly singular systems have been studied in [10] leading to robust multigrid methods for planar linear elasticity problems (see [11]). In [13] a multigrid method has been presented for a finite element discretization with $P_2 - P_0$ elements. This approach relies on a local basis for the weakly divergence-free functions.

In this setting, presently known (multilevel) iterative solution methods are optimal or nearly optimal for the pure displacement problem only, i.e., when Dirichlet boundary conditions are imposed on the entire boundary, see, e.g., [1, 4]. For pure traction or mixed boundary conditions the problem gets more involved. It is known, that standard (conforming and nonconforming) finite element methods then require certain stabilization techniques, see, e.g., [3, 6]. We employ a discretization scheme introduced in [3] which achieves the stabilization via reduced integration. Note that based on an appropriate discrete version of Korn's second inequality optimal error estimates have been shown for this method (see [3]).

The remainder of this paper is organized as follows: The formulation of the linear elasticity problem with pure traction boundary conditions and its finite element discretization are given in Sect. 2. We briefly recall some convergence results for the *Method of Successive Subspace Correction* (MSSC) in Sect. 3. In Sect. 4 we present a specific space decomposition which defines an MSSC preconditioner. Finally, we

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present a numerical test illustrating the optimal performance of the preconditioner in Sect. 5.

2 Problem Formulation

For the sake of simplicity we consider only two-dimensional problems in this paper. Let Ω be a bounded, connected and open subset of $\mathbb{R}^2$, denoting the reference configuration of an elastic body. The boundary of Ω is denoted by $\partial\Omega$. Following [3] we consider the pure traction problem of linear elasticity which reads

$$\boldsymbol{\sigma} = \mu \left[\boldsymbol{\varepsilon}(\mathbf{u}) + \frac{\nu}{1-2\nu} \operatorname{div} \mathbf{u} I \right] \qquad \text{in } \Omega, \tag{1a}$$

$$-\operatorname{div} \boldsymbol{\sigma} = \mathbf{f} \qquad \text{in } \Omega, \tag{1b}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{g} \qquad \text{on } \partial\Omega. \tag{1c}$$

where $\boldsymbol{\sigma}$ denotes the stress tensor and $\boldsymbol{\varepsilon}(\mathbf{u}) := \nabla^{(s)}\mathbf{u}$ is the symmetric gradient, i.e., $\varepsilon_{ij}(\mathbf{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. Further $\mathbf{u}$ denotes the vector of displacements, $\mathbf{f}$ denotes the body forces, $\mathbf{n}$ is the outwards pointing unit normal vector on $\Gamma = \partial\Omega$ and $\mathbf{g}$ is the applied load on Γ. The properties of the material depend on the Poisson ratio $\nu \in [0, 1/2)$, and the shear modulus $\mu := \frac{E}{1+\nu}$ where E is the modulus of elasticity.

We consider the space $\boldsymbol{V}^{\mathrm{RBM}} := \{\mathbf{v} : \mathbf{v} = (a_1 + by, a_2 - bx)^t,\ a_1, a_2, b \in \mathbb{R}\}$ of rigid body motions and define the subspace $\hat{\boldsymbol{V}}$ of H^1-functions orthogonal to $\boldsymbol{V}^{\mathrm{RBM}}$, i.e.,

$$\hat{\boldsymbol{V}} := \{\mathbf{v} \in [H^1(\Omega)]^2 : \int_\Omega \mathbf{v}\, \mathrm{d}\mathbf{x} = \mathbf{0} \quad \text{and} \quad \int_\Omega v_1 y - v_2 x\, \mathrm{d}\mathbf{x} = 0\}. \tag{2}$$

Let $\mathscr{T}_H$ be a quasi-uniform triangulation of Ω. Moreover, we subdivide each triangle $T \in \mathscr{T}_H$ into four congruent triangles by adding the midpoints of the edges to the set of vertices. The obtained refined triangulation $\mathscr{T}_h$ of Ω has a mesh size $h = H/2$. We introduce the vector space $\boldsymbol{V} := [V]^2 := [H^1(\Omega)]^2$ and the subspace $\boldsymbol{V}_h := [V_h]^2$, which consists of the vector-valued continuous piecewise linear functions on the fine mesh $\mathscr{T}_h$. Next we define $\hat{\boldsymbol{V}}_h := \boldsymbol{V}_h \cap \hat{\boldsymbol{V}}$ and denote the space of piecewise constant functions on $\mathscr{T}_H$ by S_H. Then we consider the problem: Find $\mathbf{u}_h \in \hat{\boldsymbol{V}}_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = L(\mathbf{v}_h) := (\mathbf{f}, \mathbf{v}_h)_0 + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v}_h\, \mathrm{d}s \qquad \forall \mathbf{v}_h \in \hat{\mathbf{V}}_h, \tag{3}$$

$$a(\mathbf{u}_h, \mathbf{v}_h) := \mu \left((\boldsymbol{\varepsilon}(\mathbf{u}_h), \boldsymbol{\varepsilon}(\mathbf{v}_h))_0 + \frac{\nu}{1-2\nu} (P_0 \operatorname{div} \mathbf{u}_h, P_0 \operatorname{div} \mathbf{v}_h)_0 \right), \tag{4}$$

where $\mathbf{f} \in [L_2(\Omega)]^2$ and $\mathbf{g} \in [L_2(\partial\Omega)]^2$. P_0 is the L^2-projection onto S_H, that is,

$$P_0(v)|_{T_H} = \frac{1}{|T_H|} \int_{T_H} v\, \mathrm{d}\mathbf{x} \qquad \forall T_H \in \mathscr{T}_H, \tag{5}$$

for any scalar function $v \in L^2(\Omega)$. It is known that under the compatibility condition $L(\mathbf{v}) = 0$ for all $\mathbf{v} \in \boldsymbol{V}^{\mathrm{RBM}}$ problem (3) has a unique solution $\mathbf{u}_h \in \hat{\boldsymbol{V}}_h$, see, e.g., [1]. In [3] optimal order error estimates have been shown for this approximation, which are robust with respect to the Poisson ratio ν.

3 Subspace Correction Framework

The general framework of subspace correction methods is closely related to the abstract Schwarz theory, see, e.g., [5, 14].

Let us consider the variational problem: Find $u \in V$ such that

$$a(u,v) = f(v) \qquad \forall v \in V\,, \tag{6}$$

with $V \subset H$ being a closed subset of the Hilbert space H. Moreover, we assume that the bilinear form $a(.,.) : H \times H \to \mathbb{R}$ is continuous, symmetric, and H-elliptic. If f is a continuous linear functional on H, then this problem is well-posed.

Now, let us split V into a–not necessarily direct–sum of closed subspaces $V_i \subset V$, $i = 1,\ldots,J$, i.e., $V = \sum_{i=1}^{J} V_i$. With each subspace V_i we associate a symmetric, bounded, and elliptic bilinear form $a_i(.,.)$ approximating $a(.,.)$ on V_i. The MSSC (see [16, Algorigthm 2.1]) solves the residual equation for $i = 1,\ldots,J$ with $u_l = u^l$: Find $e_i \in V_i$ such that for all $v_i \in V_i$, there holds:

$$a(e_i, v_i) = f(v_i) - a(u_{l+i-1}, v_i), \quad \text{and set} \quad u_{l+i} = u_{l+i-1} + e_i, \tag{7}$$

Finally, the next iterate is $u^{l+1} = u_{l+J}$. Let $T_i : V \to V_i$ be defined as

$$a_i(T_i v, v_i) = a(v, v_i), \quad \text{for all} \quad v_i \in V_i.$$

The assumptions on $a_i(.,.)$ imply that T_i is well-defined, $\mathscr{R}(T_i) = V_i$, and $T_i : V_i \to V_i$ is an isomorphism. The error after l iterations of the MSSC is given by $u - u^l = E(u - u^{l-1}) = \ldots = E^l(u - u^0)$, where the error propagation operator E can be represented in product form , i.e.,

$$E = (I - T_J)(I - T_{J-1}) \cdots (I - T_1)\,. \tag{8}$$

In the following we consider the case of exact subspace solves, i.e., $a_i(.,.) = a(.,.)$ on V_i, in which T_i reduces to the idempotent, a-adjoint operator P_i defined by

$$a(P_i v, v_i) = a(v, v_i) \qquad \forall v_i \in V_i\,. \tag{9}$$

For a proof of the following identity for the energy norm of the error propagation operator we refer the reader to [16].

Theorem 1. *Under the assumptions* (9) *and* $V = \sum_{i=1}^{J} V_i$ *we have*

$$\|E\|_a^2 = \|(I - P_J)(I - P_{J-1}) \cdots (I - P_1)\|_a^2 = \frac{c_0}{1 + c_0} \tag{10}$$

where $c_0 = \sup_{\|v\|_a = 1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} \|P_i \sum_{j=i+1}^{J} v_j\|_a^2 < \infty$.

Let $\mathscr{E}_H$ be the set of edges of $\mathscr{T}_H$ and $\mathscr{V}_H$ be the set of (coarse) vertices of the mesh $\mathscr{T}_H$. Then for any vertex $v_i \in \mathscr{V}_H$ we denote the set of edges sharing v_i by $\mathscr{N}_i^{\mathscr{E}}$. For any edge $E = (v_{E,1}, v_{E,2}) \in \mathscr{E}_H$ by φ_E we denote the scalar nodal basis function corresponding to the midpoint of the edge E, and by $\varphi_{E,1}$ and $\varphi_{E,2}$ the nodal basis functions corresponding to the vertices $v_{E,1}$ and $v_{E,2}$ of E. The corresponding vector-valued degrees of freedom (dof) of any function $\mathbf{v}_h \in \boldsymbol{V}_h$ are denoted by $\mathbf{v}_E$, $\mathbf{v}_{E,1}$ and $\mathbf{v}_{E,2}$, respectively. We further use φ_i and $\mathbf{v}_i$ to denote the basis functions and dof associated with the vertices from $\mathscr{V}_H$.

For any edge $E \in \mathscr{E}_H$ we assume that $v_{E,1} < v_{E,2}$ and that the globally defined tangential vector τ_E points from $v_{E,1}$ to $v_{E,2}$. The global edge normal vector $\mathbf{n}_E$ is orthogonal to τ_E and is obtained from τ_E by a clockwise rotation. By $\boldsymbol{V}_H^{RT}$ we denote the lowest order Raviart Thomas space (cf. [2]), i.e.,

$$\boldsymbol{V}_H^{RT} := \{\mathbf{v} \in [L^2(\Omega)]^2 \,:\, \mathbf{v} = \mathbf{a} + (bx, by)^t \text{ on each } T \in \mathscr{T}_H,\ \mathbf{a} \in \mathbb{R}^2,\ b \in \mathbb{R}\} \tag{11}$$

where the degrees of freedom are the normal fluxes over the edges E, i.e., $F_E^{RT}(\mathbf{v}) := \frac{1}{|E|}\int_E \mathbf{v}\cdot\mathbf{n}_E \, ds$. The basis functions φ_E^{RT} corresponding to an edge E of an element $T \in \mathscr{T}_H$ are such that $F_{E'}^{RT}(\varphi_E^{RT}) := \delta_{EE'}$. We also use the projection $\Pi^{RT} : \boldsymbol{V} \mapsto \boldsymbol{V}_H^{RT}$ defined by $\Pi^{RT}(\mathbf{v}) = \sum_{E\in\mathscr{E}_H} F_E^{RT}(\mathbf{v})\varphi_E^{RT}$, for which the commuting property $P_0 \operatorname{div} \mathbf{v}_h = \operatorname{div} \Pi^{RT}(\mathbf{v}_h)$ holds for any $\mathbf{v}_h \in \boldsymbol{V}_h$ (cf. [2, p. 131]).

4 Space Decomposition

Let us consider the following unique decomposition of any function $\mathbf{v}_h \in \boldsymbol{V}_h$:

$$\begin{aligned}
\mathbf{v}_h &= \sum_{i\in\mathscr{V}_H} \varphi_i \mathbf{v}_i + \sum_{E\in\mathscr{E}_H} \varphi_E \mathbf{v}_E \\
&= \underbrace{\sum_{i\in\mathscr{V}_H} \Big[\varphi_i \mathbf{v}_i - \frac{1}{2}\sum_{E\in\mathscr{N}_i^{\mathscr{E}}} (\mathbf{v}_i\cdot\mathbf{n}_E)\varphi_E \mathbf{n}_E\Big]}_{=:\mathbf{v}_{\mathscr{V}}} + \underbrace{\sum_{E\in\mathscr{E}_H} (\mathbf{v}_E\cdot\tau_E)\varphi_E \tau_E}_{=:\mathbf{v}_\tau} \\
&\quad + \underbrace{\sum_{E\in\mathscr{E}_H} \left(\Big[\mathbf{v}_E + \frac{1}{2}(\mathbf{v}_{E,1} + \mathbf{v}_{E,2})\Big]\cdot\mathbf{n}_E\right)\varphi_E \mathbf{n}_E}_{=:\mathbf{v}_n} .
\end{aligned}$$

Next we define the splitting $\boldsymbol{V}_h = \boldsymbol{V}_{\mathscr{V}} \oplus \boldsymbol{V}_\tau \oplus \boldsymbol{V}_n$, where

$$\boldsymbol{V}_{\mathscr{V}} := \{\mathbf{v}_h \in \boldsymbol{V}_h \,:\, \mathbf{v}_h = \sum_{i\in\mathscr{V}_H} \Big[\varphi_i \mathbf{v}_i - \frac{1}{2}\sum_{E\in\mathscr{N}_i^{\mathscr{E}}} (\mathbf{v}_i\cdot\mathbf{n}_E)\,\varphi_E \mathbf{n}_E\Big]\},$$

$$\boldsymbol{V}_\tau := \{\mathbf{v}_h \in \boldsymbol{V}_h \,:\, \mathbf{v}_h = \sum_{E\in\mathscr{E}_H} \alpha_E \varphi_E \tau_E\}, \quad \boldsymbol{V}_n := \{\mathbf{v}_h \in \boldsymbol{V}_h \,:\, \mathbf{v}_h = \sum_{E\in\mathscr{E}_H} \alpha_E \varphi_E \mathbf{n}_E\}.$$

Note that $\Pi^{RT}(\boldsymbol{V}_{\mathscr{V}}) = \Pi^{RT}(\boldsymbol{V}_\tau) = \{0\}$. Next, we introduce the spaces

$$\boldsymbol{V}_{\text{curl}} := \{\mathbf{v}_h \in \boldsymbol{V}_h \,:\, \mathbf{v}_h = \sum_{i\in\mathscr{V}_H} \beta_i \sum_{E\in\mathscr{N}_i^{\mathscr{E}}} \frac{\delta_{E,i}}{|E|}\varphi_E \mathbf{n}_E\},$$

$$\boldsymbol{V}_{\nabla_h} := \{\mathbf{v}_h \in \boldsymbol{V}_h \,:\, \mathbf{v}_h = \sum_{T\in\mathscr{T}_H} \gamma_T \sum_{E\subset T} (\mathbf{n}_E\cdot\mathbf{n}_{E,T})\,\varphi_E \mathbf{n}_E\}.$$

Here $\delta_{E,i}$ is defined by

$$\delta_{E,i} = \begin{cases} -1 & \text{if } i = \nu_{E,1} \\ 1 & \text{if } i = \nu_{E,2} \end{cases} . \tag{12}$$

Note that $\boldsymbol{V}_{\mathrm{curl}} \subset \boldsymbol{V}_n$, and $\boldsymbol{V}_{\nabla_h} \subset \boldsymbol{V}_n$, and the following properties hold:

$$\begin{aligned} P_0 \operatorname{div}(\mathbf{v}_{\mathrm{curl}}) &= \operatorname{div} \Pi^{RT}(\mathbf{v}_{\mathrm{curl}}) = 0 \qquad && \forall \mathbf{v}_{\mathrm{curl}} \in \boldsymbol{V}_{\mathrm{curl}}\,, \\ P_0 \operatorname{div}(\mathbf{v}_{\nabla_h}) &= \operatorname{div} \Pi^{RT}(\mathbf{v}_{\nabla_h}) \neq 0 \qquad && \forall \mathbf{v}_{\nabla_h} \in \boldsymbol{V}_{\nabla_h}\,. \end{aligned}$$

Moreover, $\dim(\boldsymbol{V}_{\mathrm{curl}}) = n_{v,H} - 1$ and $\dim(\boldsymbol{V}_{\nabla_h}) = n_{T,H}$, and thus, using Euler's formula, i.e., $n_{v,H} - 1 + n_{T,H} = n_{E,H}$, we find that $\boldsymbol{V}_n = \boldsymbol{V}_{\mathrm{curl}} \oplus \boldsymbol{V}_{\nabla_h}$. Hence we obtain

$$\boldsymbol{V}_h = \boldsymbol{V}_{\mathscr{V}} \oplus \boldsymbol{V}_{\tau} \oplus \boldsymbol{V}_{\mathrm{curl}} \oplus \boldsymbol{V}_{\nabla_h}. \tag{13}$$

Finally, we decompose $\boldsymbol{V}_h$ into two overlapping subspaces $\boldsymbol{V}_I$ and $\boldsymbol{V}_{II}$:

$$\boldsymbol{V}_I = \boldsymbol{V}_{\mathscr{V}} \oplus \boldsymbol{V}_{\tau} \oplus \boldsymbol{V}_{\mathrm{curl}} \tag{14}$$

$$\boldsymbol{V}_{II} = \boldsymbol{V}_{\tau} \oplus \boldsymbol{V}_{\mathrm{curl}} \oplus \boldsymbol{V}_{\nabla_h} \tag{15}$$

The overlap of $\boldsymbol{V}_I$ and $\boldsymbol{V}_{II}$ is given by $\boldsymbol{V}_{\tau} + \boldsymbol{V}_{\mathrm{curl}}$, and any element $\mathbf{v}_{II} \in \boldsymbol{V}_{II}$ can be uniquely decomposed into $\mathbf{v}_{II} = \mathbf{v}_{\tau} + \mathbf{v}_{\mathrm{curl}} + \mathbf{v}_{\nabla_h}$, with $\mathbf{v}_{\tau} \in \boldsymbol{V}_{\tau}$, $\mathbf{v}_{\mathrm{curl}} \in \boldsymbol{V}_{\mathrm{curl}}$ and $\mathbf{v}_{\nabla_h} \in \boldsymbol{V}_{\nabla_h}$. However, finding the components $\mathbf{v}_{\mathrm{curl}} \in \boldsymbol{V}_{\mathrm{curl}}$ and $\mathbf{v}_{\nabla_h} \in \boldsymbol{V}_{\nabla_h}$ for a given function $\mathbf{v}_n \in \boldsymbol{V}_n$ requires a solution of a system with an M-matrix corresponding to the lowest order mixed method for Laplace equation with lumped mass [2].

Note that since $P_0 \operatorname{div}(\boldsymbol{V}_I) = \operatorname{div} \Pi^{RT}(\boldsymbol{V}_I) = \{0\}$ the bilinear form $a(.,.)$ satisfies

$$a(\mathbf{u}_I, \mathbf{v}_I) = \mu(\boldsymbol{\varepsilon}(\mathbf{u}_I), \boldsymbol{\varepsilon}(\mathbf{v}_I))_0 \qquad \forall \mathbf{u}_I, \mathbf{v}_I \in \boldsymbol{V}_I\,, \tag{16}$$

and in the limit case $\nu = 0$ we have $a(\mathbf{u}_h, \mathbf{v}_h) = \mu(\boldsymbol{\varepsilon}(\mathbf{u}_h), \boldsymbol{\varepsilon}(\mathbf{v}_h))_0$ for all $\mathbf{u}_h, \mathbf{v}_h \in \boldsymbol{V}_h$.

In the following, we use the operator representations $A : V \to V$ and $A_\varepsilon : V \to V$ for the bilinear forms $a(.,.)$ and $\mu(\boldsymbol{\varepsilon}(.), \boldsymbol{\varepsilon}(.))_0$. If we symmetrize the MSSC, we obtain the following error propagation $\bar{E}_{MSSC}$, compare with (8) in case of $J = 2$ and exact subsolves, i.e.,

$$\bar{E}_{MSSC} = (I - P_I)(I - P_{II})(I - P_I)\,.$$

The error propagation operator can be rewritten as $\bar{E}_{MSSC} = I - \bar{B}_{MSSC}A$, with symmetric $\bar{B}_{MSSC}$. Further, $\bar{B}_{MSSC}$ is positive definite, since $\bar{E}_{MSSC}$ is non-expansive. Note that even though $\bar{B}_{MSSC} = (I - \bar{E}_{MSSC})A^{-1}$ formally involves the inverse of A, we do not need A^{-1} in order to apply $\bar{B}_{MSSC}$.

If ν is bounded away from the incompressible limit $1/2$, we know that A_ε is spectrally equivalent to A. Further, there are efficient preconditioners for A_ε. We now define the additive preconditioner B by

$$B := \frac{1 - 2\nu}{1 - \nu} A_\varepsilon^{-1} + \frac{\nu}{1 - \nu} \bar{B}_{MSSC}\,. \tag{17}$$

Note that B is a convex combination of A_ε^{-1} and $\bar{B}_{MSSC}$.

Remark 1. It has been shown in [14, 16] that an inexact solution of the subproblems (7) results in a uniform preconditioner under reasonable assumptions. The subproblems on the spaces $\boldsymbol{V}_I$ and $\boldsymbol{V}_h$ involve the bilinear form

$$\bar{a}(\mathbf{u}_i, \mathbf{v}_i) = \mu(\boldsymbol{\varepsilon}(\mathbf{u}_i), \boldsymbol{\varepsilon}(\mathbf{v}_i))_0 \qquad \forall \mathbf{u}_i, \mathbf{v}_i \in \boldsymbol{W} = \boldsymbol{V}_I, \boldsymbol{V}_h. \tag{18}$$

Any efficient preconditioning technique for the vector-Laplace equation can be employed in these steps, e.g., classical AMG (see [12]) or AMGm (see [8]).

The problem on $\boldsymbol{V}_{II} = \boldsymbol{V}_E := \{\mathbf{v}_h \in \boldsymbol{V}_h : \mathbf{v}_h(\mathbf{x}_i) = \mathbf{0} \ v_i \in \mathscr{V}_H\}$ is more involved. First, by using Korn's inequality, Poincarè's inequality and the inverse inequality one can show that

$$\|\boldsymbol{\varepsilon}(\mathbf{v}_E)\|_0^2 \approx \|\boldsymbol{\nabla}\mathbf{v}_E\|_0^2 \approx H^{-2}\|\mathbf{v}_E\|_0^2.$$

Second, any function $\mathbf{v}_E \in \boldsymbol{V}_E$ can be uniquely decomposed into $\mathbf{v}_E = \mathbf{v}_n + \mathbf{v}_\tau$ where $\mathbf{v}_n \in \boldsymbol{V}_n$ and $\mathbf{v}_\tau \in \boldsymbol{V}_\tau$. Moreover, by locally estimating the angle between $\boldsymbol{V}_n$ and $\boldsymbol{V}_\tau$ in the $a(\cdot,\cdot)$-inner product, it can be shown that

$$\|\mathbf{v}_E\|_0^2 = \|\mathbf{v}_n + \mathbf{v}_\tau\|_0^2 \approx \|\mathbf{v}_n\|_0^2 + \|\mathbf{v}_\tau\|_0^2 \tag{19}$$

holds uniformly with respect to the mesh size h. Furthermore $\Pi^{RT}(\mathbf{v}_\tau) = 0$ for all $\mathbf{v}_\tau \in \boldsymbol{V}_\tau$. Hence, the relation $a(\mathbf{u}_E, \mathbf{v}_E) \approx \tilde{a}(\mathbf{u}_E, \mathbf{v}_E)$ holds on $\boldsymbol{V}_{II}$ where

$$\begin{aligned} \tilde{a}(\mathbf{u}_E, \mathbf{v}_E) := \mu\Big\{ & H^{-2}(\mathbf{u}_\tau, \mathbf{v}_\tau)_0 \\ & + H^{-2}(\mathbf{u}_n, \mathbf{v}_n)_0 + \frac{\nu}{1-2\nu}(P_0 \operatorname{div} \mathbf{u}_n, P_0 \operatorname{div} \mathbf{v}_n)_0 \Big\}. \end{aligned} \tag{20}$$

Now, using the interpolation operator $I_{RT}^h : \boldsymbol{V}_H^{RT} \to \boldsymbol{V}_h$, defined by $I_{RT}^h(\varphi_E^{RT}) = 2\varphi_E \mathbf{n}_E \in \boldsymbol{V}_n$, one can show that $\boldsymbol{V}_n$ is isomorphic to $\boldsymbol{V}_H^{RT}$. Thus solving a variational problem with $\tilde{a}(.,.)$ on $\boldsymbol{V}_n$ is equivalent to solving a problem with the bilinear form

$$a_{RT}(\mathbf{u}_{RT}, \mathbf{v}_{RT}) := \mu\left\{H^{-2}(\mathbf{u}_{RT}, \mathbf{v}_{RT})_0 + \frac{\nu}{1-2\nu}(\operatorname{div}\mathbf{u}_{RT}, \operatorname{div}\mathbf{v}_{RT})_0\right\}, \tag{21}$$

on $\boldsymbol{V}_H^{RT}$ (see [7, 15]). An efficient solver for the latter problem can be designed by using the auxiliary space preconditioner of [7], or by using the robust algebraic multilevel iteration method developed in [9].

5 Numerical Experiment

We now perform a numerical test to show that the preconditioner (17) is an efficient and robust preconditioner. We consider the problem with homogenous Dirichlet boundary conditions on the unit square $\Omega = (0,1)^2$. The number of PCG iterations for a residual reduction by a factor 10^8 are shown in Table 1. The subproblems on V_I and V_{II} are solved exactly. Additionally, we list the estimated condition numbers $\kappa(BA)$, obtained from the Lanczos process.

Table 1. Iteration numbers (#it.) and condition numbers ($\kappa(BA)$) of the pcg-cycle.

#DOF	242		1 058		4 418		18 050		72 962		293 378	
	#it.	κ	#it.	κ	#it.	κ	#it.	κ	#it.	κ	#it.	κ
$\nu = 0$:	1	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1	1.00
$\nu = 0.25$:	8	1.41	8	1.48	8	1.53	9	1.55	9	1.57	9	1.57
$\nu = 0.4$:	10	1.90	11	2.19	12	2.38	12	2.49	13	2.57	13	2.62
$\nu = 0.45$:	11	2.11	12	2.61	14	3.01	15	3.25	15	3.41	15	3.52
$\nu = 0.49$:	10	1.90	11	2.54	14	3.31	16	3.97	17	4.39	17	4.69
$\nu = 0.499$:	9	1.98	10	1.98	11	2.13	14	2.99	15	3.83	17	4.51
$\nu = 0.4999$:	9	1.99	9	1.99	9	1.99	10	1.99	12	2.43	13	3.34
$\nu = 0.49999$:	9	1.99	9	1.99	9	2.00	9	2.00	9	2.00	10	2.00

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