Stable BETI Methods in Electromagnetics

Olaf Steinbach and Markus Windisch

Institute of Computational Mathematics, TU Graz, Steyrergasse 30, 8010 Graz, Austria, o.steinbach@tugraz.at, mwind82@gmx.at

Summary. In this paper we present a stable boundary element tearing and interconnecting domain decomposition method for the parallel solution of the electromagnetic wave equation with piecewise constant wave numbers. In particular we consider stable boundary integral formulations and generalized Robin type transmission conditions to ensure unique solvability of the local subproblems. Numerical results confirm the robustness of the proposed approach.

1 Introduction

The application of standard finite and boundary element tearing and interconnecting domain decomposition methods [4, 5] may fail in the case of the acoustic or electromagnetic wave equation due to a possible occurence of spurious modes which are related to local Dirichlet or Neumann boundary value problems. For the acoustic wave equation we have introduced in [9, 10] a boundary element tearing and interconnecting domain decomposition approach which is stable for all local wave numbers. The aim of this paper is to extend these results when considering the electromagnetic wave equation. Although the general concept is rather similar in both cases, the numerical analysis of boundary integral equations and boundary element methods for the Maxwell system requires advanced techniques, in particular appropriate space splitting approaches. For the definition of Sobolev spaces which are related to the Maxwell equation, see, e.g., [2], for the analysis of Maxwell boundary integral equations, see, for example, [7], and for related boundary element methods, see, e.g., [1].

2 Formulation of the Domain Decomposition Approach

As a model problem we consider the Neumann boundary value problem of the electromagnetic wave equation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) - [k(x)]^2\mathbf{U}(x) = \mathbf{0} \qquad \text{for } x \in \Omega, \tag{1}$$

$$\gamma_N \mathbf{U}(x) := \mathbf{curl}\,\mathbf{U}(x) \times \mathbf{n} = \mathbf{f}(x) \qquad \text{for } x \in \Gamma, \tag{2}$$

R. Bank et al. (eds.), *Domain Decomposition Methods in Science and Engineering XX*, Lecture Notes in Computational Science and Engineering 91, DOI 10.1007/978-3-642-35275-1_25, © Springer-Verlag Berlin Heidelberg 2013

where $\Omega \subset \mathbb{R}^3$ is a Lipschitz polyhedron with boundary $\Gamma = \partial\Omega$. We assume that the boundary value problems (1) and (2) admits a unique solution. Since the wave number $k(x)$ is assumed to be piecewise constant, i.e. $k(x) = k_i$ for $x \in \Omega_i$, instead of (1) and (2) we consider local boundary value problems to find $\mathbf{U}_i = \mathbf{U}_{|\Omega_i}$ satisfying

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}_i(x) - k_i^2\mathbf{U}_i(x) = \mathbf{0} \text{ for } x \in \Omega_i,\ \gamma_N \mathbf{U}_i(x) = \mathbf{g}(x) \text{ for } x \in \Gamma_i \cap \Gamma$$

with respect to a non–overlapping domain decomposition

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j, \quad \Gamma_i = \partial\Omega_i,$$

together with the transmission or interface boundary conditions

$$\gamma_{D,i}\mathbf{U}_i(x) = \gamma_{D,j}\mathbf{U}_j(x) \quad \text{for } x \in \Gamma_{ij} = \Gamma_i \cap \Gamma_j, \tag{3}$$

$$\gamma_{N,i}\mathbf{U}_i(x) + \gamma_{N,j}\mathbf{U}_j(x) = \mathbf{0} \quad \text{for } x \in \Gamma_{ij}, \tag{4}$$

where the Dirichlet trace operator is given by

$$\gamma_D \mathbf{U} = \mathbf{n} \times (\mathbf{U}_{|\Gamma} \times \mathbf{n}).$$

Since the local Dirichlet or Neumann boundary value problems may exhibit spurious modes, instead of the Neumann transmission condition in (4) we consider a generalized Robin interface condition

$$\gamma_{N,i}\mathbf{U}_i(x) + \gamma_{N,j}\mathbf{U}_j(x) + i\eta_{ij}\mathsf{R}_{ij}[\gamma_{D,i}\mathbf{U}_i(x) - \gamma_{D,j}\mathbf{U}_j(x)] = \mathbf{0} \quad \text{for } x \in \Gamma_{ij}, i < j. \tag{5}$$

The operators R_{ij} are assumed to be strictly positive, i.e. $\langle \mathsf{R}_{ij}\mathbf{u}, \mathbf{u}\rangle_{\Gamma_{ij}} > 0$ for all $\mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_{ij})$, and $\eta_{ij} \in \mathbb{R}\backslash\{0\}$. We define

$$(\mathsf{R}_i u_{|\Gamma_i})(x) := (\mathsf{R}_{ij} u_{|\Gamma_{ij}})(x) \quad \text{for } x \in \Gamma_{ij}$$

and

$$\eta_i(x) := \begin{cases} \eta_{ij} & \text{for } x \in \Gamma_{ij},\ i < j, \\ -\eta_{ij} & \text{for } x \in \Gamma_{ij},\ i > j, \\ 0 & \text{for } x \in \Gamma_i \cap \Gamma, \end{cases}$$

where we assume that $\eta_i(x)$ for $x \in \Gamma_i$ does not change its sign, see also [9]. In this case we can ensure unique solvability [11] of the local Robin boundary value problems

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}_i(x) - k_i^2\mathbf{U}_i(x) = \mathbf{0} \qquad \text{for } x \in \Omega_i, \tag{6}$$

$$\gamma_N \mathbf{U}_i(x) + i\eta_i \mathsf{R}\gamma_D \mathbf{U}_i(x) = \mathbf{g}(x) \quad \text{for } x \in \Gamma_i \cap \Gamma. \tag{7}$$

For the solution of local Dirichlet and Robin boundary value problems we will apply boundary element methods which are based on the use of the Stratton–Chu representation formula for $x \in \Omega$, see [3],

$$\mathbf{U}(x) = \Psi_k^M(\gamma_D \mathbf{U})(x) + \Psi_k^A(\gamma_N \mathbf{U})(x) + \frac{1}{k^2}\mathbf{grad}\,\Psi_k^S \mathrm{div}_\Gamma\,(\gamma_N \mathbf{U})(x).$$

Here,

$$\Psi_k^A(\lambda)(x) := \int_\Gamma g_k(x,y)\lambda(y)ds_y \quad \text{for } x \notin \Gamma, \quad g_k(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|},$$

is the vector–valued single layer potential with the fundamental solution of the Helmholtz equation, and

$$\Psi_k^M(\lambda)(x) := \mathbf{curl}\,\Psi_k^A(\lambda \times \mathbf{n})(x) \quad \text{for } x \notin \Gamma$$

is the Maxwell double layer potential. In addition,

$$\Psi_k^V(\lambda)(x) := \int_\Gamma g_k(x,y)\lambda(y)ds_y \quad \text{for } x \notin \Gamma$$

is the scalar single layer potential. By introducing the Maxwell single layer potential

$$\Psi_k^S(\lambda)(x) := \Psi_k^A(\lambda)(x) + \frac{1}{k^2}\mathbf{grad}\,\Psi_k^S \mathrm{div}_\Gamma\,(\lambda)(x) \quad \text{for } x \notin \Gamma,$$

we can write the Straton–Chu representation formula as

$$\mathbf{U}(x) = \Psi_k^M(\gamma_D \mathbf{U}(x)) + \Psi_k^S(\gamma_N \mathbf{U}(x)) \quad \text{for } x \in \Omega. \tag{8}$$

The application of the Maxwell trace operators gives the boundary integral equations [7, 11]

$$\begin{aligned} \gamma_N \mathbf{U} &= \mathsf{N}_k(\gamma_D \mathbf{U}) + (\frac{1}{2}I + \mathsf{B}_k)(\gamma_N \mathbf{U}), \\ \gamma_D \mathbf{U} &= (\frac{1}{2}I + \mathsf{C}_k)(\gamma_D \mathbf{U}) + \mathsf{S}_k(\gamma_N \mathbf{U}). \end{aligned} \tag{9}$$

Now we are in a position to derive different approaches to solve local boundary value problems with generalized Robin boundary conditions. Here we consider an approach which is based on the use of the Steklov–Poincaré operator

$$\mathsf{T}_k = \mathsf{N} + (\frac{1}{2}I + \mathsf{B}_k)\mathsf{S}_k^{-1}(\frac{1}{2}I + \mathsf{C}_k) = \mathsf{S}_k^{-1}(\frac{1}{2}I + \mathsf{C}_k) \tag{10}$$

which requires the invertibility of the single layer operator S_k. Since S_k is not invertible for all wave numbers k, instead of (10) we consider a system of boundary integral equations to find $\mathbf{u} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{t} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ such that

$$\begin{pmatrix} \mathsf{N}_k + i\eta\mathsf{R} & \frac{1}{2}I + \mathsf{B}_k \\ -\frac{1}{2}I + \mathsf{C}_k & \mathsf{S}_k \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ \mathbf{0} \end{pmatrix} \tag{11}$$

is satisfied. The unique solvability of (11) follows from a generalized Garding inequality

$$\mathrm{Re}\left(\left\langle \begin{pmatrix} \mathsf{N}_k + i\eta \mathsf{R} & \frac{1}{2}I + \mathsf{B}_k \\ -\frac{1}{2}I + \mathsf{C}_k & \mathsf{S}_k \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}, \begin{pmatrix} \mathscr{Y}\mathbf{u} \\ \mathscr{X}\mathbf{t} \end{pmatrix} \right\rangle_\Gamma + C((\mathbf{u},\mathbf{t}),(\mathbf{u},\mathbf{t}))\right) \\ \geq c\left(\|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} + \|\mathbf{t}\|^2_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \right)$$

for some appropriate bijective operators $\mathscr{X}$ and $\mathscr{Y}$, and from injectivity which is in fact related to the unique solvability of the local Robin boundary value problems (6) and (7), see [11]. Since the proof of the generalized Grarding inequality requires a comprehensive study of the trace spaces $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ and $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$, and of the corresponding Hodge–type splittings, we refer to [2, 11] for a detailed presentation.

By summing up all local boundary integral equation systems with respect to the transmission conditions (5) we finally obtain the following variational formulation to find $\mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma_S)$ and $\mathbf{t}_i \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma_i)$ satisfying

$$\sum_{i=1}^p \left[\langle \mathsf{N}_i \mathbf{u}_{|\Gamma_i}, \mathbf{v}_{|\Gamma_i}\rangle_{\Gamma_i} + \langle (\frac{1}{2}I + \mathsf{B}_i)\mathbf{t}_i, \mathbf{v}_{|\Gamma_i}\rangle_{\Gamma_i} + i\eta_i \langle \mathsf{R}_i \mathbf{u}_{|\Gamma_i}, \mathbf{v}_{|\Gamma_i}\rangle_{\Gamma_i} \right] = \langle \mathbf{f}, \mathbf{v}\rangle_\Gamma \tag{12}$$

for all $\mathbf{v} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma_S)$ and

$$\langle \mathsf{S}_i \mathbf{t}_i, \mu_i\rangle_{\Gamma_i} + \langle (-\frac{1}{2}I + \mathsf{C}_i)\mathbf{u}_{|\Gamma_i}, \mu_i\rangle_{\Gamma_i} = 0 \tag{13}$$

for all $\mu_i \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma_i), i = 1,\ldots,p$. The variational formulation (12), (13) admits a unique solution iff the orginal problems (1) and (2) has a unique solution, see [11].

A boundary element discretization of the Sobolev spaces $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma_S)$ and $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma_i)$ by using Raviart–Thomas elements [8, 11], i.e.

$$\mathscr{E}_h := \mathscr{E}_h(\Gamma_S) = \mathrm{span}\{\phi_k\}_{k=1}^{M_S} \subset \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma_S)$$

and

$$\mathscr{F}_{i,h} = \mathrm{span}\{\psi_k^i\}_{k=1}^{N_i} \subset \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma_i),$$

then results in a linear system of algebraic equations,

$$\begin{pmatrix} \mathsf{S}_{1,h} & & & \widetilde{\mathsf{C}}_{1,h}A_i \\ & \cdots & & \vdots \\ & & \mathsf{S}_{p,h} & \widetilde{\mathsf{C}}_{p,h}A_p \\ A_1^\top \widetilde{\mathsf{B}}_{1,h} & \ldots & A_p^\top \widetilde{\mathsf{B}}_{p,h} & \sum\limits_{i=1}^p A_i^\top [\mathsf{N}_{i,h} + i\eta_i \mathsf{R}_{i,h}]A_i \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \vdots \\ \underline{t}_p \\ \underline{u} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \vdots \\ \underline{0} \\ \sum\limits_{i=1}^p A_i^\top \underline{f}_i \end{pmatrix}, \tag{14}$$

where the block matrices are given by

$$
\begin{aligned}
\mathsf{S}_{i,h}[\ell,k] &= \langle \mathsf{S}_i \psi^i_k, \psi^i_\ell \rangle_{\Gamma_i},\\
\widetilde{\mathsf{C}}_{i,h}[\ell,n] &= \langle (-\frac{1}{2}I + \mathsf{C}_i)\phi^i_n, \psi^i_\ell \rangle_{\Gamma_i},\\
\widetilde{\mathsf{B}}_{i,h}[m,k] &= \langle (\frac{1}{2}I + \mathsf{B}_i)\psi^i_k, \phi^i_m \rangle_{\Gamma_i},\\
\mathsf{N}_{i,h}[m,n] &= \langle \mathsf{N}_i \phi^i_n, \phi^i_m \rangle_{\Gamma_i},\\
\mathsf{R}_{i,h}[m,n] &= \langle \mathsf{R}_i \phi^i_n, \phi^i_m \rangle_{\Gamma_i}
\end{aligned}
$$

for $k,\ell = 1,\ldots,N_i$, $m,n = 1,\ldots,M_i$, and $i = 1,\ldots,p$.

In what follows we will discuss an efficient and parallel solution of the linear system (14). Although the computation of all block matrices can be done in parallel, the construction of an appropriate preconditioner is more challenging. A possible approach is to design preconditioners as in tearing and interconnecting methods which are well established for a wide range of applications. A first step into this direction is the formulation of stable tearing and interconnecting methods.

The idea of the tearing and interconnecting approach is to tear the global degrees of freedom, which are given by $\underline{u}$, into local degrees of freedom $\underline{u}_i$. To ensure global continuity, we need to glue them together by using Langrange multipliers [10, 11], see also Fig. 1. Note, that instead of Neumann transmission condition we use the generalized Robin transmission conditions as given in (5). As in the standard tearing and interconnecting approach this leads to the extended linear system

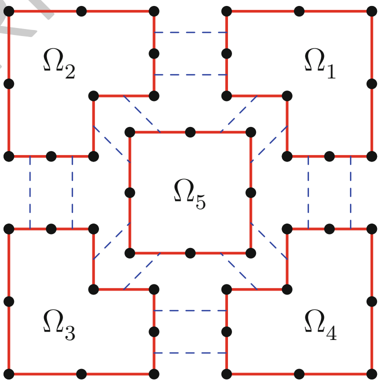

Fig. 1. Tearing and Interconnecting for edge based trial functions

$$\begin{pmatrix} \mathsf{N}_{1,h}+i\eta_1\mathsf{R}_{i,h} & \widetilde{\mathsf{B}}_{1,h} & & & & -B_1^\top \\ \widetilde{\mathsf{C}}_{1,h} & \mathsf{S}_{1,h} & & & & \\ & & \ddots & & & \vdots \\ & & & \mathsf{N}_{p,h}+i\eta_p\mathsf{R}_{p,h} & \widetilde{\mathsf{B}}_{p,h} & -B_p^\top \\ & & & \widetilde{\mathsf{C}}_{p,h} & \mathsf{S}_{p,h} & \\ B_1 & & \dots & B_p & & \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \underline{t}_1 \\ \vdots \\ \underline{u}_p \\ \underline{t}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \underline{0} \\ \vdots \\ \underline{f}_p \\ \underline{0} \\ \underline{0} \end{pmatrix} \tag{15}$$

where the sparse and Boolean matrices B_i ensure the continuity of the global solution. Since the local Robin boundary value problems (6) and (7) are uniquely solvable, the local block matrices are invertible, and we can consider the Schur complement system

$$\begin{aligned} \sum_{i=1}^{p} (0\ B_i) \begin{pmatrix} \mathsf{N}_{i,h}+i\eta_i\mathsf{R}_{i,h} & \widetilde{\mathsf{B}}_{i,h} \\ \widetilde{\mathsf{C}}_{i,h} & \mathsf{S}_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} B_i^\top \underline{\lambda} \\ \underline{0} \end{pmatrix} \\ = -\sum_{i=1}^{p} (B_i\ 0) \begin{pmatrix} \mathsf{N}_{i,h}+i\eta_i\mathsf{R}_{i,h} & \widetilde{\mathsf{B}}_{i,h} \\ \widetilde{\mathsf{C}}_{i,h} & \mathsf{S}_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} \underline{f}_i \\ \underline{0} \end{pmatrix}. \end{aligned} \tag{16}$$

Note that (16) corresponds to the adjoint system of standard tearing and interconnecting approaches [4, 5].

3 Numerical Results

As a first example we consider the Neumann boundary value problem

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} &= \mathbf{0} && \text{in } \Omega, \\ \gamma_N \mathbf{U} &= \mathbf{f} && \text{on } \Gamma \end{aligned} \tag{17}$$

where the domain Ω is given by $(-1.0, 1.5)\times(0.0, 1.0)\times(0.0, 1.0)$, and Ω is divided into two subdomains Ω_i by the yz–plane, see Fig. 2.

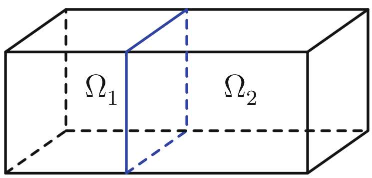

Fig. 2. Computational domain Ω and domain decomposition

As an analytical solution for both examples we use

$$\mathbf{U}(x) = \left[\frac{1+ikr-k^2r^2}{r^3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{3+3ikr-k^2r^2}{r^5}(x_1-\hat{x}_1) \begin{pmatrix} x_1-\hat{x}_1 \\ x_2-\hat{x}_2 \\ x_3-\hat{x}_3 \end{pmatrix} \right] e^{ikr}$$

with $r = |x - \hat{x}|$ and $\hat{x} = (-3.0, 2.1, 1.1)^\top$. The boundary element discretization of the coupled variational formulation (12) and (13) is done with respect to a globally uniform boundary element mesh with E_i edges per subdomain Ω_i, and by using first order Raviart–Thomas elements. The number of Lagrange multipliers is denoted by Λ. The linear system (16) is solved by a GMRES method with a relative residuum reduction of $\varepsilon = 10^{-7}$. For our numerical tests we consider two different wave numbers: The first one is $k = 1.0$ and the second one is the first Dirichlet and Neumann eigenfrequency of the unit cube Ω_1, $k = \sqrt{2}\pi \approx 4.44288$. The results are given in Table 1, where the error is the relative $L_2(\Gamma_1)$ error of the lowest order Raviart–Thomas approximation of the local Dirichlet datum $\mathbf{u}_1$.

E_i	Λ	iter	error	E_i	Λ	iter	error
36	8	5	0.1824189	36	8	5	0.7042192
144	28	17	0.0895037	144	28	19	0.3055468
576	104	49	0.0440296	576	104	47	0.1472184
2304	400	142	0.0234164	2304	400	104	0.0772003

Table 1. Iteration numbers and errors for $k = 1$ (left) and $k = \sqrt{2}\pi$ (right).

In a second example we consider the Neumann boundary value problem (17) for the unit cube $\Omega = (0,1)^3$ which is divided into eight subcubes Ω_i. The results for two different wave numbers $k = 1.0, 8.0$ are given in Table 2.

E_i	Λ	iter	error	E_i	Λ	iter	error
36	90	60	0.1133393	36	90	60	0.9432815
144	324	147	0.0550944	144	324	153	0.3776120
576	1224	476	0.0266769	576	1224	397	0.1769975

Table 2. Iteration numbers and errors for $k = 1$ (left) and $k = 8$ (right).

Both numerical experiments confirm the stability and robustness of the proposed approach, and the theoretical error estimate as given in [11], i.e. we expect a linear order of convergence when using lowest order Raviart–Thomas elements. Note that the linear system (16) is solved by a GMRES method without preconditioner. Hence we observe a rapidly increasing number of required iterations. Therefore, the use of local and global preconditioners is mandatory for the solution of problems of practical interest. Probably, possible preconditioners can be constructed as in the acoustic scattering case see [11]. Another possibility is to consider a dual–primal approach as in [6].

Acknowledgments This work was supported by the Austrian Science Fund (FWF) within the project *Data sparse boundary and finite element domain decomposition methods in electromagnetics* under the grant P19255.

Bibliography

[1] A. Buffa, R. Hiptmair, T. von Petersdorff, and C. Schwab. Boundary element methods for Maxwell transmission problems in Lipschitz domains. *Numer. Math.*, 95:459–485, 2003.

[2] A. Buffa and P. Ciarlet Jr. On traces for functional spaces related to Maxwell's equations. I. An integration by parts formula in Lipschitz polyhedra. *Math. Methods Appl. Sci.*, 24:9–30, 2001.

[3] L. J. Chu and J. A. Stratton. Diffraction theory of electromagnetic waves. *Phys. Rev.*, 56:99–107, 1939.

[4] C. Farhat and F.-X. Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Int. J. Numer. Meth. Engrg.*, 32:1205–1227, 1991.

[5] U. Langer and O. Steinbach. Boundary element tearing and interconnecting methods. *Computing*, 71:205–228, 2003.

[6] Y. Li and J.-M. Jin. A vector dual-primal finite element tearing and interconnecting method for solving 3-D large-scale electromagnetic problems. *IEEE Trans. Antennas Propag.*, 54:3000–3009, 2006.

[7] J.-C. Nédélec. *Acoustic and electromagnetic equations*, volume 144 of *Applied Mathematical Sciences*. Springer, New York, 2001.

[8] P.-A. Raviart and J. M. Thomas. A mixed finite element method for 2nd order elliptic problems. In *Mathematical aspects of finite element methods*, volume 606 of *Lecture Notes in Mathematics*, pages 292–315. Springer, Berlin, 1977.

[9] O. Steinbach and M. Windisch. Robust boundary element domain decomposition solvers in acoustics. In Y. Huang et al., editor, *Domain Decomposition Methods in Science and Engineering XIX*, volume 78 of *Lecture Notes in Computational Science and Engineering*, pages 277–284. Springer, Berlin, Heidelberg, 2011.

[10] O. Steinbach and M. Windisch. Stable boundary element domain decomposition methods for the Helmholtz equation. *Numer. Math.*, 118:171–195, 2011.

[11] M. Windisch. *Boundary element tearing and interconnecting methods for acoustic and electromagnetic scattering*. PhD thesis, Institute of Computational Mathematics, TU Graz, 2010.