A Continuous Approach to FETI-DP Mortar Methods: Application to Dirichlet and Stokes Problem

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Summary. In this contribution we extend the FETI-DP mortar method for elliptic problems introduced by Bernardi et al. [2] and Chacón Vera [3] to the case of the incompressible Stokes equations showing that the same results hold in the two dimensional setting. These ideas extend easily to three dimensional problems. Finally some numerical tests are shown as a conclusion. This contribution is a condensed version of a more detailed forthcoming paper. We use standard notation, see for instance [1].

1 Incompressible Stokes Equations

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. We look for $u \in \mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^2$ and $p \in L^2(\Omega)$ such that $\int_\Omega p = 0$ and

$$\begin{aligned} (\nabla u, \nabla v)_\Omega - (p, div(v))_\Omega &= (f,v)_\Omega, \quad \forall v \in \mathbf{H}_0^1(\Omega) \\ -(q, div(u))_\Omega &= 0, \qquad \forall q \in L^2(\Omega). \end{aligned}$$

We better accomodate the restriction on the pressure by adding a new scalar unknown: we look for a pair of values $(u,\tau) \in \mathbf{H}_0^1(\Omega) \times \mathbb{R}$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} (\nabla u, \nabla v)_\Omega - (p, div(v))_\Omega + t\,(\tau - \int_\Omega p) &= (f,v)_\Omega, \quad \forall (v,t) \in \mathbf{H}_0^1(\Omega) \times \mathbb{R} \\ -(q, div(u))_\Omega - \tau \int_\Omega q &= 0, \qquad \forall q \in L^2(\Omega). \end{aligned}$$

Set $W = \mathbf{H}_0^1(\Omega) \times \mathbb{R}$ normed by $\|\underline{v}\|_W^2 = \|(v,t)\|_W^2 = \|\nabla v\|_{0,\Omega}^2 + t^2$ for any $\underline{v} = (v,t) \in W$, let $(\cdot,\cdot)_W$ be the scalar product on W and $b : W \times L^2(\Omega) \mapsto \mathbb{R}$ given by

$$b(q,(v,t)) = -(q, div(v))_\Omega - t \int_\Omega q.$$

Then, we look for $\underline{u} = (u,\tau) \in W$ and $p \in L^2(\Omega)$ such that

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$$(\underline{u},\underline{v})_W + b(p,\underline{v}) = (f,v)_\Omega, \quad \forall \underline{v} \in W \tag{1}$$
$$b(q,\underline{u}) = 0, \quad \forall q \in L^2(\Omega). \tag{2}$$

It is quite straightforward to see that:

Lemma 1. *There exists a positive constant $\beta > 0$ such that for all $p \in L^2(\Omega)$*

$$\sup_{(v,t)\in W} \frac{b(p,(v,t))}{\|(v,t)\|_W} \geq \sup_{v\in \mathbf{H}_0^1(\Omega),t\in\mathbb{R}} \frac{b(p,(v,t))}{(\|\nabla v\|_{0,\Omega}^2 + t^2)^{1/2}} \geq \beta \|p\|_{0,\Omega}. \tag{3}$$

As a consequence, problem (1)–(2) is well posed and its unique solution is the one of the original Stokes problem with Dirichlet homogeneous boundary conditions.

Next, we split $\Omega = \cup_{s=1}^S \Omega^s$ with nonoverlaping polygonal subdomains, suppose that

$$\Gamma_{s,t} = \partial\Omega^s \cap \partial\Omega^t$$

is either an edge (i.e., a segment), a crosspoint or empty and, finally, consider $\mathscr{E}_0 = \{\Gamma_e\}_{e=1,..,E}$ the sorted set of all edges inside Ω. We suppose that each Ω^s is of area $\mathscr{O}(H^2)$ and shape regular while each Γ_e is of length $\mathscr{O}(H)$ for some fixed $H > 0$. The set of all vertices of the polygonal subdomains Ω^s that are not on $\partial\Omega$ will be called **cross points** and denoted by $\mathscr{C}$. Finally, we denote by $[v]_{\Gamma_e}$ the jump across any interface Γ_e.

We take

$$X_\delta = \{v \in L^2(\Omega); v^s = v_{|\Omega^s} \in H^1(\Omega^s) \cap H_0^1(\Omega),\ 1 \leq s \leq S\},$$
$$X = \{v \in X_\delta,\ [v]_{\Gamma_e} \in H_{00}^{1/2}(\Gamma_e),\ \forall \Gamma_e \in \mathscr{E}_0\}.$$

With $\mathbf{X} = X \times X$ we construct $\mathbf{V} = \mathbf{X} \times \mathbb{R}$ and represent by $\underline{v} = (v,t)$ any element of $\mathbf{V}$ where $v \in \mathbf{X}$ and $t \in \mathbb{R}$. $\mathbf{V}$ is Hilbert space with norm $\|\underline{v}\|_{\mathbf{V}}^2 = |v|_{\mathbf{X}}^2 + t^2$ where, thanks to Poincaré's inequality, the norm of v is

$$|v|_{\mathbf{X}} = \{\sum_{s=1}^S \|\nabla v^s\|_{0,\Omega^s}^2 + \sum_{e=1}^E \|[v]_{\Gamma_e}\|_{1/2,00,\Gamma_e}^2\}^{1/2}.$$

Here, $\|\cdot\|_{1/2,00,\Gamma_e}$ is the norm induced by the scalar product $(\cdot,\cdot)_{1/2,00,\Gamma_e}$ on $H_{00}^{1/2}(\Gamma_e)$, see [5]. To simplify, let $\{\cdot,\cdot\}_{\Gamma_e} = (\cdot,\cdot)_{1/2,00,\Gamma_e}$. For the pressure space we consider $\mathbf{M} = \prod_{s=1}^S L^2(\Omega^s) (\approx L^2(\Omega))$ and define the continuous bilinear form $b : \mathbf{M} \times \mathbf{V} \mapsto \mathbb{R}$ given by

$$b(q,\underline{v}) = -\sum_{s=1}^S (q^s, div(v^s))_{\Omega^s} - t\sum_{s=1}^S \int_{\Omega^s} q^s, \quad \forall q^s \in L^2(\Omega^s).$$

Next, for each $\Gamma_e \in \mathscr{E}_0$ we take $\mathbf{H}_{00}^{1/2}(\Gamma_e) = (H_{00}^{1/2}(\Gamma_e))^2$, and handle the Lagrange multipliers for the jumps with the space $\mathbf{N} = \prod_{e=1}^E \mathbf{H}_{00}^{1/2}(\Gamma_e)$.

We propose to look for $\underline{u} = (u,\tau) \in \mathbf{V}$, $p = \{p^s\}_s \in \mathbf{M}$ and $\lambda = \{\lambda_e\}_e \in \mathbf{N}$ such that

$$
\begin{aligned}
&\sum_{s=1}^{S} (\nabla u^s, \nabla v^s)_{\Omega^s} + \sum_{e=1}^{E} \{[u]_{\Gamma_e}, [v]_{\Gamma_e}\}_{\Gamma_e} + \tau t \\
&- \sum_{s=1}^{S} (p^s, div(v^s))_{\Omega^s} - t \sum_{s=1}^{S} \int_{\Omega^s} p^s + \sum_{e=1}^{E} \{\lambda_e, [v]_{\Gamma_e}\}_{\Gamma_e} = \sum_{s=1}^{S} (f, v^s)_{\Omega^s}, \\
&\qquad - \sum_{s=1}^{S} (q^s, div(u^s))_{\Omega^s} - \tau \sum_{s=1}^{S} \int_{\Omega^s} q^s = 0, \\
&\qquad\qquad \sum_{e=1}^{E} \{\mu_e, [u]_{\Gamma_e}\}_{\Gamma_e} = 0
\end{aligned}
$$

for all $\underline{v} = (v,t) \in \mathbf{V}$, $q = \{q^s\}_s \in \mathbf{M}$ and $\mu = \{\mu_e\}_e \in \mathbf{N}$.

We see that we added the jumps to the elliptic terms and replaced the pairings $H_{00}^{-1/2}(\Gamma) - H_{00}^{1/2}(\Gamma)$ for the normal fluxes on the edges by the scalar product in $H_{00}^{1/2}(\Gamma)$. As a consequence, we have made a regularization of order 1 for the Lagrange multipliers and now all terms are suitable to compute in a Galerkin approach. Moreover, the solution to this problem is that of the incompressible Stokes equations on Ω.

Next, we elliminate via a standard Schur process the primal variables $\underline{u}$ and p in terms of the dual variable λ, and obtain a dual problem that once solved will give the correct boundary data for the primal variables. Thanks to the fact that the elliptic part is the scalar product on $\mathbf{V}$, that the inf-sup condition for the bilinear form b is achieved with velocities without jumps and that the inf-sup condition for c is achieved with velocities with jumps, our dual problem is a well posed symmetric positive definite problem.

2 Finite Dimensional Approach

We consider a conforming triangulation $\mathscr{T}_h$, h is the mesh size, of $\overline{\Omega}$ that contains the skeleton $\mathscr{E}_0$ as union of edges of triangles and such that on each edge only one partition is inherited from both sides. As $\mathscr{T}_h$ is also compatible with the subdivision of Ω, its restriction to each $\overline{\Omega_s}$ gives a mesh $\mathscr{T}_h^s$ on $\overline{\Omega^s}$. We use the Taylor-Hood finite element for the velocity and pressure pair on each subdomain. Define the family of subspaces $\{Y_h\}_h \subset H_0^1(\Omega)$ and $\{Q_h\}_h \subset H^1(\Omega)$ given by

$$
\begin{aligned}
Y_h &= \{v \in H_0^1(\Omega);\ v_{|_\kappa} \in \mathbb{P}_2(\kappa),\ \forall \kappa \in \mathscr{T}_h\}, \\
Q_h &= \{p \in H^1(\Omega);\ p_{|_\kappa} \in \mathbb{P}_1(\kappa),\ \forall \kappa \in \mathscr{T}_h\}
\end{aligned}
$$

where $\mathbb{P}_r(\kappa)$ is the space of polynomials of degree less or equal to r in the two variables x and y. On each subdomain, we take also

$$Y_h(\Omega^s) = Y_h \cap H^1(\Omega^s), \quad Q_h(\Omega^s) = Q_h \cap H^1(\Omega^s),\ s \le S.$$

Consider now $\mathbf{X}_h = X_h \times X_h$, where X_h is the broken version of Y_h given by

$$\begin{aligned} X_h = \{&v \in L^2(\Omega);\ v^s \in Y_h^s,\ \forall\, s = 1,2,\ldots,S, \\ &\text{and } v \text{ is continuous at every cross point in } \mathscr{C}\} \subset X, \end{aligned}$$

define $\mathbf{V}_h = \mathbf{X}_h \times \mathbb{R}$, $\mathbf{M}_h = \prod_{s=1}^{S} Q_h(\Omega^s)$ and finally $\mathbf{N}_h \subset \mathbf{N}$ is given by the restriction of functions in $\mathbf{X}_h$ to the skeleton $\mathscr{E}_0$.

The discrete uniform inf-sup condition for c on the pair $\mathbf{V}_h$ and $\mathbf{N}_h$ is by now a well known result and the discrete uniform inf-sup condition for b is a consequence of Theorem 1.12 pp. 130 in [4]. The idea is to use locally on each subdomain Ω^s the stability of the pair $\mathbb{P}_2 - \mathbb{P}_1$ and that of the pair $\mathbb{P}_2 - \mathbb{P}_0$ globally on the substructures Ω^s of Ω. This inf-sup condition is achieved with a discrete continuous function in the wohle of Ω and, as a consequence, the continuous setting is replicated and the equation for the multiplier can be solved via Conjugate Gradient Method (CG) without preconditioner. Then, we have

1. An external computational cicle, the CG for the Lagrange multiplier with a fixed number of iterations independent of the discretization parameter h and
2. At each iteration of this external cicle, the resolution of a primal problem of the form:
   Find $(\underline{w}_h, q_h) \in \mathbf{V}_h \times \mathbf{M}_h$ such that

$$\begin{aligned} (\underline{w}_h, \underline{v}_h)_{\mathbf{V}} + b(q_h, \underline{v}_h) &= (\xi, \underline{v}_h) \quad \forall \underline{v}_h \in \mathbf{V}_h, \\ b(p, \underline{w}_h) &= 0 \quad \forall p \in \mathbf{M}_h \end{aligned}$$

   where for the initial residuous r_0 we have $(\xi, \underline{v}_h) = \sum_{s=1}^{S} (f, v_h^s)_{\Omega^s}$ and for the iteration $m \ge 0$ we have $(\xi, \underline{v}_h) = \sum_{e=1}^{E} \{\{d_m\}_e, [v_h]_{\Gamma_e}\}_{\Gamma_e} = 0$

A closer inspection to the general form of this saddle point problem for the primal variables shows that the solution can be obtained by means of independent solves per subdomain. Ordering the unknows per subdomains, $x^s = (u^s, p^s)$ and $x^C = u^C$, the linear system for the primal variables is

$$\begin{pmatrix} M_{11} & M_{1,2} & \ldots & \ldots & \ldots & M_{1,S} & M_{1,C} & D_1 \\ M_{21} & M_{2,2} & M_{2,3} & \ldots & \ldots & \ldots & M_{2,C} & D_2 \\ M_{31} & M_{3,2} & M_{3,3} & M_{3,4} & \ldots & \ldots & M_{3,C} & D_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ldots & \ldots & \ldots & M_{S,S-1} & M_{S,S} & M_{S,C} & D_S \\ M_{1,C}^t & M_{2,C}^t & \ldots & \ldots & M_{S-1,C}^t & M_{S,C}^t & M_{C,C} & 0 \\ D_1^t & D_2^t & \ldots & \ldots & D_{S-1}^t & D_S^t & 0^t & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ \vdots \\ \vdots \\ x^S \\ x^C \\ \tau \end{pmatrix} = \begin{pmatrix} b^1 \\ b^2 \\ b^3 \\ \vdots \\ \vdots \\ b^S \\ b^C \\ 0 \end{pmatrix}$$

where the different blocks are of the form

$$M_{s,s} = \begin{pmatrix} A_{s,s} & B_{s,s} \\ B^t_{s,s} & 0 \end{pmatrix},\ M_{s,s'} = \begin{pmatrix} A_{s,s'} & 0 \\ 0 & 0 \end{pmatrix},\ M_{s,C} = \begin{pmatrix} A_{s,C} \\ B^t_{s,C} \end{pmatrix},\ M_{C,C} = A_{C,C}$$

here each block $M_{s,s}$ is similar to a standard Stokes matrix on the subdomain Ω^s, but with our interface contributions, each block $M_{s,s'}$ is sparse and contains the interaction through interfaces of the domain Ω^s with $\Omega^{s'}$, the rectangular blocks $M_{s,C}$ contains the interaction with the crosspoints and $M_{C,C}$ contains the interaction of the crosspoints with themselves. Although this linear system couples all the subdomains it can be solved by means of the Preconditioned Conjugate Gradient Method using as a preconditioner the matrix P formed by the main blocks

$$P = \begin{pmatrix} M_{11} & 0 & \dots & \dots & 0 & M_{1,C} & D_1 \\ 0 & M_{2,2} & 0 & \dots & 0 & M_{2,C} & D_2 \\ 0 & 0 & M_{3,3} & 0 & \ddots & M_{3,C} & D_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \dots & \dots & \dots & 0 & M_{S,S} & M_{S,C} & D_S \\ M^t_{1,C} & M^t_{2,C} & \dots & M^t_{S-1,C} & M^t_{S,C} & M_{C,C} & 0 \\ D^t_1 & D^t_2 & \dots & D^t_{S-1} & D^t_S & 0^t & 1 \end{pmatrix}.$$

Therefore, the main task here is the resolution of a linear system of the form $Px = b$ which is done using a Schur complement process for the variables x^C and τ. The equations are

$$(M_{C,C} - \sum_{s=1}^{S} M^t_{s,C} M^{-1}_{s,s} M_{s,C}) x^C - \sum_{s=1}^{S} M^t_{s,C} M^{-1}_{s,s} D_s \tau = b^C - \sum_{s=1}^{S} M^t_{s,C} M^{-1}_{s,s} b^s,$$

$$\sum_{s=1}^{S} D^t_s M^{-1}_{s,s} M_{s,C} x^C + (\sum_{s=1}^{S} D^t_s M^{-1}_{s,s} D_s - 1)\tau = \sum_{s=1}^{S} D^t_s M^{-1}_{s,s} b^s.$$

We finally write x^C in terms of τ and solve first for τ, next x^C and finally compute all the x^s. As a consequence, the main job is performed with independent solves of the matrices $M_{s,s}$ that can be performed independently, i.e., computations of the form

$$M^{-1}_{s,s} b^s, \qquad M^{-1}_{s,s} M_{s,C}, \qquad M^{-1}_{s,s} D_s.$$

3 Some Numerical Tests

For $L = 1, 2, 3, \dots$ integer we consider on $\Omega_L = [0, L] \times [0, 1]$ the exact solution

$$u(x,y) = \begin{pmatrix} -\sin^3(\pi x L^{-1}) \sin^2(\pi y) \cos(\pi y) \\ -L^{-1} \sin^2(\pi x L^{-1}) \sin^3(\pi y) \cos(\pi x L^{-1}) \end{pmatrix}, \quad p(x,y) = \frac{x^2}{L^2} - y^2$$

and partition Ω_L into $\Omega^s_L = (s-1, s) \times (0, 1)$ for $s = 1, 2, \dots, L$. For the dual problem we start our iteration process with $\lambda_{0,e} = 0$ on each Γ_e and stop all iterations according to a relative residual less than 10^{-6}. In this example the gradients control the jumps and there is no need to introduce them in the elliptic part; then the blocks $M_{s,t}$ are null for $s \neq t$. Then, there is no need for a PCG in the internal cycle. The following Table 1 shows that the iteration count for the dual problem is mesh independent on different configurations Table 2 shows relative errors with respect to the true solution

	h = 1/24	h = 1/48	h = 1/96
L = 4	17	17	17
L = 8	23	24	24
L = 16	37	39	39

Table 1. Mesh independent iteration count for the dual problem on different configurations and for different values of h on $\Omega_L = [0,L] \times [0,1]$. The number of subdomains is L given by $\Omega^s = [s-1,s] \times [0,1]$ for $s = 1,2,3,\ldots,L$

u and p on Ω_L Finally, we take on $\Omega = (0,1)^2$ the exact solution

eu(h)	h = 1/24	h = 1/48	h = 1/96
L = 4	2.1e−04	2.6e−05	3.5e−06
L = 8	1.8e−04	2.3e−05	3.0e−06
L = 16	1.7e−04	2.2e−05	2.9e−06

ep(h)	h = 1/24	h = 1/48	h = 1/96
L = 4	6.7e−04	1.6e−04	4.0e−05
L = 8	6.8e−04	1.6e−04	4.2e−05
L = 16	6.8e−04	1.7e−04	4.3e−05

Table 2. Relative errors in velocity field and pressure for different values of h on $\Omega_L = [0,L] \times [0,1]$ and with the same configuration as in Table 1

$$u(x,y) = \begin{pmatrix} -\sin^3(\pi x)\sin^2(\pi y)\cos(\pi y) \\ -\sin^2(\pi x)\sin^3(\pi y)\cos(\pi x) \end{pmatrix}, \quad p(x,y) = (x-0.25)^2(y-0.25)^2$$

and partition Ω into 4 equal subdomains with a cross point at $(0.5,0.5)$. Table 3 shows the results and we see that the number of iterations is independent of the mesh size again (Fig. 1).

AQ1

	Dual	Initial PCG	Final PCG		
h	# Iters	# Iters	# Iters	eu(h)	ep(h)
1/12	7	22	20	6.9e−4	4.2e−03
1/24	7	21	20	8.8e−5	1.0e−03
1/48	7	23	21	1.2e−5	2.5e−04
1/96	7	23	23	1.4e−6	8.3e−05

Table 3. Results obtained when subdividing the domain $\Omega = (0,1)^2$ into four subdomains with a cross point at $(0.5,0.5)$

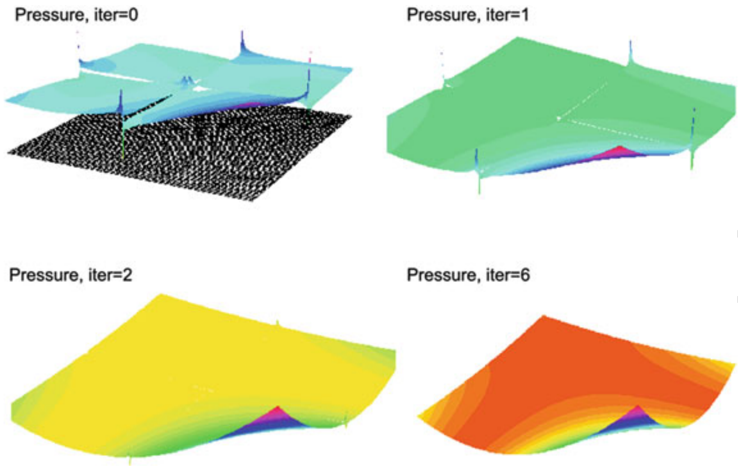

Fig. 1. Inital iteration with the underlying mesh and some contiguous iterations for the computed pressure

4 Conclusions

We presented a FETI-DP Mortar method applied to incompressible Stokes equations. Continuity at crosspoints is retained and the jumps across interfaces are included in the continuous formulation. The Lagrange multipliers are represented by their Riesz-canonical isometry, which improves their regularity from $H_{00}^{-1/2}(\Gamma)$ to $H_{00}^{1/2}(\Gamma)$, and the mortaring is performed using the $H_{00}^{1/2}(\Gamma)$ scalar product for each interface Γ. As a consequence, continuous bounds are replicated at the discrete level and no stabilization is required. In this setting we solve a dual problem by a CG that has a mesh independent condition number. The primal problems involved include the effect of the coupling between neighboring subdomains at interfaces and are solved by PCG. Still independent solves per subdomains are possible.

The advantage of the continuous framework introduced is the clear sight of the effect of condensing all information on subdomains and interfaces before the discrete work starts and the use of, to our belief, the most appropriated norms on subdomains and interfaces that make no necessary the use of mesh dependent norms for obtaining stability.

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