Heterogeneous Domain Decomposition Methods for Eddy Current Problems

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Summary. The usual setting of an eddy current problem distinguishes between a conducting region and an air region (non-conducting) surrounding the conductor. For the numerical approximation of this heterogeneous problem it is very natural to use iterative substructuring methods based on transmission conditions at the interface. We analyze the convergence of the Dirichlet-Neumann iterative method for two different formulations of the eddy current problem: the one that consider as main unknown the electric field and the one based on the magnetic field.

1 Introduction

To model the electromagnetic phenomena concerning alternating currents at low frequencies it is often used the time-harmonic eddy current model (see e.g. [2]). The main equations of this model are Faraday's law

$$\operatorname{curl} \mathbf{E} = -i\omega\mu\mathbf{H} \quad \text{in } \Omega\,, \tag{1}$$

and Ampère's law

$$\operatorname{curl} \mathbf{H} = \sigma\mathbf{E} + \mathbf{J}_e \quad \text{in } \Omega\,, \tag{2}$$

where $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{J}_e$ denote the electric field, the magnetic field and the applied current density respectively. For the sake of simplicity we assume that the computational domain $\Omega \subset \mathbb{R}^3$ is a simply connected Lipschitz polyhedron with connected boundary that contains a conducting region $\Omega_C \subset\subset \Omega$ and that both Ω_C and its complement $\Omega_I := \Omega \setminus \overline{\Omega_C}$ are connected Lipschitz polyhedra. Let us denote $\Gamma := \overline{\Omega_C} \cap \overline{\Omega_I}$. The magnetic permeability μ is assumed to be a symmetric uniformly positive definite 3×3 matrix with entries in $L^\infty(\Omega)$, whereas the electric conductivity σ is supposed to be a bounded symmetric positive definite matrix in the conducting regions, and to be null in non-conducting regions. The real scalar constant $\omega \neq 0$ is a given angular frequency. In $\partial\Omega$ suitable boundary conditions must be assigned. Most often the tangential component of either the electric field $\mathbf{E} \times \mathbf{n}$ or the magnetic field $\mathbf{H} \times \mathbf{n}$ are given (here $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$).

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Let us introduce some notations that will be used in the following. The space $H(\mathrm{curl}\,;\Omega)$ indicates the set of real or complex vector valued functions $\mathbf{v} \in (L^2(\Omega))^3$ such that $\mathrm{curl}\,\mathbf{v} \in (L^2(\Omega))^3$ and $H^0(\mathrm{curl}\,;\Omega)$ its subspace constituted by curl-free functions. Given a certain subset $\Lambda \subset \partial\Omega$, we denote by $H_{0,\Lambda}(\mathrm{curl}\,;\Omega)$ the subspace of functions in $H(\mathrm{curl}\,;\Omega)$ such that their tangential trace is null on Λ, and in particular we write $H_0(\mathrm{curl}\,;\Omega) := H_{0,\partial\Omega}(\mathrm{curl}\,;\Omega)$.

We recall the spaces $H^{-1/2}(\mathrm{curl}_\tau;\partial\Omega) := \left\{(\mathbf{n}\times\mathbf{v}\times\mathbf{n})_{|\partial\Omega}\,|\,\mathbf{v}\in H(\mathrm{curl}\,;\Omega)\right\}$, and $H^{-1/2}(\mathrm{div}_\tau;\partial\Omega) := \left\{(\mathbf{v}\times\mathbf{n})_{|\partial\Omega}\,|\,\mathbf{v}\in H(\mathrm{curl}\,;\Omega)\right\}$, (see [4]). These two spaces are in duality and the following formula of integration by parts holds true

$$\int_\Omega (\mathbf{w}\cdot\mathrm{curl}\,\overline{\mathbf{v}} - \mathrm{curl}\,\mathbf{w}\cdot\overline{\mathbf{v}}) = \langle \mathbf{w}\times\mathbf{n}, \mathbf{n}\times\overline{\mathbf{v}}\times\mathbf{n}\rangle_{\partial\Omega} \quad \forall\,\mathbf{w},\,\mathbf{v}\in H(\mathrm{curl}\,;\Omega)\,.$$

2 One Field Formulations

First we notice that Eqs. (1) and (2) do not completely determine the electric field in Ω_I and it is necessary to require the gauge condition

$$\mathrm{div}\mathbf{E}_I = 0 \ \text{ in } \Omega_I\,. \tag{3}$$

(Here and in the sequel, given any vector field $\mathbf{v}$ defined in Ω, we denote $\mathbf{v}_L$ its restriction to Ω_L, $L = C, I$.) When imposing electric boundary conditions, $\mathbf{E}\times\mathbf{n} = \mathbf{0}$ on $\partial\Omega$, in order to have a unique solution we need to impose the additional gauge condition $\int_\Gamma \mathbf{E}_I\cdot\mathbf{n} = 0$.

From Faraday law $\mu^{-1}\mathrm{curl}\,\mathbf{E} = -i\omega\mathbf{H}$ and replacing in Ampère law one has $\mathrm{curl}\,(\mu^{-1}\mathrm{curl}\,\mathbf{E}) = -i\omega(\sigma\mathbf{E}+\mathbf{J}_e)$. So the $\mathbf{E}$-based formulation of the eddy current problem with electric boundary conditions reads

$$\begin{array}{ll}
\mathrm{curl}\,(\mu^{-1}\mathrm{curl}\,\mathbf{E}) + i\omega\sigma\mathbf{E} = -i\omega\mathbf{J}_e & \text{in } \Omega\\
\mathrm{div}\mathbf{E}_I = 0 & \text{in } \Omega_I\\
\int_\Gamma \mathbf{E}_I\cdot\mathbf{n} = 0 & \\
\mathbf{E}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,.
\end{array}$$

Since $\sigma \equiv 0$ in the non-conducting region, the generator current has to satisfy the compatibility conditions $\mathrm{div}\mathbf{J}_{e,I} = 0$ in Ω_I and, when imposing $\mathbf{E}\times\mathbf{n} = 0$ on $\partial\Omega$, $\int_\Gamma \mathbf{J}_{e,I}\cdot\mathbf{n} = 0$.

Notice that the two gauge conditions $\mathrm{div}\mathbf{E}_I = 0$ and $\int_\Gamma \mathbf{E}_I\cdot\mathbf{n} = 0$ are equivalent to $\int_{\Omega_I} \mathbf{E}_I\cdot\nabla\overline{\phi}_I = 0$ for all $\phi_I \in H^1_*(\Omega_I)$ being $H^1_*(\Omega_I) = \{\phi_I \in H^1(\Omega_I) : \phi_{I|\partial\Omega} \equiv 0$ and $\phi_{I|\Gamma}$ is constant$\}$. Hence the weak form of the $\mathbf{E}$-based formulation is

$$\begin{array}{l}
\text{Find } \mathbf{E}\in W \text{ such that}\\
\quad \int_\Omega (\mu^{-1}\mathrm{curl}\,\mathbf{E}\cdot\mathrm{curl}\,\overline{\mathbf{w}} + i\omega\sigma\mathbf{E}\cdot\overline{\mathbf{w}}) = -i\omega\int_\Omega \mathbf{J}_e\cdot\overline{\mathbf{w}}\\
\text{for all } \mathbf{w}\in W
\end{array}$$

where $W := \{\mathbf{w}\in H_0(\mathrm{curl}\,;\Omega) : \int_{\Omega_I}\mathbf{w}_I\cdot\nabla\overline{\phi}_I = 0\ \forall\,\phi_I\in H^1_*(\Omega_I)\}$.

Remark 1. The gauge conditions can be imposed by means of a Lagrange multiplier. (See [2], Sect. 4.6.)

Due to the heterogeneous nature of the problem, it is natural to consider an iterative procedure by subdomains in order to deal with homogeneous problem. A procedure of this kind is the following:

$$
\begin{aligned}
&\text{Given } \boldsymbol{\lambda}^{(0)} \in H^{-1/2}(\operatorname{curl}_\tau;\Gamma) \text{ for } n \geq 0\\
&\text{find } \mathbf{E}_I^{(n+1)} \in W_I \text{ such that}\\
&\quad \mathbf{n}\times\mathbf{E}_I^{(n+1)}\times\mathbf{n} = \boldsymbol{\lambda}^{(n)} \text{ on } \Gamma\\
&\quad \int_{\Omega_I}\mu^{-1}\operatorname{curl}\mathbf{E}_I^{(n+1)}\cdot\operatorname{curl}\overline{\mathbf{w}}_I = -i\omega\int_{\Omega_I}\mathbf{J}_{e,I}\cdot\overline{\mathbf{w}}_I \quad \forall\,\mathbf{w}_I\in W_I\cap H_0(\operatorname{curl};\Omega_I);\\
&\text{find } \mathbf{E}_C^{(n+1)} \in H(\operatorname{curl};\Omega_C) \text{ such that}\\
&\quad \int_{\Omega_C}(\mu^{-1}\operatorname{curl}\mathbf{E}_C^{(n+1)}\cdot\operatorname{curl}\overline{\mathbf{w}}_C + i\omega\sigma\mathbf{E}_C^{(n+1)}\cdot\overline{\mathbf{w}}_C) = -i\omega\int_{\Omega_C}\mathbf{J}_{e,C}\cdot\overline{\mathbf{w}}_C\\
&\qquad\quad -\langle\mu^{-1}\operatorname{curl}\mathbf{E}_I^{(n+1)}\times\mathbf{n}_I,\mathbf{n}\times\mathbf{w}_C\times\mathbf{n}\rangle_\Gamma \quad \forall\,\mathbf{w}_C\in H(\operatorname{curl};\Omega_C);\\
&\text{set}\\
&\quad \boldsymbol{\lambda}^{(n+1)} = (1-\theta)\boldsymbol{\lambda}^{(n)} + \theta(\mathbf{n}\times\mathbf{E}_C^{(n+1)}\times\mathbf{n})_{|\Gamma},
\end{aligned}
$$

where $W_I := \{\mathbf{w}_I \in H_{0,\partial\Omega}(\operatorname{curl};\Omega_I) : \int_{\Omega_I}\mathbf{w}_I\cdot\nabla\overline{\phi}_I = 0\ \forall\,\phi_I\in H^1_*(\Omega_I)\}$, $\mathbf{n}_I$ denotes the unit normal vector on Γ pointing outwards Ω_I and θ is a positive acceleration parameter.

Another possibility is to eliminate the electric field. Multiplying Faraday law by a function $\mathbf{v}\in H_0(\operatorname{curl};\Omega)$ with $\operatorname{curl}\mathbf{v}_I = 0$;

$$
\begin{aligned}
i\omega\int_\Omega\mu\mathbf{H}\cdot\overline{\mathbf{v}} &= -\int_\Omega\operatorname{curl}\mathbf{E}\cdot\overline{\mathbf{v}} = -\int_\Omega\mathbf{E}\cdot\operatorname{curl}\overline{\mathbf{v}}\\
&= -\int_{\Omega_C}\sigma^{-1}(\operatorname{curl}\mathbf{H}_C - \mathbf{J}_{e,C})\cdot\operatorname{curl}\overline{\mathbf{v}}_C\,.
\end{aligned}
$$

Given $\mathbf{g}_I \in (L^2(\Omega_I))^3$ let $V(\mathbf{g}_I)$ denotes the space $V(\mathbf{g}_I) := \{\mathbf{v}\in H_0(\operatorname{curl};\Omega) : \operatorname{curl}\mathbf{v}_I = \mathbf{g}_I\}$. The weak form of $\mathbf{H}$-based formulation of the eddy current problem with magnetic boundary conditions $\mathbf{H}\times\mathbf{n} = \mathbf{0}$ on $\partial\Omega$ reads

$$
\begin{aligned}
&\text{Find } \mathbf{H}\in V(\mathbf{J}_{e,I}) \text{ such that}\\
&\quad \int_{\Omega_C}\sigma^{-1}\operatorname{curl}\mathbf{H}\cdot\operatorname{curl}\overline{\mathbf{v}} + i\omega\int_\Omega\mu\mathbf{H}\cdot\overline{\mathbf{v}} = \int_{\Omega_C}\sigma^{-1}\mathbf{J}_{e,C}\cdot\operatorname{curl}\overline{\mathbf{v}}_C\\
&\text{for all } \mathbf{v}\in V(\mathbf{0})\,.
\end{aligned}
\tag{4}
$$

Since $\sigma\equiv 0$ in the non-conducting region, when imposing $\mathbf{H}\times\mathbf{n} = 0$ on $\partial\Omega$ the generator current has to satisfy the compatibility conditions $\operatorname{div}\mathbf{J}_{e,I} = 0$ in Ω_I and $\mathbf{J}_{e,I}\cdot\mathbf{n} = 0$ on $\partial\Omega$. Hence there exists $\mathbf{H}^*_{e,I}\in H_{0,\partial\Omega}(\operatorname{curl};\Omega_I)$ such that $\operatorname{curl}\mathbf{H}^*_{e,I} = \mathbf{J}_{e,I}$. Then we can write $\mathbf{H}_I = \mathbf{H}^*_{e,I} + \mathbf{Z}_I$ with $\mathbf{Z}_I\in H^0_{0,\partial\Omega}(\operatorname{curl};\Omega_I)$. Let $\mathbf{H}^*_e$ be a function in $H(\operatorname{curl};\Omega)$ such that $\mathbf{H}^*_{e|\Omega_I} = \mathbf{H}^*_{e,I}$ and let us denote $\mathbf{Z} := \mathbf{H} - \mathbf{H}^*_e \in V(\mathbf{0})$. Multiplying Eq. (4) by $-i\omega^{-1}$ and setting $\widehat{F}(\mathbf{v}) := \int_\Omega\mu\mathbf{H}^*_e\cdot\overline{\mathbf{v}} - i\omega^{-1}\int_{\Omega_C}\sigma^{-1}\operatorname{curl}\mathbf{H}^*_e\cdot\operatorname{curl}\overline{\mathbf{v}}$, we can consider the equivalent problem

Find $\mathbf{Z} \in V(\mathbf{0})$ such that
$$\int_\Omega \mu \mathbf{Z} \cdot \overline{\mathbf{v}} - i\omega^{-1} \int_{\Omega_C} \sigma^{-1} \operatorname{curl} \mathbf{Z} \cdot \operatorname{curl} \overline{\mathbf{v}} = -i\omega^{-1} \int_{\Omega_C} \sigma^{-1} \mathbf{J}_{e,C} \cdot \operatorname{curl} \overline{\mathbf{v}}_C - \widehat{F}(\mathbf{v})$$
for all $\mathbf{v} \in V(\mathbf{0})$.

For the sake of simplicity we will assume that $\mathbf{J}_{e,I} \cdot \mathbf{n} = 0$ on Γ. Then it is possible to take $\mathbf{H}^*_{e,I} \in H_0(\operatorname{curl}; \Omega_I)$ and $\mathbf{H}^*_{e,C}$ equal zero.

Remark 2. Notice that $H^0_{0,\partial\Omega}(\operatorname{curl}; \Omega_I) = \nabla H^1_{0,\partial\Omega}(\Omega_I) \oplus \mathscr{H}(\Omega_I)$ where $\mathscr{H}(\Omega_I) := \{\mathbf{v}_I \in H^0_{0,\partial\Omega}(\operatorname{curl}; \Omega_I) \,:\, \operatorname{div} \mathbf{v}_I = 0 \text{ and } \mathbf{v}_I \cdot \mathbf{n} = 0 \text{ on } \Gamma\}$ that is a space of finite dimension. In this geometrical setting the dimension of $\mathscr{H}(\Omega_I)$ coincides with the first Betti number of Ω_I. (See [2], Sect. 5.1.)

We propose an iterative procedure for the solution of the $\mathbf{H}$-based formulation that start from a data in the trace space

$$H_0^{-1/2}(\operatorname{curl}_\tau; \Gamma) := \{(\mathbf{n} \times \mathbf{w}_I \times \mathbf{n})_{|\Gamma} \,:\, \mathbf{w}_I \in H^0_{0,\partial\Omega}(\operatorname{curl}; \Omega_I)\}.$$

It reads:

Given $\boldsymbol{\lambda}^{(0)} \in H_0^{-1/2}(\operatorname{curl}_\tau; \Gamma)$ for $n \geq 0$

find $\mathbf{H}_C^{(n+1)} \in H(\operatorname{curl}; \Omega_C)$ such that
$$\mathbf{n} \times \mathbf{H}_C^{(n+1)} \times \mathbf{n} = \boldsymbol{\lambda}^{(n)} \quad \text{on } \Gamma$$
$$\int_{\Omega_C} (\mu \mathbf{H}_C^{(n+1)} \cdot \overline{\mathbf{v}}_C - i\omega^{-1} \sigma^{-1} \operatorname{curl} \mathbf{H}_C^{(n+1)} \cdot \operatorname{curl} \overline{\mathbf{v}}_C) = -i\omega^{-1} \int_{\Omega_C} \sigma^{-1} \mathbf{J}_{e,C} \cdot \operatorname{curl} \overline{\mathbf{v}}_C \quad \forall \mathbf{v}_C \in H_0(\operatorname{curl}; \Omega_C);$$

find $\mathbf{Z}_I^{(n+1)} \in H^0_{0,\partial\Omega}(\operatorname{curl}; \Omega_I)$ such that
$$\int_{\Omega_I} \mu \mathbf{Z}_I^{(n+1)} \cdot \overline{\mathbf{v}}_I = i\omega^{-1} \langle \sigma^{-1} (\operatorname{curl} \mathbf{H}_C^{(n+1)} - \mathbf{J}_{e,C}) \times \mathbf{n}_C, \mathbf{n} \times \mathbf{v}_I \times \mathbf{n} \rangle_\Gamma - \int_{\Omega_I} \mu \mathbf{H}^*_{e,I} \cdot \overline{\mathbf{v}}_I \quad \forall \mathbf{v}_{I,h} \in H^0_{0,\partial\Omega}(\operatorname{curl}; \Omega_I);$$

set
$$\boldsymbol{\lambda}^{(n+1)} = (1-\theta) \boldsymbol{\lambda}^{(n)} + \theta (\mathbf{n} \times \mathbf{Z}_I^{(n+1)} \times \mathbf{n})_{|\Gamma},$$

being $\mathbf{n}_C$ the unit normal vector on Γ pointing outwards Ω_C and θ a positive acceleration parameter.

3 Convergence Analysis

Both the $\mathbf{H}$-based formulation and the $\mathbf{E}$-based formulation are of the form: find $\mathbf{u} \in V \subset H(\operatorname{curl}; \Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \quad \forall \mathbf{v} \in V, \tag{5}$$

where $a(\cdot,\cdot)$ is a sesquilinear form continuous and coercive in $V \times V$ and $F(\cdot)$ is a continuous linear functional on the Hilbert space V. The proposed iterative procedures are preconditioned Richardson methods for the Steklov-Poincare equation obtained in the following way (see e.g. [8]): for $L = C, I$ let us define the spaces $V_L := \{\mathbf{v}_{|\Omega_L} : \mathbf{v} \in V\}$, $X := \{(\mathbf{n} \times \mathbf{v} \times \mathbf{n})_\Gamma : \mathbf{v} \in V\}$ and $V_{L,0} := \{\mathbf{v}_L \in V_L : (\mathbf{n} \times \mathbf{v}_L \times \mathbf{n})_\Gamma = \mathbf{0}\}$; the sesquilinear forms $a_L(\cdot,\cdot) : V_L \times V_L \to \mathbb{C}$ and the linear functionals $F_L : V_L \to \mathbb{C}$ such that $a(\mathbf{v},\mathbf{w}) = a_C(\mathbf{v}_C,\mathbf{w}_C) + a_I(\mathbf{v}_I,\mathbf{w}_I)$ and $F(\mathbf{v}) = F_C(\mathbf{v}_C) + F_I(\mathbf{v}_I) \quad \forall \mathbf{v},\ \mathbf{w} \in V$. If *the sesquilinear forms* $a_L(\cdot,\cdot)$ *are continuous and coercive in* $V_{L,0}$ *for both* $L = C, I$ we can define the extension operators $\mathbf{R}_L : X \to V_L$ in the following way: for any $\boldsymbol{\eta} \in X$, $\mathbf{R}_L\boldsymbol{\eta}$ is the unique function in V_L such that

$$(\mathbf{n} \times \mathbf{R}_L\boldsymbol{\eta} \times \mathbf{n})_{|\Gamma} = \boldsymbol{\eta}$$
$$a_L(\mathbf{R}_L\boldsymbol{\eta}, \mathbf{v}_L) = 0 \quad \forall \mathbf{v}_L \in V_{L,0}\,.$$

Let us consider the Steklov-Poincare operators $S_L : X \to X'$ given by

$$\langle S_L\boldsymbol{\eta}, \boldsymbol{\nu}\rangle_\Gamma = a_L(\mathbf{R}_L\boldsymbol{\eta}, \mathbf{R}_L\boldsymbol{\nu}) \quad \forall \boldsymbol{\eta},\ \boldsymbol{\nu} \in X\,.$$

Moreover we can define the functions $\hat{\mathbf{u}}_L \in V_{L,0}$ such that

$$a_L(\hat{\mathbf{u}}_L, \mathbf{v}_L) = F_L(\mathbf{v}_L) \quad \forall \mathbf{v}_L \in V_{L,0}$$

and $\boldsymbol{\chi}_L \in X'$ given by $\langle \boldsymbol{\chi}_L, \boldsymbol{\eta}\rangle_\Gamma = F_L(\mathbf{R}_L\boldsymbol{\eta}) - a_L(\hat{\mathbf{u}}_L, \mathbf{R}_L\boldsymbol{\eta}) \quad \forall \boldsymbol{\eta} \in X$. Let us denote $\boldsymbol{\chi} = \boldsymbol{\chi}_I + \boldsymbol{\chi}_C$. The Steklov-Poincare equation reads: find $\boldsymbol{\lambda} \in X$ such that

$$(S_I + S_C)\boldsymbol{\lambda} = \boldsymbol{\chi}\,. \tag{6}$$

If $\boldsymbol{\lambda}$ is solution of (6) then $\mathbf{u} = \begin{cases} \mathbf{R}_C\boldsymbol{\lambda} + \hat{\mathbf{u}}_C & \text{in } \Omega_C \\ \mathbf{R}_I\boldsymbol{\lambda} + \hat{\mathbf{u}}_I & \text{in } \Omega_I \end{cases}$ is solution of (5).

If *for one of the two subdomains the sesquilinear form* $a_L(\cdot,\cdot)$ *is also continuous and coercive in* V_L then for each $\boldsymbol{\xi} \in X'$ there exist a unique $\mathbf{F}_L\boldsymbol{\xi} \in V_L$ such that $a_L(\mathbf{F}_L\boldsymbol{\xi}, \mathbf{w}_L) = \langle \boldsymbol{\xi}, \mathbf{n} \times \mathbf{w}_L \times \mathbf{n}\rangle_\Gamma \quad \forall \mathbf{w}_L \in V_L$. It is easy to see that $\langle S_L(\mathbf{n} \times \mathbf{F}_L\boldsymbol{\xi} \times \mathbf{n}), \boldsymbol{\eta}\rangle_\Gamma = \langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle_\Gamma$ for all $\boldsymbol{\eta} \in X$ hence $S_L^{-1}(\boldsymbol{\xi}) = \mathbf{n} \times \mathbf{F}_L\boldsymbol{\xi} \times \mathbf{n}$. It is well known that the Dirichlet-Neumann iterative method is equivalent to the preconditioned Richardson method for the Steklov-Poincare equation

$$\boldsymbol{\lambda}^{(n+1)} = \boldsymbol{\lambda}^{(n)} + \theta S_L^{-1}\left[\boldsymbol{\chi} - (S_I + S_C)\boldsymbol{\lambda}^{(n)}\right].$$

In the $\mathbf{H}$-based formulation the preconditioner is S_I while in the $\mathbf{E}$-based formulation the preconditioner is S_C.

We are interested in the finite element approximation of these problems using the Nédélec curl-conforming edge elements of degree k, $N_{L,h}^k \subset H(\mathrm{curl}\,;\Omega_L)$ (see [7]) for $L = C, I$. Let us denote $\mathbb{P}_k$, $k \geq 0$, the space of polynomials of degree less than or equal k in the three variables x_1, x_2, x_3, and by $\widetilde{\mathbb{P}}_k$ the space of homogeneous polynomials of degree k. For $k \geq 1$ we define the polynomial spaces $M_k := \{\mathbf{q} \in (\widetilde{\mathbb{P}}_k)^3 \,|\, \mathbf{q}(\mathbf{x}) \cdot \mathbf{x} = 0\}$ and $R_k := (\mathbb{P}_{k-1})^3 \oplus M_k$. Let us consider a tetrahedral triangulation of Ω, $\mathcal{T}_h$, such that its restriction to Ω_L, $\mathcal{T}_{L,h}$, induces a triangulation of Ω_L. Then

$$N_{L,h} := \{\mathbf{w}_h \in H(\mathrm{curl}\,;\Omega_L) \,|\, \mathbf{w}_{h|K} \in R_k \quad \forall K \in \mathcal{T}_{L,h}\}\,.$$

We want to show that in the discrete setting the iterative procedure converges and that the convergence rate is independent of h.

The discrete **H**-based formulation is stated in the space

$$V_h(\mathbf{0}) := \{\mathbf{v}_h \in N_h^k \, : \, \mathbf{v}_{I,h} \in H^0_{0,\partial\Omega}(\mathrm{curl}\,;\Omega_I)\} \subset V(\mathbf{0})\,.$$

The space X for the Dirichlet-Neumann procedure is

$$\chi_h^0 = \{(\mathbf{n}\times\mathbf{v}_h\times\mathbf{n})_{|\Gamma} \, : \, \mathbf{v}_h \in V_h(\mathbf{0})\} \subset H_0^{-1/2}(\mathrm{curl}\,_\tau;\Gamma)\,.$$

In Ω_C we use the standard Nédélec finite elements $N^k_{C,h}$, while in Ω_I we have the finite element space

$$V_{I,h}(\mathbf{0}) = N^k_{I,h} \cap H^0_{0,\partial\Omega}(\mathrm{curl}\,;\Omega_I)\,.$$

Remark 3. Let $L^k_{I,h} \subset H^1(\Omega_I)$ be the space of standard Lagrange finite elements of degree k and $H_{I,h,0} = L^k_{I,h} \cap H^1_{0,\partial\Omega}(\Omega_I)$. Then

$$V_{I,h}(\mathbf{0}) = \nabla H_{I,h,0} + \mathscr{H}_{I,h}$$

where $\mathscr{H}_{I,h}$ is a space whose dimension coincides with n_Γ, the first Betti number of Ω_I. More precisely, there exits a system of cutting surfaces Ξ_l, $l = 1,\ldots,n_\Gamma$ with $\partial\Xi_l \subset \Gamma$ such that every function $\mathbf{v}_I \in H_{0,\partial\Omega}(\mathrm{curl}\,;\Omega_I)$ restricted to $\Omega_I \setminus \cup_{l=1}^{n_\Gamma}\Xi_l$ is the gradient of a function belonging to $H^1(\Omega_I \setminus \cup_{l=1}^{n_\Gamma}\Xi_l)$ (see e.g. [3, 5, 6]). If the triangulation $\mathscr{T}_{I,h}$ induces a triangulation on each surface Ξ_l the space $\mathscr{H}_{I,h}$ is the one generated by the $(L^2(\Omega_I))^3$-extension of the gradient of the piecewise linear function taking value one at the node on one side of Ξ_l and value zero at all the other nodes including those on the other side of Ξ_l (see [2], Sect. 5.4).

Concerning the **E**-based formulation, for its finite element approximation we consider the space

$$W_h := \{\mathbf{w}_h \in N_h^k \, : \, \int_{\Omega_I} \mathbf{w}_h \cdot \nabla\overline{\phi}_{I,h} = 0 \quad \forall\, \phi_{I,h} \in H^k_{I,h,*}\}$$

where $H^k_{I,h,*} = L^k_{I,h} \cap H^1_*(\Omega_I)$. (Notice that W_h is not a subspace of W.) The space X where the Steklov-Poincare operators are defined is the space of discrete traces

$$\chi_h = \{(\mathbf{n}\times\mathbf{w}_h\times\mathbf{n})_{|\Gamma} \, : \, \mathbf{w}_h \in N_h^k\} \subset H^{-1/2}(\mathrm{curl}\,_\tau;\Gamma)\,.$$

Also in this case we use the standard Nédélec finite elements $N^k_{C,h}$ in Ω_C while in Ω_I we consider the finite element space

$$W_{I,h} := \{\mathbf{w}_{I,h} \in N^k_{I,h} \, : \, \int_{\Omega_I} \mathbf{w}_{I,h} \cdot \nabla\overline{\phi}_{I,h} = 0 \quad \forall\, \phi_{I,h} \in H^k_{I,h,*}\}\,.$$

In order to prove the convergence of the iterative procedure let us proceed as in [1]. If $k \in \mathbb{C}$ is an eigenvalue of the map $T_L : X \to X$, $T_L\boldsymbol{\eta} := \boldsymbol{\eta} - \theta S_L^{-1}(S_I + S_C)\boldsymbol{\eta}$

with $L=I$ or $L=C$, then $k = 1-\theta\frac{\langle (S_I+S_C)\boldsymbol{\eta},\boldsymbol{\eta}\rangle_\Gamma}{\langle S_L\boldsymbol{\eta},\boldsymbol{\eta}\rangle_\Gamma} = (1-\theta)-\theta\frac{\langle S_M\boldsymbol{\eta},\boldsymbol{\eta}\rangle_\Gamma}{\langle S_L\boldsymbol{\eta},\boldsymbol{\eta}\rangle_\Gamma}$ for any eigenvector $\boldsymbol{\eta}\in X$. Here $M=I$ or $M=C$ but $M\neq L$. If

$$\mathrm{Re}[\langle S_I\boldsymbol{\eta},\boldsymbol{\eta}\rangle_\Gamma]\mathrm{Re}[\langle S_C\boldsymbol{\eta},\boldsymbol{\eta}\rangle_\Gamma]+\mathrm{Im}[\langle S_I\boldsymbol{\eta},\boldsymbol{\eta}\rangle_\Gamma]\mathrm{Im}[\langle S_C\boldsymbol{\eta},\boldsymbol{\eta}\rangle_\Gamma]\geq 0 \qquad (7)$$

and $0\leq\theta\leq 1$ then

$$|k|^2\leq(1-\theta)^2+\theta^2\frac{|\langle S_M\boldsymbol{\eta},\boldsymbol{\eta}\rangle_\Gamma|^2}{|\langle S_L\boldsymbol{\eta},\boldsymbol{\eta}\rangle_\Gamma|^2}\leq(1-\theta)^2+\theta^2\frac{\beta_M^2}{\alpha_L^2}$$

being β_M the continuity constant of S_M and α_L the coercivity constant of S_L. Choosing $0<\theta<\min\left(1,\frac{2\alpha_L^2}{\alpha_L^2+\beta_M^2}\right)$ on has $|k|<1$ for each k eigenvalue of T, hence in the discrete setting the Dirichlet-Neumann procedures converges and, if α_L and β_M are independent of the mesh size, h, also the convergence rate is independent of h.

In the **H**-based formulation we have $L=I$ and $M=C$. The sesquilinear form

$$a_C(\mathbf{v}_C,\mathbf{w}_C):=\int_{\Omega_C}\left(-i\omega^{-1}\sigma^{-1}\mathrm{curl}\,\mathbf{v}_C\cdot\mathrm{curl}\,\overline{\mathbf{w}}_C+\mu\mathbf{v}_C\cdot\overline{\mathbf{w}}_C\right)$$

is clearly continuous and coercive in $H(\mathrm{curl}\,;\Omega_C)$ hence in $N^k_{C,h}$. In the insulator $a_I(\mathbf{v}_I,\mathbf{w}_I):=\int_{\Omega_I}\mu\mathbf{v}_I\cdot\overline{\mathbf{w}}_I$ is continuous and coercive in $H^0(\mathrm{curl}\,;\Omega_I)$ then also in $V^0_{I,h}$. The coercivity of S_I with a constant α_I independent of h follows from the coercivity of $a_I(\cdot,\cdot)$ and the continuity of the trace operator while the continuity of S_C with a constant β_C independent of h follows from the continuity of $a_C(\cdot,\cdot)$ and the existence of a continuous extension operator $\mathscr{E}_{C,h}:\chi_h\to N^k_{C,h}$ with continuity constant independent of h. Such an extension has been constructed in [1]. Moreover (7) clearly holds because it reduces to $\left(\int_{\Omega_C}\mu\mathbf{R}_C\boldsymbol{\eta}\cdot\overline{\mathbf{R}_C\boldsymbol{\eta}}\right)\left(\int_{\Omega_I}\mu\mathbf{R}_I\boldsymbol{\eta}\cdot\overline{\mathbf{R}_I\boldsymbol{\eta}}\right)\geq 0$. Hence taking θ small enough the iterative Dirichlet-Neumann procedure for the **H**-based formulation converges with a rate independent of the mesh size.

On the other hand for the **E**-based formulation we have $L=C$ and $M=I$. Again the sesquilinear form

$$a_C(\mathbf{v}_C,\mathbf{w}_C):=\int_{\Omega_C}\left(\mu^{-1}\mathrm{curl}\,\mathbf{v}_C\cdot\mathrm{curl}\,\overline{\mathbf{w}}_C+i\omega\sigma\mathbf{v}_C\cdot\overline{\mathbf{w}}_C\right)$$

is clearly continuous and coercive in $H(\mathrm{curl}\,;\Omega_C)$ hence in $N^k_{C,h}$. The coercivity of S_C (the preconditioner in this case) with a constant α_C independent of h follows from the uniform coercivity of $a_C(\cdot,\cdot)$ and the continuity of the trace operator. In the insulator we have $a_I(\mathbf{v}_I,\mathbf{w}_I):=\int_{\Omega_I}\mu^{-1}\mathrm{curl}\,\mathbf{v}_I\cdot\mathrm{curl}\,\overline{\mathbf{w}}_I$ that is continuous in $H(\mathrm{curl}\,;\Omega_I)$, hence in $W_{I,h}$. Proceeding as in [2], Sect. 5.5, it can be proved that it is coercive in $W_{I,h}\cap H_0(\mathrm{curl}\,;\Omega_I)$. In order to prove the continuity of S_I with a constant β_I independent of h we need a continuous extension operator $\mathscr{E}_{I,h}:\chi_h\to W_{I,h}\cap H_{0,\partial\Omega}(\mathrm{curl}\,;\Omega_I)$. We know that there exists a continuous extension $\widehat{\mathscr{E}}_{I,h}:\chi_h\to N^k_{I,h}\cap H_{0,\partial\Omega}(\mathrm{curl}\,;\Omega_I)$ (see again [1]). Given $\boldsymbol{\eta}_h\in\chi_h$ let $\boldsymbol{\Phi}_{I,h}\in H^k_{I,h,*}$ be such that

$$\int_{\Omega_I} \nabla \Phi_{I,h} \cdot \nabla \psi_{I,h} = \int_{\Omega_I} \widehat{\mathscr{E}}_{I,h} \boldsymbol{\eta}_h \cdot \nabla \psi_{I,h} \quad \forall\, \psi_{I,h} \in H^k_{I,h,*}\,.$$

Then $\mathscr{E}_{I,h}\boldsymbol{\eta}_h := \widehat{\mathscr{E}}_{I,h}\boldsymbol{\eta}_h - \nabla \Phi_{I,h}$ is a continuous extension from χ_h in the space $W_{I,h} \cap H_{0,\partial\Omega}(\mathrm{curl}\,;\Omega_I)$ with continuity constant independent of h. Condition (7) reduce in this case to $\left(\int_{\Omega_C} \mu^{-1} \mathrm{curl}\, \mathbf{R}_C \boldsymbol{\eta} \cdot \mathrm{curl}\, \overline{\mathbf{R}_C \boldsymbol{\eta}}\right) \left(\int_{\Omega_I} \mu^{-1} \mathrm{curl}\, \mathbf{R}_I \boldsymbol{\eta} \cdot \mathrm{curl}\, \overline{\mathbf{R}_I \boldsymbol{\eta}}\right) \geq 0$ that clearly holds true.

4 Conclusion

We proposed two iterative substructuring methods for two different formulations of the eddy current problem based on the electric field and magnetic field, respectively, and provided the convergence analysis. Both formulations use a constrained space in the insulator. In the **E**-based formulation the constrain is imposed introducing a Lagrange multiplier while in the **H**-based formulation a finite element approximation $V_{I,h}(\mathbf{0})$ of the constrained space $H_{0,\partial\Omega}(\mathrm{curl}\,;\Omega_I)$ is used. The dimension of $V_{I,h}(\mathbf{0})$ is equal to n_Γ, the dimension of the $\mathscr{H}_{I,h}$, plus the dimension of $H_{I,h,0}$, that is a space of scalar functions. So the subproblem in the insulator is smaller for the **H**-based formulation than for the **E**-based formulation. However the construction of a base of $\mathscr{H}_{I,h}$ requires the determination of a system of cutting surfaces. This procedure can be cumbersome in complex geometry configurations (for instance if the conductor is a trefoil knot) an the **E** based formulation avoids this difficult.

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