A Posteriori Error Estimates for a Neumann-Neuman Domain Decomposition Algorithm Applied to Contact Problems

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1 Introduction

Contact problems are frequent in structural analysis. They are characterized by inequality constraints such as non-interpenetration conditions, sign condition on the normal constraints, and an active contact, an area that is a priori unknown. Several approaches exist for solving the non linear equations issued from the finite element discretization of frictionless contact problems. Recently, many efficient error estimates for solving frictionless contact problems have been proposed, see for example [1] and with domain decomposition techniques combined with adapative finite element methods, see [8, 5].

In this work, we consider a natural Neuman-Neumann domain decomposition (NNDD) algorithm, in which each iterative step consists of a Dirichlet problem for the one body, a contact problem for the other one and two Neumann problems to coordinate contact stresses. Two main approximation errors are introduced by this algorithm: a discretization error due to the finite element method (FEM) and an algebraic error due to the NNDD algorithm.

In [5] an error estimator in the constitutive relation for contact problems solved by a Neumann-Dirichlet domain decomposition algorithm has been proposed. The objective of this paper is to extend this error estimator for a frictionless contact problem, solved by a NNDD algorithm and to present two errors indicators which allow us to estimate the part of the error due to the spatial discretization and the part of the error due to the domain decomposition algorithm. Numerical results are presented, showing the pratical efficiency of the proposed error estimators.

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2 A contact problem, notations and conventions

Two plane bounded domains Ω_1 and Ω_2 representing two linear elastic bodies are considered. Their Lipschitz boundaries are composed of distinct parts Γ_D^α, Γ_N^α and Γ_C^α :

$$\partial\Omega_\alpha = \overline{\Gamma_D^\alpha} \cup \overline{\Gamma_N^\alpha} \cup \overline{\Gamma_C^\alpha} \quad \alpha = 1,2.$$

The indices D, N, C of the boundary parts indicate respectively Dirichlet, Neumann and contact imposed boundary conditions, see problem (2)–(5). For the sake of simplicity, we suppose that $\Gamma_C^1 = \Gamma_C^2 = \partial\Omega_1 \cap \partial\Omega_2 = \Gamma_C$ is a common part of $\partial\Omega_\alpha$ along which the bodies Ω_α are in unilateral contact. On the presumed contact boundary Γ_C, we define

$$\mathbf{n} = \quad \mathbf{n}^1 = -\mathbf{n}^2 \quad \text{and} \quad \mathbf{t} = \quad \mathbf{t}^1 = -\mathbf{t}^2,$$

where $\mathbf{n}^\alpha$ and $\mathbf{t}^\alpha$ denote, respectively, the unit external normal and tangential vectors to $\partial\Omega_\alpha$.

On each domain Ω_α, $\alpha = 1,2$, the stress tensor is $\overline{\sigma}^\alpha$ and $\overline{\mathscr{E}}(\mathbf{u}^\alpha)$ is the linearized strain tensor associated with the displacement $\mathbf{u}^\alpha$. With the elasticity tensors $\mathbb{E}^\alpha$, characterizing the materials of Ω_α, we have the linear strain-stress relation :

$$\overline{\sigma}^\alpha = \mathbb{E}^\alpha \overline{\mathscr{E}}(\mathbf{u}^\alpha). \tag{1}$$

The bilinear energy forms, of linear elastic deformation, are then defined as

$$a^\alpha(\mathbf{u}^\alpha, \mathbf{u}^*) = \int_{\Omega^\alpha} \overline{\sigma}^\alpha : \overline{\mathscr{E}}(\mathbf{u}^*).$$

The external loads (surfacic tractions of density $\mathbf{F}_\alpha$ on Γ_N^α) are represented, in their weak form, as the linear forms b^α:

$$b^\alpha(\mathbf{u}^*) = \int_{\Gamma_N^\alpha} \mathbf{F}_\alpha . \mathbf{u}^*.$$

3 Unilateral Contact problem and 'Neumann-Neumann' domain decomposition algorithm (NNDD)

We consider a unilateral frictionless contact problem between Ω_1 and With volumic forces neglected and tractions of density $\mathbf{F}^\alpha$ imposed on Γ_N^α, the equilibrium equations can be written for $\alpha = 1,2$:

$$\mathrm{div}\overline{\sigma}^\alpha = 0 \quad \text{in } \Omega^\alpha \tag{2}$$

$$\overline{\sigma}^\alpha . \mathbf{n}^\alpha = \mathbf{F}^\alpha \quad \text{on } \Gamma_N^\alpha \tag{3}$$

with the kinematic boundary condition and unilateral frictionless contact conditions :

$$\mathbf{u}^\alpha = \mathbf{u}_D^\alpha \text{ on } \Gamma_D^\alpha \tag{4}$$

$$\left.\begin{array}{ll} (\mathbf{u}^1-\mathbf{u}^2).\mathbf{n} & \leq 0 \\ \sigma_{TN}^1 = \sigma_{TN}^2 & = 0 \\ \sigma_{NN}^1 = \sigma_{NN}^2 & = \sigma_N \\ \sigma_N & \leq 0 \\ \sigma_N.(\mathbf{u}^1-\mathbf{u}^2).\mathbf{n} & = 0 \end{array}\right\} \text{ on } \Gamma_C \tag{5}$$

with

$$\sigma_{NN}^\alpha = \mathbf{n}^\alpha.\overline{\sigma}^\alpha\mathbf{n}^\alpha \tag{6}$$
$$\sigma_{NT}^\alpha = \mathbf{t}^\alpha.\overline{\sigma}^\alpha\mathbf{t}^\alpha. \tag{7}$$

We now define a Neumann-Neumann domain decomposition (NNDD) algorithm. First, for any given normal displacement λ_p on Γ_C, we define the functional spaces

$$\begin{array}{ll} \mathbf{V}^1 & = \{\mathbf{u}\in\mathbf{H}^1(\Omega^1);\mathbf{u}|_{\Gamma_D^1}=\mathbf{u}_D^1\} \\ \mathbf{U}_C^1(\lambda_p) & = \{\mathbf{u}\in\mathbf{V}^1;\mathbf{u}|_{\Gamma_C^1}.\mathbf{n}=\lambda_p\} \\ \mathbf{V}^2 & = \{\mathbf{u}\in\mathbf{H}^2(\Omega^2);\mathbf{u}|_{\Gamma_D}=\mathbf{u}_D^2\} \\ \mathbf{K}_C^2(\lambda_p) & = \{\mathbf{u}\in\mathbf{V}^2;\mathbf{u}|_{\Gamma_C^2}.\mathbf{n}\geq\lambda_p\}. \end{array}$$

Given a non-negative parameter θ and an initial arbitrary λ_1, we define two sequences of displacements $\mathbf{u}_p^\alpha$ on each solid Ω^α, $\alpha = 1,2$. Each iteration p of the NNDD algorithm is divided in two successive steps.

- Step 1 – Two independent elasticity problems (hence parallelizable) are solved on Ω_1 and Ω_2:

  1. In Ω_1, the variational problem writes

$$\left\{\begin{array}{l} \text{Find } \mathbf{u}_p^1\in\mathbf{U}_C^1(\lambda_p) \text{ such that} \\ a^1(\mathbf{u}_p^1,\mathbf{u}^*-\mathbf{u}_p^1) = b^1(\mathbf{u}^*-\mathbf{u}_p^1) \quad \forall\mathbf{u}^*\in\mathbf{U}_C^1(\lambda_p) \end{array}\right. \tag{8}$$

  2. In Ω_2, with the given λ_p normal displacement defined on Γ_C, we solve the following variational problem corresponding to a unilateral frictionless contact problem on Γ_C^2 :

$$\left\{\begin{array}{l} \text{Find } \mathbf{u}_p^2\in\mathbf{K}_C^2(\lambda_p) \text{ such that} \\ a^2(\mathbf{u}_p^2,\mathbf{u}^*-\mathbf{u}_p^2) \geq b^2(\mathbf{u}^*-\mathbf{u}_p^2) \quad \forall\mathbf{u}^*\in\mathbf{K}_C^2(\lambda_p) \end{array}\right. \tag{9}$$

     From the respective unique solutions $\mathbf{u}_p^1$ and $\mathbf{u}_p^2$ of (8) and (9) we deduce $\mathbf{r}_p^1$ and $\mathbf{r}_p^2$, defined on the contact Γ_C as

$$\mathbf{r}_p^1 = \overline{\sigma}_p^1 \mathbf{n}^1$$

$$\mathbf{r}_p^2 = \overline{\sigma}_p^2 \mathbf{n}^2.$$

where $\overline{\sigma}_p^1$ and $\overline{\sigma}_p^2$ are the stress tensor associated with the respective solutions $\mathbf{u}_p^1$ and of $\mathbf{u}_p^2$ of problems (8) and (9).

- Step 2 – With $\mathbf{r}_p^1$ and $\mathbf{r}_p^2$ obtained in step 1, we solve two independent "Neumann type" problems (hence the name NNDD):
  In Ω_1, we solve

$$\left\{ \begin{array}{l} \text{Find } \mathbf{w}_p^1 \in \mathbf{V}^1 \text{ such that} \\ a^1(\mathbf{w}_p^1, \mathbf{u}^* - \mathbf{w}_p^1) = -\int_{\Gamma_C} \frac{1}{2}(\mathbf{r}_p^1 + \mathbf{r}_p^2).(\mathbf{u}^* - \mathbf{w}_p^1) \quad \forall \mathbf{u}^* \in \mathbf{V}^1. \end{array} \right. \tag{10}$$

  In Ω_2, we solve

$$\left\{ \begin{array}{l} \text{Find } \mathbf{w}_p^2 \in \mathbf{V}^2 \text{ such that} \\ a^2(\mathbf{w}_p^2, \mathbf{u}^* - \mathbf{w}_p^2) = \quad \int_{\Gamma_C} \frac{1}{2}(\mathbf{r}_p^1 + \mathbf{r}_p^2).(\mathbf{u}^* - \mathbf{w}_p^2) \quad \forall \mathbf{u}^* \in \mathbf{V}^2. \end{array} \right. \tag{11}$$

  Let ε_t be the precision of the algorithm, we have the alternative :

  1. If ε_τ is small enough, the algorithm stops.
  2. Else, the normal displacement λ_p is updated :

$$\lambda_{p+1} := \lambda_p + \theta(\mathbf{w}_p^1 - \mathbf{w}_p^2).\mathbf{n}$$

  and we return to step 1 for iteration $p+1$.

If $\mathbf{r}_p^1 + \mathbf{r}_p^2 = 0$, it means that the equilibrium is satisfied on the contact interface, in other words the solutions $\mathbf{u}_p^1$ and $\mathbf{u}_p^2$ of step 1 constitute the unique solution of the reference problem (2)–(5). The proof of convergence of the NNDD algorithm (8)–(11) is given in [6] for any sufficiently small $\theta > 0$:

Theorem 1. *There is a $\theta_0 > 0$ such that for any $0 < \theta \leq \theta_0$, the NNDD algorithm for unilateral frictionless contact converges.*

4 Error estimates

The NNDD algorithm introduces two error sources. The first one is introduced by the solution of the FE problems (8)–(9). The second is introduced by the iterative NNDD algorithm. The global error is defined as the difference between the solution of the weak form of the reference problem $\mathbf{u}^\alpha$ and the finite element solution computed from the NNDD algorithm $\mathbf{u}_h^\alpha$. Let

$$e_h = \sqrt{\sum_{\alpha=1}^{2} \|\mathbf{u}^\alpha - \mathbf{u}_h^\alpha\|_{u,\Omega^\alpha}^2} \text{ where } \|\mathbf{u}\|_{u,\Omega^\alpha}^2 = \int_{\Omega^\alpha} \mathbb{E}^\alpha \overline{\mathscr{E}}(\mathbf{u}) \,.\, \overline{\mathscr{E}}(\mathbf{u})\, d\Omega^\alpha$$

In the next section, we will define an a posteriori global error estimator, which is an adaptation to the NNDD algorithm of the error estimator proposed in [5], [4]. Moreover, we propose here two error indicators that allow us to estimate separately the part of the error due to the FE discretization and that due to the NNDD algorithm.

4.1 Global error estimator

The global error estimator is based on the concept of error in the constitutive relation [7]. Let us consider kinematically admissible displacements, i.e those satisfying (4), $\hat{\mathbf{v}} = (\mathbf{v}^1, \mathbf{v}^2, v_N)$ and statically admissible stress tensor fields $\hat{c} = (\overline{\tau}^1, \overline{\tau}^2, \mathbf{t_c})$, i.e. those satisfying (5), where on Γ_c, with $\mathbf{w}^\alpha = \mathbf{v}^\alpha|_{\Gamma_c}$:

$$\mathbf{w}_c = \mathbf{w}^1 - \mathbf{w}^2, \text{ and } \mathbf{t_c} = \overline{\tau}^\alpha \mathbf{n}^\alpha.$$

We define a global error estimator for any admissible $\hat{s} = (\hat{c}, \hat{\mathbf{v}})$:

$$e_{CRE}(\hat{s}) = \left[\sum_{\alpha=1}^{2} \| \overline{\tau}^\alpha - \mathbb{E}^\alpha \overline{\mathscr{E}}(\mathbf{v}^\alpha) \|^2_{\overline{\tau},\Omega^\alpha} + 2 \int_{\Gamma_c} [\phi(-\mathbf{w}_c) + \phi^*(\mathbf{t_c}) + \mathbf{w}_c.\mathbf{t}_c] \, dS \right]^{1/2},$$

with

$$\| \overline{\tau}^\alpha \|^2_{\overline{\tau},\Omega^\alpha} = \int_{\Omega^\alpha} \overline{\tau}^\alpha : (\mathbb{E}^\alpha)^{-1} (\overline{\tau}^\alpha),$$

and where ϕ and ϕ^* are the conjugate convex potentials introduced in [2] to model the Coulomb's contstitutive law in a frictionless case:

$$\phi(\mathbf{v}) = \begin{cases} 0 & \text{if } v_N \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$
$$\phi^*(\mathbf{t}) = \begin{cases} 0 & \text{if } t_N \leq 0 \text{ and } t_T = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

where the indices N and T indicate respectively the normal and the tangential component.
From [2, 3] the unilateral frictionless contact condition is equivalent to

$$\phi(-\mathbf{w}_c) + \phi^*(\mathbf{t_c}) + \mathbf{w}_c.\mathbf{t}_c = 0 \text{ on } \Gamma_C. \tag{12}$$

$e_{CRE}(\hat{s})$ is the constitutive relation error estimator for the admissible solution $\hat{s}$. It is equal to zero if and only if $\hat{s}$ is the exact solution of the unilateral frictionless contact problem (5)–(2). From [1], we have the upper bound,

$$e_{CRE}(\hat{s}) \geq e_h = \sqrt{\sum_{\alpha=1}^{2} \| \mathbf{u}_h^\alpha - \mathbf{u}^\alpha \|^2_{u,\Omega^\alpha}}.$$

4.2 Error indicators

The discretization error is estimated through a discretization error indicator computed for a second reference problem defined by (8)–(9) for a given λ_p. The only approximation used to solve this problem is the Finite Element approximation.

Let $\hat{s}_p = (\hat{u}_p, \hat{c}_p)$ be an admissible pair for this new reference problem, then the *discretization error* indicator is defined by

$$\eta_{h,p}^{dis} = e_{CRE}(\hat{s}_p).$$

To define an algorithm error indicator, we consider a third reference problem obtained with the Finite Element discretization of equations (2)–(5) (It is also necessary to introduce a discretized contact constitutive relations), the only approximation used to solve this problem is the Neuman-Neuman domain decomposition algorithm. Let $\hat{s}_h = (\hat{u}_h, \hat{c}_h)$ be an admissible pair for this third reference problem, then the *algorithm error* indicator is defined by

$$\eta_h^{NNDD} = e_{CRE}(\hat{s}_h)$$

To build the admissible fields $\hat{s}_p$ and $\hat{s}_h$, we use an adaptation of the techniques developed in [5].

5 Numerical results

We consider a test problem illustrating the reference problem (2)–(5). The domain Ω^1 is subject to a non-zero imposed displacement on a part Γ_D^1 of its boundary and to a rigid frictionless contact on another part $\Gamma_{D'}^1$. The domain Ω^2 has zero displacement imposed on Γ_D^2. Some surface forces $\mathbf{F}_N$ are imposed on Γ_N^1 to illustrate some loss of contact at the interface, see Figure 1. The two domains are in contact on Γ_C.

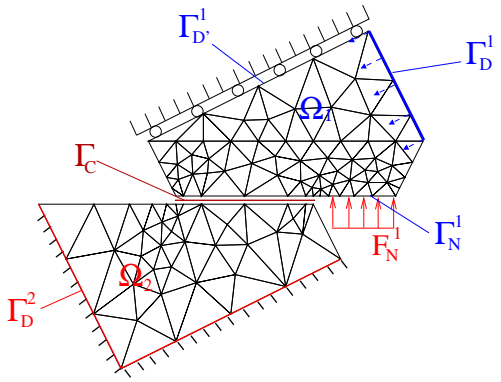

Fig. 1 A test problem for NNDD algorithm: frictionless unilateral contact between 2 elastic bodies.

In our implementation of the NNDD Algorithm, we define the precision of the algorithm ε_t as

$$\varepsilon_\tau = \frac{2\max_{\Gamma_C}|\mathbf{r}_p^1 + \mathbf{r}_p^2|}{\max_{\Gamma_C}|\mathbf{r}_p^1| + \max_{\Gamma_C}|\mathbf{r}_p^2|}$$

where $\mathbf{r}_p^1$ and $\mathbf{r}_p^2$ are obtained from step 1 of the NNDD algorithm at iteration p.

We first test the a posteriori error estimates of the NNDD algorithm (8)–(11) for different values of θ, and two meshes, one coarse mesh with 380 nodes and one finer mesh with 5994 nordes, see Figure 2. For both meshes, we notice an apparently optimal value near $0.4 \leq \theta \leq 0.5$ after 3 iterations of the NNDD algorithm. We also remark that the algorithm errors are very similar for both the fine and coarse meshes. The discretisation errors are naturally greater for the coarse mesh, but it doesn't change much with θ.

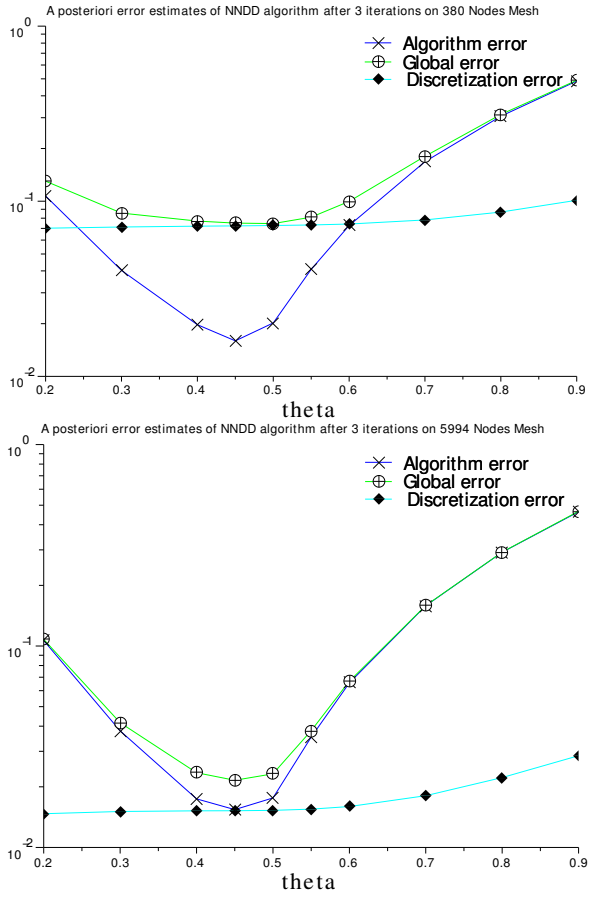

Fig. 2 NN error indicators for different values of θ, coarse 380 nodes mesh (up) and finer 5994 nodes mesh (down) after 3 iterations.

In Figure 3, we show the evolution of the algorithm error and the precision ε_t for an increasing number of iterations for a fixed value $\theta = 0.4$ and a fixed coarse mesh (380 nodes). While both decrease towards zero, the slopes of each appear very different. It means that the precision ε_t may not be a very good stopping criterion and can be deceiving as it appears much smaller than the algorithm error, which constitutes the largest part of the global error when using finer mesh, see the previuos figure 2.

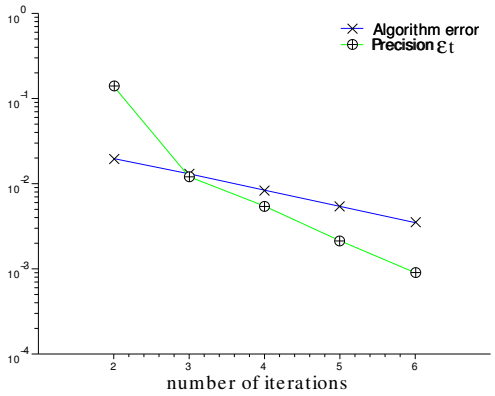

Fig. 3 Algorithm error and precision ε_t on fixed coarse mesh per number of iterations, with $\theta = 0.4$

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