Penalty Robin-Robin Domain Decomposition Schemes for Contact Problems of Nonlinear Elastic Bodies

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1 Introduction

Many domain decomposition techniques for contact problems have been proposed on discrete level, particularly substructuring and FETI methods [1, 4].

Domain decomposition methods (DDMs), presented in [2, 10, 11, 16] for unilateral two-body contact problems of linear elasticity, are obtained on continuous level. All of them require the solution of nonlinear one-sided contact problems for one or both of the bodies in each iteration.

In works [6, 14, 15] we have proposed a class of penalty parallel Robin–Robin domain decomposition schemes for unilateral multibody contact problems of linear elasticity, which are based on penalty method and iterative methods for nonlinear variational equations. In each iteration of these schemes we have to solve in a parallel way some linear variational equations in subdomains.

In this contribution we generalize domain decomposition schemes, proposed in [6, 14, 15] to the solution of unilateral and ideal contact problems of nonlinear elastic bodies. We also present theorems about the convergence of these schemes.

2 Formulation of Multibody Contact Problem

Consider a contact problem of N nonlinear elastic bodies $\Omega_\alpha \subset \mathbb{R}^3$ with sectionally smooth boundaries Γ_α, $\alpha = 1,2,\ldots,N$ (Fig. 1). Denote $\Omega = \bigcup_{\alpha=1}^{N} \Omega_\alpha$.

A stress-strain state in point $\mathbf{x} = (x_1,x_2,x_3)^\top$ of each body Ω_α is defined by the displacement vector $\mathbf{u}_\alpha = u_{\alpha i}\,\mathbf{e}_i$, the tensor of strains $\hat{\boldsymbol{\varepsilon}}_\alpha = \varepsilon_{\alpha ij}\,\mathbf{e}_i\,\mathbf{e}_j$ and the tensor of stresses $\hat{\boldsymbol{\sigma}}_\alpha = \sigma_{\alpha ij}\,\mathbf{e}_i\,\mathbf{e}_j$. These quantities satisfy Cauchy relations, equilibrium equations and nonlinear stress-strain law [8]:

$$\sigma_{\alpha ij} = \lambda_\alpha\,\delta_{ij}\,\Theta_\alpha + 2\,\mu_\alpha\,\varepsilon_{\alpha ij} - 2\,\mu_\alpha\,\omega_\alpha(e_\alpha)\,e_{\alpha ij}\,,\ i,j = 1,2,3\,, \quad (1)$$

where $\Theta_\alpha = \varepsilon_{\alpha 11} + \varepsilon_{\alpha 22} + \varepsilon_{\alpha 33}$ is the volume strain, $\lambda_\alpha(\mathbf{x}) > 0$, $\mu_\alpha(\mathbf{x}) > 0$ are bounded Lame parameters, $e_{\alpha ij} = \varepsilon_{\alpha ij} - \delta_{ij}\,\Theta_\alpha/3$ are the components of the strain

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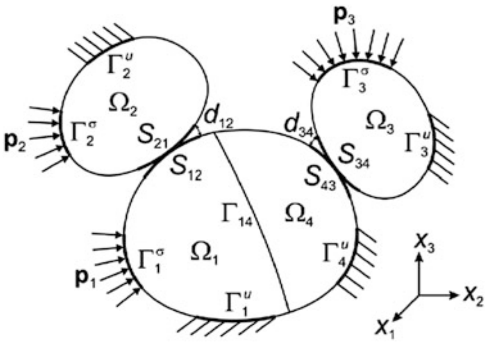

Fig. 1. Contact of several bodies

deviation tensor, $e_\alpha = \sqrt{2 g_\alpha}/3$ is the deformation intensity, $g_\alpha = (\varepsilon_{\alpha 11} - \varepsilon_{\alpha 22})^2 + (\varepsilon_{\alpha 22} - \varepsilon_{\alpha 33})^2 + (\varepsilon_{\alpha 33} - \varepsilon_{\alpha 11})^2 + 6\,(\varepsilon_{\alpha 12}^2 + \varepsilon_{\alpha 23}^2 + \varepsilon_{\alpha 31}^2)$, and $\omega_\alpha(z)$ is nonlinear differentiable function, which satisfies the following properties:

$$0 \le \omega_\alpha(z) \le \partial\,(z\,\omega_\alpha(z))\,/\partial z < 1,\;\; \partial\,(\omega_\alpha(z))\,/\partial z \ge 0\,. \tag{2}$$

On the boundary Γ_α let us introduce the local orthonormal basis $\boldsymbol{\xi}_\alpha$, $\boldsymbol{\eta}_\alpha$, $\mathbf{n}_\alpha$, where $\mathbf{n}_\alpha$ is the outer unit normal to Γ_α. Then the vectors of displacements and stresses on the boundary can be written in the following way: $\mathbf{u}_\alpha = u_{\alpha\xi}\,\boldsymbol{\xi}_\alpha + u_{\alpha\eta}\,\boldsymbol{\eta}_\alpha + u_{\alpha n}\,\mathbf{n}_\alpha$, $\boldsymbol{\sigma}_\alpha = \hat{\boldsymbol{\sigma}}_\alpha \cdot \mathbf{n}_\alpha = \sigma_{\alpha\xi}\,\boldsymbol{\xi}_\alpha + \sigma_{\alpha\eta}\,\boldsymbol{\eta}_\alpha + \sigma_{\alpha n}\,\mathbf{n}_\alpha\,.$

Suppose that the boundary Γ_α of each body consists of four disjoint parts: $\Gamma_\alpha = \Gamma_\alpha^u \bigcup \Gamma_\alpha^\sigma \bigcup \Gamma_\alpha^I \bigcup S_\alpha$, $\Gamma_\alpha^u \neq \emptyset$, $\Gamma_\alpha^u = \overline{\Gamma_\alpha^u}$, $\Gamma_\alpha^I \bigcup S_\alpha \neq \emptyset$, where $S_\alpha = \bigcup_{\beta \in B_\alpha} S_{\alpha\beta}$, and $\Gamma_\alpha^I = \bigcup_{\beta' \in I_\alpha} \Gamma_{\alpha\beta'}$. Surface $S_{\alpha\beta}$ is the possible unilateral contact area of body Ω_α with body Ω_β, and $B_\alpha \subset \{1, 2, \ldots, N\}$ is the set of the indices of all bodies in unilateral contact with body Ω_α. Surface $\Gamma_{\alpha\beta'} = \Gamma_{\beta'\alpha}$ is the ideal contact area between bodies Ω_α and $\Omega_{\beta'}$, and $I_\alpha \subset \{1, 2, \ldots, N\}$ is the set of the indices of all bodies which have ideal contact with Ω_α.

We assume that the areas $S_{\alpha\beta} \subset \Gamma_\alpha$ and $S_{\beta\alpha} \subset \Gamma_\beta$ are sufficiently close ($S_{\alpha\beta} \approx S_{\beta\alpha}$), and $\mathbf{n}_\alpha(\mathbf{x}) \approx -\mathbf{n}_\beta(\mathbf{x}')$, $\mathbf{x} \in S_{\alpha\beta}$, $\mathbf{x}' = P(\mathbf{x}) \in S_{\beta\alpha}$, where $P(\mathbf{x})$ is the projection of $\mathbf{x}$ on $S_{\alpha\beta}$ [12]. Let $d_{\alpha\beta}(\mathbf{x}) = \pm \|\mathbf{x} - \mathbf{x}'\|_2$ be a distance between bodies Ω_α and Ω_β before the deformation. The sign of $d_{\alpha\beta}$ depends on a statement of the problem.

We consider homogenous Dirichlet boundary conditions on the part Γ_α^u, and Neumann boundary conditions on the part Γ_α^σ:

$$\mathbf{u}_\alpha(\mathbf{x}) = 0,\;\; \mathbf{x} \in \Gamma_\alpha^u;\;\; \boldsymbol{\sigma}_\alpha(\mathbf{x}) = \mathbf{p}_\alpha(\mathbf{x}),\;\; \mathbf{x} \in \Gamma_\alpha^\sigma\,. \tag{3}$$

On the possible contact areas $S_{\alpha\beta}$, $\beta \in B_\alpha$, $\alpha = 1, 2, \ldots, N$ the following nonlinear unilateral contact conditions hold:

$$\sigma_{\alpha n}(\mathbf{x}) = \sigma_{\beta n}(\mathbf{x}') \le 0\,,\;\; \sigma_{\alpha\,\xi}(\mathbf{x}) = \sigma_{\beta\,\xi}(\mathbf{x}') = \sigma_{\alpha\,\eta}(\mathbf{x}) = \sigma_{\beta\,\eta}(\mathbf{x}') = 0\,, \tag{4}$$

$$u_{\alpha n}(\mathbf{x}) + u_{\beta n}(\mathbf{x}') \le d_{\alpha\beta}(\mathbf{x})\,, \tag{5}$$

$$\big(u_{\alpha n}(\mathbf{x})+u_{\beta n}(\mathbf{x}')-d_{\alpha\beta}(\mathbf{x})\big)\,\sigma_{\alpha n}(\mathbf{x})=0\,,\ \mathbf{x}\in S_{\alpha\beta}\,,\ \mathbf{x}'=P(\mathbf{x})\in S_{\beta\alpha}\,. \qquad (6)$$

On ideal contact areas $\Gamma_{\alpha\beta'}=\Gamma_{\beta'\alpha}$, $\beta'\in I_\alpha$, $\alpha=1,2,\dots,N$ we consider ideal mechanical contact conditions:

$$\mathbf{u}_\alpha(\mathbf{x})=\mathbf{u}_{\beta'}(\mathbf{x})\,,\ \boldsymbol{\sigma}_\alpha(\mathbf{x})=-\boldsymbol{\sigma}_{\beta'}(\mathbf{x}),\ \mathbf{x}\in\Gamma_{\alpha\beta'}\,. \qquad (7)$$

3 Penalty Variational Formulation of the Problem

For each body Ω_α consider Sobolev space $V_\alpha=[H^1(\Omega_\alpha)]^3$ and the closed subspace $V_\alpha^0=\{\mathbf{u}_\alpha\in V_\alpha:\ \mathbf{u}_\alpha=0 \text{ on } \Gamma_\alpha^u\}$. All values of the elements $\mathbf{u}_\alpha\in V_\alpha$, $\mathbf{u}_\alpha\in V_\alpha^0$ on the parts of boundary Γ_α should be understood as traces [9].

Define Hilbert space $V_0=V_1^0\times\ldots\times V_N^0$ with the scalar product $(\mathbf{u},\mathbf{v})_{V_0}=\sum_{\alpha=1}^N(\mathbf{u}_\alpha,\mathbf{v}_\alpha)_{V_\alpha}$ and norm $\|\mathbf{u}\|_{V_0}=\sqrt{(\mathbf{u},\mathbf{u})_{V_0}}$, $\mathbf{u},\mathbf{v}\in V_0$. Introduce the closed convex set of all displacements in V_0, which satisfy nonpenentration contact conditions (5) and ideal kinematic contact conditions:

$$K=\big\{\mathbf{u}\in V_0:\ u_{\alpha n}+u_{\beta n}\le d_{\alpha\beta} \text{ on } S_{\alpha\beta},\ \mathbf{u}_{\alpha'}=\mathbf{u}_{\beta'} \text{ on } \Gamma_{\alpha'\beta'}\big\}, \qquad (8)$$

where $\{\alpha,\beta\}\in Q$, $Q=\{\{\alpha,\beta\}:\ \alpha\in\{1,2,\dots,N\},\ \beta\in B_\alpha\}$, $\{\alpha',\beta'\}\in Q^I$, $Q^I=\{\{\alpha',\beta'\}:\ \alpha'\in\{1,2,\dots,N\},\ \beta'\in I_\alpha\}$, and $d_{\alpha\beta}\in H_{00}^{1/2}(\Xi_\alpha)$, $\Xi_\alpha=\operatorname{int}(\Gamma_\alpha\setminus\Gamma_\alpha^u)$.

Let us introduce bilinear form $A(\mathbf{u},\mathbf{v})=\sum_{\alpha=1}^N a_\alpha(\mathbf{u}_\alpha,\mathbf{v}_\alpha)$, $\mathbf{u},\mathbf{v}\in V_0$, which represents the total elastic deformation energy of the system of bodies, linear form $L(\mathbf{v})=\sum_{\alpha=1}^N l_\alpha(\mathbf{v}_\alpha)$, $\mathbf{v}\in V_0$, which is equal to the external forces work, and nonquadratic functional $H(\mathbf{v})=\sum_{\alpha=1}^N h_\alpha(\mathbf{v}_\alpha)$, $\mathbf{v}\in V_0$, which represents the total nonlinear deformation energy:

$$a_\alpha(\mathbf{u}_\alpha,\mathbf{v}_\alpha)=\int_{\Omega_\alpha}\big[\lambda_\alpha\Theta_\alpha(\mathbf{u}_\alpha)\Theta_\alpha(\mathbf{v}_\alpha)+2\mu_\alpha\sum_{i,j}\varepsilon_{\alpha ij}(\mathbf{u}_\alpha)\varepsilon_{\alpha ij}(\mathbf{v}_\alpha)\big]\,d\Omega\,, \qquad (9)$$

$$l_\alpha(\mathbf{v}_\alpha)=\int_{\Omega_\alpha}\mathbf{f}_\alpha\cdot\mathbf{v}_\alpha\,d\Omega+\int_{\Gamma_\alpha^\sigma}\mathbf{p}_\alpha\cdot\mathbf{v}_\alpha\,dS\,, \qquad (10)$$

$$h_\alpha(\mathbf{v}_\alpha)=3\int_{\Omega_\alpha}\mu_\alpha\int_0^{e_\alpha(\mathbf{v}_\alpha)}z\,\omega_\alpha(z)\,dz\,d\Omega\,, \qquad (11)$$

where $\mathbf{p}_\alpha\in[H_{00}^{-1/2}(\Xi_\alpha)]^3$, and $\mathbf{f}_\alpha\in[L_2(\Omega_\alpha)]^3$ is the vector of volume forces.

Using [12], we have shown that the original contact problem has an alternative weak formulation as the following minimization problem on the set K:

$$F(\mathbf{u})=A(\mathbf{u},\mathbf{u})/2-H(\mathbf{u})-L(\mathbf{u})\to\min_{\mathbf{u}\in K}. \qquad (12)$$

Bilinear form A is symmetric, continuous with constant $M_A>0$ and coercive with constant $B_A>0$, and linear form L is continuous. Nonquadratic functional H is doubly Gateaux differentiable in V_0:

$$H'(\mathbf{u},\mathbf{v}) = \sum_{\alpha} h'_\alpha(\mathbf{u}_\alpha,\mathbf{v}_\alpha),\ \ H''(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{\alpha} h''_\alpha(\mathbf{u}_\alpha,\mathbf{v}_\alpha,\mathbf{w}_\alpha),\ \ \mathbf{u},\mathbf{v},\mathbf{w}\in V_0, \tag{13}$$

$$h'_\alpha(\mathbf{u}_\alpha,\mathbf{v}_\alpha) = 2\int_{\Omega_\alpha} \mu_\alpha\, \omega_\alpha(e_\alpha(\mathbf{u}_\alpha)) \sum_{i,j} e_{\alpha ij}(\mathbf{u}_\alpha)\, e_{\alpha ij}(\mathbf{v}_\alpha)\, d\Omega. \tag{14}$$

Moreover, we have proved that the following conditions hold:

$$(\exists C>0)\,(\forall \mathbf{u}\in V_0)\,\{\,(1-C)A\,(\mathbf{u},\mathbf{u}) \geq 2H\,(\mathbf{u})\,\}, \tag{15}$$

$$(\forall \mathbf{u}\in V_0)\,(\exists R>0)\,(\forall \mathbf{v}\in V_0)\left\{\,\left|H'(\mathbf{u},\mathbf{v})\right| \leq R\,\|\mathbf{v}\|_{V_0}\right\}, \tag{16}$$

$$(\exists D>0)\,(\forall \mathbf{u},\mathbf{v},\mathbf{w}\in V_0)\left\{\,\left|H''(\mathbf{u},\mathbf{v},\mathbf{w})\right| \leq D\,\|\mathbf{v}\|_{V_0}\,\|\mathbf{w}\|_{V_0}\right\}, \tag{17}$$

$$(\exists B>0)\,(\forall \mathbf{u},\mathbf{v}\in V_0)\left\{A\,(\mathbf{v},\mathbf{v}) - H''(\mathbf{u},\mathbf{v},\mathbf{v}) \geq B\,\|\mathbf{v}\|^2_{V_0}\right\}. \tag{18}$$

From these properties, it follows that there exists a unique solution $\bar{\mathbf{u}}\in K$ of minimization problem (12), and this problem is equivalent to the following variational inequality, which is nonlinear in $\mathbf{u}$:

$$A\,(\mathbf{u},\mathbf{v}-\mathbf{u}) - H'(\mathbf{u},\mathbf{v}-\mathbf{u}) - L\,(\mathbf{v}-\mathbf{u}) \geq 0,\ \ \forall \mathbf{v}\in K,\ \mathbf{u}\in K. \tag{19}$$

To obtain a minimization problem in the whole space V_0, we apply a penalty method [3, 7, 9, 13] to problem (12). We use a penalty in the form

$$J_\theta(\mathbf{u}) = \frac{1}{2\theta}\sum_{\{\alpha,\,\beta\}\in Q} \left\|\left(d_{\alpha\beta} - u_{\alpha n} - u_{\beta n}\right)^-\right\|^2_{L_2(S_{\alpha\beta})} + \\ + \frac{1}{2\theta}\sum_{\{\alpha',\,\beta'\}\in Q^I} \left\|\mathbf{u}_{\alpha'} - \mathbf{u}_{\beta'}\right\|^2_{[L_2(\Gamma_{\alpha'\beta'})]^3}, \tag{20}$$

where $\theta>0$ is a penalty parameter, and $y^- = \min\{0,y\}$.

Now, consider the following unconstrained minimization problem in V_0:

$$F_\theta(\mathbf{u}) = A\,(\mathbf{u},\mathbf{u})/2 - H\,(\mathbf{u}) - L\,(\mathbf{u}) + J_\theta(\mathbf{u}) \to \min_{\mathbf{u}\in V_0}. \tag{21}$$

The penalty term J_θ is nonnegative and Gateaux differentiable in V_0, and its differential $J'_\theta(\mathbf{u},\mathbf{v}) = -\frac{1}{\theta}\sum_{\{\alpha,\,\beta\}\in Q}\int_{S_{\alpha\beta}}\left(d_{\alpha\beta} - u_{\alpha n} - u_{\beta n}\right)^-\,\left(v_{\alpha n} + v_{\beta n}\right)dS +$ $\frac{1}{\theta}\sum_{\{\alpha',\,\beta'\}\in Q^I}\int_{\Gamma_{\alpha'\beta'}}\left(\mathbf{u}_{\alpha'} - \mathbf{u}_{\beta'}\right)\cdot\left(\mathbf{v}_{\alpha'} - \mathbf{v}_{\beta'}\right)dS$ satisfy the following properties [15]:

$$(\forall \mathbf{u}\in V_0)(\exists \tilde{R}>0)(\forall \mathbf{v}\in V_0)\left\{\left|J'_\theta(\mathbf{u},\mathbf{v})\right| \leq \tilde{R}\,\|\mathbf{v}\|_{V_0}\right\}, \tag{22}$$

$$(\exists \tilde{D}>0)(\forall \mathbf{u},\mathbf{v},\mathbf{w}\in V_0)\left\{\left|J'_\theta(\mathbf{u}+\mathbf{w},\mathbf{v}) - J'_\theta(\mathbf{u},\mathbf{v})\right| \leq \tilde{D}\,\|\mathbf{v}\|_{V_0}\|\mathbf{w}\|_{V_0}\right\}, \tag{23}$$

$$(\forall \mathbf{u},\mathbf{v}\in V_0)\left\{J'_\theta(\mathbf{u}+\mathbf{v},\mathbf{v}) - J'_\theta(\mathbf{u},\mathbf{v}) \geq 0\right\}. \tag{24}$$

Using these properties and the results in [3], we have shown that problem (21) has a unique solution $\bar{\mathbf{u}}_\theta\in V_0$ and is equivalent to the following nonlinear variational equation in the space V_0:

$$F'_\theta(\mathbf{u},\mathbf{v}) = A\,(\mathbf{u},\mathbf{v}) - H'(\mathbf{u},\mathbf{v}) + J'_\theta(\mathbf{u},\mathbf{v}) - L\,(\mathbf{v}) = 0,\ \ \forall \mathbf{v}\in V_0,\ \ \mathbf{u}\in V_0. \tag{25}$$

Using the results of works [7, 13], we have proved that $\|\bar{\mathbf{u}}_\theta - \bar{\mathbf{u}}\|_{V_0} \underset{\theta\to 0}{\longrightarrow} 0$.

4 Iterative Methods for Nonlinear Variational Equations

In arbitrary reflexive Banach space V_0 consider an abstract nonlinear variational equation

$$\Phi(\mathbf{u},\mathbf{v}) = L(\mathbf{v}),\ \forall \mathbf{v} \in V_0,\ \mathbf{u} \in V_0 \tag{26}$$

where $\Phi : V_0 \times V_0 \to \mathbb{R}$ is a functional, which is linear in $\mathbf{v}$, but nonlinear in $\mathbf{u}$, and L is linear continuous form. Suppose that this variational equation has a unique solution $\bar{\mathbf{u}}_* \in V_0$.

For the numerical solution of (26) we use the next iterative method [5, 6, 15]:

$$G(\mathbf{u}^{k+1},\mathbf{v}) = G(\mathbf{u}^k,\mathbf{v}) - \gamma\left[\Phi(\mathbf{u}^k,\mathbf{v}) - L(\mathbf{v})\right],\ \forall \mathbf{v} \in V_0,\ \ k = 0,1,\ldots, \tag{27}$$

where G is some given bilinear form in $V_0 \times V_0$, $\gamma \in \mathbb{R}$ is fixed parameter, and $\mathbf{u}^k \in V_0$ is the k-th approximation to the exact solution of problem (26).

We have proved the next theorem [5, 15] about the convergence of this method.

Theorem 1. *Suppose that the following conditions hold*

$$(\forall \mathbf{u} \in V_0)\,(\exists R_\Phi > 0)\,(\forall \mathbf{v} \in V_0)\left\{|\Phi(\mathbf{u},\mathbf{v})| \le R_\Phi \|\mathbf{v}\|_{V_0}\right\}, \tag{28}$$

$$(\exists D_\Phi > 0)(\forall \mathbf{u},\mathbf{v},\mathbf{w} \in V_0)\left\{|\Phi(\mathbf{u}+\mathbf{w},\mathbf{v}) - \Phi(\mathbf{u},\mathbf{v})| \le D_\Phi \|\mathbf{v}\|_{V_0}\|\mathbf{w}\|_{V_0}\right\}, \tag{29}$$

$$(\exists B_\Phi > 0)\,(\forall \mathbf{u},\mathbf{v} \in V_0)\left\{\Phi(\mathbf{u}+\mathbf{v},\mathbf{v}) - \Phi(\mathbf{u},\mathbf{v}) \ge B_\Phi \|\mathbf{v}\|^2_{V_0}\right\}, \tag{30}$$

bilinear form G is symmetric, continuous with constant $M_G > 0$ and coercive with constant $B_G > 0$, and $\gamma \in (0; 2\gamma^*)\,, \gamma^* = B_\Phi B_G / D_\Phi^2$.

Then $\left\|\mathbf{u}^k - \bar{\mathbf{u}}_*\right\|_{V_0} \underset{k\to\infty}{\longrightarrow} 0$, where $\{\mathbf{u}^k\} \subset V_0$ is obtained by method (27). Moreover, the convergence rate in norm $\|\cdot\|_G = \sqrt{G(\cdot,\cdot)}$ is linear, and the highest convergence rate in this norm reaches as $\gamma = \gamma^*$.

In addition, we have proposed nonstationary iterative method to solve (26), where bilinear form G and parameter γ are different in each iteration:

$$G^k(\mathbf{u}^{k+1},\mathbf{v}) = G^k(\mathbf{u}^k,\mathbf{v}) - \gamma^k\left[\Phi(\mathbf{u}^k,\mathbf{v}) - L(\mathbf{v})\right],\ \forall \mathbf{v} \in V_0,\ \ k = 0,1,\ldots. \tag{31}$$

A convergence theorem for this method is proved in [15].

5 Domain Decomposition Schemes for Contact Problems

Now let us apply iterative methods (27) and (31) to the solution of nonlinear penalty variational equation (25) of multibody contact problem. This penalty equation can be written in form (26), where

$$\Phi(\mathbf{u},\mathbf{v}) = A(\mathbf{u},\mathbf{v}) - H'(\mathbf{u},\mathbf{v}) + J'_\theta(\mathbf{u},\mathbf{v}),\ \ \mathbf{u},\mathbf{v} \in V_0. \tag{32}$$

We consider such variants of methods (27) and (31), which lead to the domain decomposition.

Let us take the bilinear form G in iterative method (27) as follows [6, 15]:

$$G(\mathbf{u},\mathbf{v}) = A(\mathbf{u},\mathbf{v}) + X(\mathbf{u},\mathbf{v}), \;\; \mathbf{u},\mathbf{v} \in V_0, \tag{33}$$

$$X(\mathbf{u},\mathbf{v}) = \frac{1}{\theta}\sum_{\alpha=1}^{N}\left[\sum_{\beta\in B_\alpha}\int_{S_{\alpha\beta}} u_{\alpha n} v_{\alpha n}\, \psi_{\alpha\beta}\, dS + \sum_{\beta'\in I_\alpha}\int_{\Gamma_{\alpha\beta'}} \mathbf{u}_\alpha \cdot \mathbf{v}_\alpha\, \phi_{\alpha\beta'}\, dS\right],$$

where $\psi_{\alpha\beta}(\mathbf{x}) = \{1, \mathbf{x}\in S^1_{\alpha\beta}\} \vee \{0, \mathbf{x}\in S_{\alpha\beta}\backslash S^1_{\alpha\beta}\}$ and $\phi_{\alpha\beta'}(\mathbf{x}) = \{1, \mathbf{x}\in \Gamma^1_{\alpha\beta'}\} \vee \{0, \; \mathbf{x}\in \Gamma_{\alpha\beta'}\backslash\Gamma^1_{\alpha\beta'}\}$ are characteristic functions of arbitrary subsets $S^1_{\alpha\beta} \subseteq S_{\alpha\beta}$, $\Gamma^1_{\alpha\beta'} \subseteq \Gamma_{\alpha\beta'}$ of possible unilateral and ideal contact areas respectively.

Introduce a notation $\tilde{\mathbf{u}}^{k+1} = [\mathbf{u}^{k+1} - \mathbf{u}^k]/\gamma + \mathbf{u}^k \in V_0$. Then iterative method (27) with bilinear form (33) can be written in such way:

$$A\left(\tilde{\mathbf{u}}^{k+1},\mathbf{v}\right) + X\left(\tilde{\mathbf{u}}^{k+1},\mathbf{v}\right) = L(\mathbf{v}) + X\left(\mathbf{u}^k,\mathbf{v}\right) + H'(\mathbf{u}^k,\mathbf{v}) - J'_\theta(\mathbf{u}^k,\mathbf{v}), \tag{34}$$

$$\mathbf{u}^{k+1} = \gamma\,\tilde{\mathbf{u}}^{k+1} + (1-\gamma)\,\mathbf{u}^k, \;\; k = 0,1,\ldots\,. \tag{35}$$

Bilinear form X is symmetric, continuous with constant $M_X > 0$, and nonnegative [15]. Due to these properties, and due to the properties of bilinear form A, it follows that the conditions of Theorem 1 hold. Therefore, we obtain the next proposition:

Theorem 2. *The sequence $\{\mathbf{u}^k\}$ of the method (34)–(35) converges strongly to the solution of penalty variational equation (25) for $\gamma \in (0; 2B_\Phi B_G/D^2_\Phi)$, where $B_G = B_A$, $B_\Phi = B$, $D_\Phi = M_A + D + \tilde{D}$. The convergence rate in norm $\|\cdot\|_G$ is linear.*

As the common quantities of the subdomains are known from the previous iteration, variational equation (34) splits into N separate equations for each subdomain Ω_α, and method (34)–(35) can be written in the following equivalent form:

$$a_\alpha(\tilde{\mathbf{u}}^{k+1}_\alpha, \mathbf{v}_\alpha) + \sum_{\beta\in B_\alpha}\int_{S_{\alpha\beta}} \frac{\psi_{\alpha\beta}}{\theta}\, \tilde{u}^{k+1}_{\alpha n}\, v_{\alpha n}\, dS + \sum_{\beta'\in I_\alpha}\int_{\Gamma_{\alpha\beta'}} \frac{\phi_{\alpha\beta'}}{\theta}\, \tilde{\mathbf{u}}^{k+1}_\alpha \cdot \mathbf{v}_\alpha\, dS$$

$$= l_\alpha(\mathbf{v}_\alpha) + \frac{1}{\theta}\sum_{\beta\in B_\alpha}\int_{S_{\alpha\beta}}\left[\psi_{\alpha\beta}\, u^k_{\alpha n} + \left(d_{\alpha\beta} - u^k_{\alpha n} - u^k_{\beta n}\right)^-\right] v_{\alpha n}\, dS$$

$$+ \frac{1}{\theta}\sum_{\beta'\in I_\alpha}\int_{\Gamma_{\alpha\beta'}}\left[\phi_{\alpha\beta'}\,\mathbf{u}^k_\alpha + \left(\mathbf{u}^k_{\beta'} - \mathbf{u}^k_\alpha\right)\right]\cdot \mathbf{v}_\alpha\, dS + h'_\alpha(\mathbf{u}^k_\alpha, \mathbf{v}_\alpha), \;\; \forall \mathbf{v}_\alpha \in V^0_\alpha, \tag{36}$$

$$\mathbf{u}^{k+1}_\alpha = \gamma\,\tilde{\mathbf{u}}^{k+1}_\alpha + (1-\gamma)\,\mathbf{u}^k_\alpha, \;\; \alpha = 1,2,\ldots,N, \;\; k = 0,1,\ldots\,. \tag{37}$$

In each iteration k of method (36)–(37), we have to solve N linear variational equations in parallel, which correspond to some linear elasticity problems in subdomains with additional volume forces in Ω_α, and with Robin boundary conditions on contact areas. Therefore, this method refers to parallel Robin–Robin type domain decomposition schemes.

Taking different characteristic functions $\psi_{\alpha\beta}$ and $\phi_{\alpha'\beta'}$, we can obtain different particular cases of penalty domain decomposition method (36)–(37).

Thus, taking $\psi_{\alpha\beta}(\mathbf{x}) \equiv 0$, $\beta \in B_\alpha$, $\phi_{\alpha\beta'}(\mathbf{x}) \equiv 0$, $\beta' \in I_\alpha$, $\alpha = 1,2,\dots,N$, we get parallel Neumann–Neumann domain decomposition scheme.

Other borderline case is when $\psi_{\alpha\beta}(\mathbf{x}) \equiv 1$, $\beta \in B_\alpha$, $\phi_{\alpha\beta'}(\mathbf{x}) \equiv 1$, $\beta' \in I_\alpha$, $\alpha = 1,2,\dots,N$, i.e. $S^1_{\alpha\beta} = S_{\alpha\beta}$, $\Gamma^1_{\alpha\beta'} = \Gamma_{\alpha\beta'}$.

Moreover, we can choose functions $\psi_{\alpha\beta}$ and $\phi_{\alpha\beta'}$ differently in each iteration k. Then we obtain nonstationary domain decomposition schemes, which are equivalent to iterative method (31) with bilinear forms

$$G^k(\mathbf{u},\mathbf{v}) = A(\mathbf{u},\mathbf{v}) + X^k(\mathbf{u},\mathbf{v}), \ \ \mathbf{u},\mathbf{v} \in V_0, \ \ k = 0,1,\dots, \tag{38}$$

$$X^k(\mathbf{u},\mathbf{v}) = \frac{1}{\theta}\sum_{\alpha=1}^{N}\left[\sum_{\beta\in B_\alpha}\int_{S_{\alpha\beta}} u_{\alpha n} v_{\alpha n}\, \psi^k_{\alpha\beta}\, dS + \sum_{\beta'\in I_\alpha}\int_{\Gamma_{\alpha\beta'}} \mathbf{u}_\alpha\cdot\mathbf{v}_\alpha\, \phi^k_{\alpha\beta'}\, dS\right].$$

If we take characteristic functions $\psi^k_{\alpha\beta}$ and $\phi^k_{\alpha\beta'}$ as follows [6, 14, 15]:

$$\psi^k_{\alpha\beta}(\mathbf{x}) = \chi^k_{\alpha\beta}(\mathbf{x}) = \begin{cases} 0, \ d_{\alpha\beta}(\mathbf{x}) - u^k_{\alpha n}(\mathbf{x}) - u^k_{\beta n}(\mathbf{x}') \geq 0 \\ 1, \ d_{\alpha\beta}(\mathbf{x}) - u^k_{\alpha n}(\mathbf{x}) - u^k_{\beta n}(\mathbf{x}') < 0 \end{cases}, \ \mathbf{x}' = P(\mathbf{x}), \ \mathbf{x} \in S_{\alpha\beta},$$

$$\phi^k_{\alpha\beta'}(\mathbf{x}) \equiv 1, \ \ \mathbf{x} \in \Gamma_{\alpha\beta'}, \ \ \beta \in B_\alpha, \ \ \beta' \in I_\alpha, \ \ \alpha = 1,2,\dots,N,$$

then we shall get the method, which can be conventionally named as nonstationary parallel Dirichlet–Dirichlet domain decomposition scheme.

In addition to methods (27), (33) and (31), (38), we have proposed another family of DDMs for the solution of (25), where the second derivative of functional $H(\mathbf{u})$ is used. These domain decomposition methods are obtained from (31), if we choose bilinear forms $G^k(\mathbf{u},\mathbf{v})$ as follows

$$G^k(\mathbf{u},\mathbf{v}) = A(\mathbf{u},\mathbf{v}) - H''(\mathbf{u}^k,\mathbf{u},\mathbf{v}) + X^k(\mathbf{u},\mathbf{v}), \ \ \mathbf{u},\mathbf{v} \in V_0, \ \ k = 0,1,\dots . \tag{39}$$

Numerical analysis of presented penalty Robin–Robin DDMs has been made for plane unilateral two-body and three-body contact problems of linear elasticity ($\omega_\alpha \equiv 0$) using finite element approximations [6, 14, 15]. Numerical experiments have confirmed the theoretical results about the convergence of these methods.

Among the positive features of proposed domain decomposition schemes are the simplicity of the algorithms and the regularization of original contact problem because of the use of penalty method. These domain decomposition schemes have only one iteration loop, which deals with domain decomposition, nonlinearity of the stress-strain relationship, and nonlinearity of unilateral contact conditions.

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