A smooth transition approach between the Vlasov-Poisson and the Euler-Poisson system

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1 Introduction

Plasma dynamics is characterized by a wide range of spatial and temporal scales. Typical examples include plasmas produced around hypersonic bodies, ion wind of corona discharges, magnetic fusion processes. Depending on conditions, kinetic models of Boltzmann type or macroscopic models are commonly used for plasma physics simulations. The most common kinetic model for plasmas is the Vlasov equation, coupled with the electromagnetic field equations. On the other hand, Euler or Navier-Stokes based models coupled with the Maxwell equations are used for describing equilibrium plasma flows. Even if fluid models are sufficiently accurate to describe many observed phenomena, however, for some of them, this choice is inadequate. In these cases, it turns out that a kinetic description is strictly necessary to correctly represent the solutions. In these circumstances, the most widely used numerical methods for solving the Vlasov equation are Particle-In-Cell (PIC) approaches [1]. They have many advantages in terms of computational cost for large dimensional problems, for enforcing physical properties such as conservation laws and in terms of flexibility when handling with complex geometries. On the other hand, these methods involve a significant level of numerical noise and the convergence rate is in general quite slow. Moreover, in situations close to thermodynamical equilibrium, the cost of PIC methods or, more in general, direct Monte Carlo simulations increases. For this reason, domain decomposition techniques have been proposed in the recent past (see [2, 4, 5, 6, 8, 9]). Indeed, in many situations, the resolution of the kinetic equations in the whole computational domain is unnecessary because the fluid equations coupled with suitable equations for the electromagnetic fields provide a sufficiently accurate solution, except in small zones like shock layers or extremely rarefied regions where departure from thermodynamical equilibrium is strong.

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In this paper, we focus on an adaptive kinetic-fluid approach which incorporates kinetic phenomena in selected regions of phase space where they play a fundamental role. More in detail, we propose a numerical method for the resolution of the collisional Vlasov-Poisson equation coupled with the compressible Euler-Poisson equations through a domain decomposition technique.

The present paper represents an extension of two our earlier works [2, 4], in which we coupled the BGK equation and the compressible Euler equations. The key point on which the method relies is the introduction of a buffer zone in which the transition from the Vlasov-BGK-Poisson equations and the Euler-Poisson equations and vice-versa is gradual. Therefore, in the buffer zone, both models are solved and the solution of the full problem is obtained as the combination of the kinetic and fluid solutions. The introduction of the intermediate zone makes each of the models degenerate at the interfaces. In this way, no interface condition is needed. Finally, in this work we consider a constant in time coupling function and refer to [7] for the time dependent case.

2 The Vlasov-BGK-Poisson equation

We consider the collisional Vlasov equation for describing the ions evolution in a plasmas. In this work we assume that the electrons form a uniform neutralizing background. The binary interactions between particles are substituted by relaxation towards the equilibrium. The rescaled equation reads

$$\partial_t f + v \cdot \nabla_x f + \mathrm{E} \cdot \nabla_v f = \frac{1}{\tau}(M_f - f), \tag{1}$$

with the initial condition

$$f(x,v,t=0) = f_0(x,v), \tag{2}$$

where $f = f(x,v,t)$ is a non negative function describing the time evolution of the distribution of particles which move with velocity $v \in \mathbb{R}^d$ in the position $x \in \Omega \subset \mathbb{R}^d$ at time $t > 0$. In the general case, the relaxation time τ is a function of the macroscopic quantities. For our scopes, in the present paper, the relaxation frequency will be fixed and given at the beginning of the simulations. We refer to [7] for more physical cases. The electric field E is given as a gradient of a potential function $\mathrm{E} = \nabla_x \Phi$, where Φ is obtained from the solution of the Poisson equation

$$\lambda^2 \Delta \Phi = \int_{\mathbb{R}^d} f dv - \rho_0, \tag{3}$$

with λ the so called Debye length and ρ_0 the background electrons density. The local thermodynamical equilibrium is defined by

$$M_f = M_f[\rho,u,T](v) = \frac{\rho}{(2\pi\theta)^{d/2}} \exp\left(\frac{-|u-v|^2}{2\theta}\right), \tag{4}$$

where ρ and u are the density and mean velocity while $\theta = RT$ with T the temperature of the ions and R the gas constant.

Formally as $\varepsilon \to 0$ the function f tends to the local Maxwellian. In this limit, multiplying the Vlasov-BGK equation (1) by 1, v, $\frac{1}{2}|v^2|$ (the so-called collision invariants), and integrating with respect to v, leads to the following system of balance laws

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho u) = 0, \tag{5}$$

$$\frac{\partial \rho u}{\partial t} + \nabla_x \cdot (\rho u \otimes u + pI) - \rho \mathrm{E} = 0, \tag{6}$$

$$\frac{\partial \mathscr{E}}{\partial t} + \nabla_x \cdot ((\mathscr{E} + p)u) - \rho u E = 0, \tag{7}$$

$$p = \rho\theta, \quad \mathscr{E} = \frac{d}{2}\rho\theta + \frac{1}{2}\rho|u|^2. \tag{8}$$

where p is the pressure and $\mathscr{E}$ the total energy.

3 The coupling method

In this section we present the coupling strategy between the Vlasov-BGK-Poisson equations and the Euler-Poisson system. We will deal here with a constant in time coupling between the micro and macroscopic models. However, our final scope is to derive a time dependent coupling strategy, we refer to [7] for this case. We also refer to [7] for more details on the numerical discretization, the treatment of the boundary conditions and for the general theory about the time-dependent case.

3.1 Decomposition of the kinetic equation

The coupling strategy is inspired by two recent works ([2, 4]) in which the rarefied gas dynamic case was considered. For sake of simplicity we describe the method in one space and velocity dimensions. It can be easily extended to a generic N-dimensional setting. Also different meshes for the cut-off function and for the other variables can be used.

We denote the buffer interval by $[a,b]$, and we introduce a cut-off function $h(x)$ such that

$$h(x) = \begin{cases} 1, & \text{for } x \le a \\ 0, & \text{for } x \ge b \\ 0 \le h(x) \le 1, & \text{for } x \in [a,b] \end{cases} \tag{9}$$

For instance, h can be chosen piecewise linear in $[a,b]$:

$$h(x) = \frac{x-b}{a-b} \quad \text{for} \quad x \in [a,b].$$

We define two distribution functions such that $f_R = hf$ while $f_L = (1-h)f$. We look now for an evolution equation for f_R and for f_L. We write

$$\partial_t f_R = \partial_t(hf) = h\partial_t f,$$
$$\partial_t f_L = \partial_t((1-h)f) = (1-h)\partial_t f.$$

Thus multiplying the Vlasov-BGK equation (1) by h and $1-h$ respectively, we obtain the following equations for the time evolution of the distributions f_R and f_L

$$\partial_t f_R = h\Big(-v\partial_x f - \mathrm{E}\cdot\nabla_v f + \frac{1}{\tau}(M_f - f)\Big),$$
$$\partial_t f_L = (1-h)\Big(-v\partial_x f - \mathrm{E}\cdot\nabla_v f + \frac{1}{\tau}(M_f - f)\Big),$$

which finally leads to the following system

$$\partial_t f_R + hv\partial_x f_R + hv\partial_x f_L + \mathrm{E}\partial_v f_R = \frac{h}{\tau}(M_f - f), \tag{10}$$
$$\partial_t f_L + (1-h)v\partial_x f_L + (1-h)v\partial_x f_R + \mathrm{E}\partial_v f_L = \frac{1-h}{\tau}(M_f - f), \tag{11}$$
$$f = f_R + f_L, \tag{12}$$

with initial data

$$f_R(x,v,0) = h(x,0)f(x,v,0)\,, \quad f_L(x,v,0) = (1-h(x,0))f(x,v,0). \tag{13}$$

It is important to note that if $f = f_L + f_R$ is the solution of (1) with initial data (2), then (f_L, f_R) is the solution of (10-11) with initial data (13) and conversely.

3.2 Kinetic-Hydrodynamic coupling

Now, let us assume that the domain can be subdivided into two regions: in one of the regions, the distribution function is close to a local Maxwellian while in the other, it is far from it. We choose to set $h = 0$ in the region where f is close to the Maxwellian. Therefore, f_L is close to its associated Maxwellian M_{f_L} and we can replace the Vlasov-BGK equation (1) by its macroscopic limit equations without making any significant error. We also suppose that in the buffer zone, f_L remains close to the equilibrium and thus, it can be replaced by M_{f_L} in the whole interval $x < b$.

Replacing f_L by M_{f_L} in (11) and taking the hydrodynamic moments (mass, momentum and energy), leads to the following modified Euler system defined in the interval $x \leq b$

$$\begin{aligned}
&\frac{\partial \rho_L}{\partial t} + (1-h)\partial_x(\rho_L u_L) = -(1-h)\partial_x\left(\int_{\mathbb{R}} v f_R\, dv\right),\\
&\frac{\partial \rho_L u_L}{\partial t} + (1-h)\partial_x(\rho_L u_L^2 + p_L) - \mathrm{E}\rho_L = -(1-h)\partial_x\left(\int_{\mathbb{R}} v^2 f_R\, dv\right),\\
&\frac{\partial \mathscr{E}_L}{\partial t} + (1-h)\partial_x((\mathscr{E}_L + p_L)u_L) - \rho_L u_L \mathrm{E} = -(1-h)\partial_x\left(\int_{\mathbb{R}} v\frac{|v|^2}{2} f_R\, dv\right),
\end{aligned} \tag{14}$$

with initial data

$$(\rho_L, u_L, \theta_L)|_{(x,0)} = (1-h|_{(x,0)})(\rho, u, \theta)|_{(x,0)}.$$

Under these assumptions, we have $f = f_R + M_{f_L}$, where f_R is a solution of:

$$\partial_t f_R + hv\partial_x f_R + hv\partial_x M_{f_L} + \mathrm{E}\partial_v f_R = \frac{h}{\tau}(M_f - f), \tag{15}$$

in the interval $x \geq a$. Thus, the coupling model consists of system (14) for the hydrodynamic moments in the region $x \leq b$ and of equation (15) for the kinetic distribution function in the region $x \geq a$.

When $h = 0$, system (14) coincides with system (8) because $f_R = 0$ and $f_L = M_{f_L}$. Moreover no boundary conditions are needed at the boundary $x = b$ because the spatial derivatives are degenerate at $x = b$ for the fluid model. A similar remark is true for f_R. Indeed, when $h = 0$, $f_R = 0$ and no boundary conditions are needed for the kinetic equation at $x = a$ because the spatial derivatives are degenerate in equation (15). In the buffer zone $[a,b]$, the solution of the full kinetic problem f is computed as the sum of the Maxwellian M_{f_L} and of the function f_R. To summarize, the solution of the full kinetic problem is given by f_R if $x > b$, by M_{f_L} if $x < a$ and by $M_{f_L} + f_R$ if $x \in [a,b]$.

An important feature of the method is that it is very easy to divide the domain in more than two zones. Thus we can define as many buffers and as many kinetic regions as necessary if the macroscopic model fails to give the correct solution in different parts of the domain which are far apart from each other. In this latter case, the function h is still a piecewise linear function but there are multiple buffer zones $[a_j, b_j]$. Additionally, we can create new buffer zones and new kinetic zones during the simulation. Such strategy is presented in [4] for the Boltzmann-BGK and in [7] for the Vlasov-Poisson equations.

4 Numerical test

The numerical example we present is a one-dimensional plasma expansion problem. This is a two-species problem composed by free ions and fixed electrons. Ions initially occupy a small region of thickness D of the space where they have an high density while in the rest of the domain they have very small density. Background electrons are initialized by a Maxwell-Boltzmann equilibrium with a self-consistent

potential and their density is constant everywhere. The test problem consists in observing the expansion of the ions.

This kind of phenomena are well described by the Vlasov-Poisson system in rarefied regions and by the Euler-Poisson system in dense regions. For this test case, we consider all our equation in their adimensional form. for dimensional test cases we refer to [7].

The numerical physical domain goes from the left boundary $x_L = -20$ to the right boundary at $x_R = 20$ while the velocity domain goes from $v_{\min} = -200$ to $v_{\max} = 200$. There are 1000 cells in physical space and 140 cells in velocity space. The slab where ions are initialized with high density is $[x_L, D]$ with $D = -8$. Initial conditions are as follows: for ions, the density is $\rho = 1$, mean velocity $u = 0$ and temperature $T = 10$ in the high-density slab $[x_L, D]$ while in the remaining part of the domain the density is $\rho = 5 \times 10^{-2}$, mean velocity $u = 0$ and temperature $T = 8$. Electrons are initialized with density $\rho_0 = 1$ everywhere.

The collision frequency is given by $1/\tau$ where $\tau = 5\ 10^{-6}$ in the hydrodynamic part and $\tau = 10^{-1}$ in the kinetic part. The Debye length takes the value $\lambda^2 = 10^{-2}$ and Dirichlet boundary conditions are imposed for the electric potential as $\Phi(x_L) = 0$ and $\Phi(x_R) = 10.0$.

The cut-off function h is initialized as $h = 0$ for x ranging from -20 to $a = -1.0$ (fluid region), $h = \frac{x-a}{b-a}$ with $b = 0.5$ (buffer zone) and $h = 1$ for $x > b$ (kinetic region).

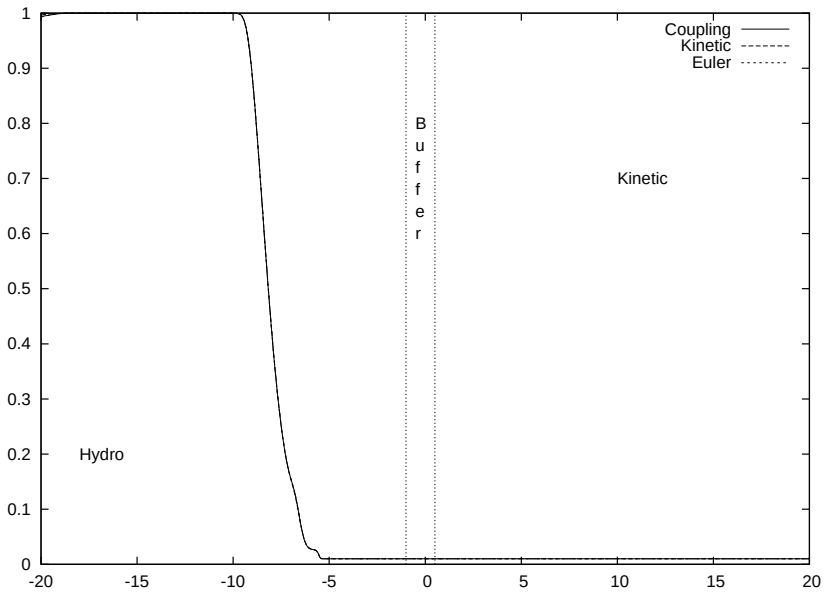

Fig. 1 Density profile for the ions at time $t = 3.6 \times 10^{-3}$. Continuous line coupling method, dashed line Vlasov-BGK-Poisson model, dotted line Euler-Poisson system.

Boundary conditions are treated as a constant incoming maxwellian injection at the left boundary and as free Neumann conditions at the right one.

When the simulation begins, ions start to expand. We plot the solution for the density after few time steps $t_i = 3.6 \times 10^{-3}$ s (Fig. 1), at an intermediate state before reaching the buffer zone at $t_m = 8.4 \times 10^{-3}$ s (Fig. 2) and at the end of the simulation at $t_f = 1.32 \times 10^{-2}$ s (Fig. 3). In the figures we report the results of the domain decomposition strategy together with the results obtained by employing a scheme which solves the Vlasov-BGK-Poisson equation everywhere. We also report the results obtained by solving the Euler-Poisson system in all the domain.

The figures show that during all the simulation the coupling strategy is able to capture the good solution which is the one furnished by solving the kinetic equation everywhere, while the scheme which solves the Euler-Poisson system fails in describing the good solution where the collision frequency is very small. For more realistic problems, it will be necessary to follow the discontinuities and the regions where the rarefaction is high during the time evolution of the problem. This can be accomplished by constructing some adaptive and dynamic decomposition which are the subject of a future work [7].

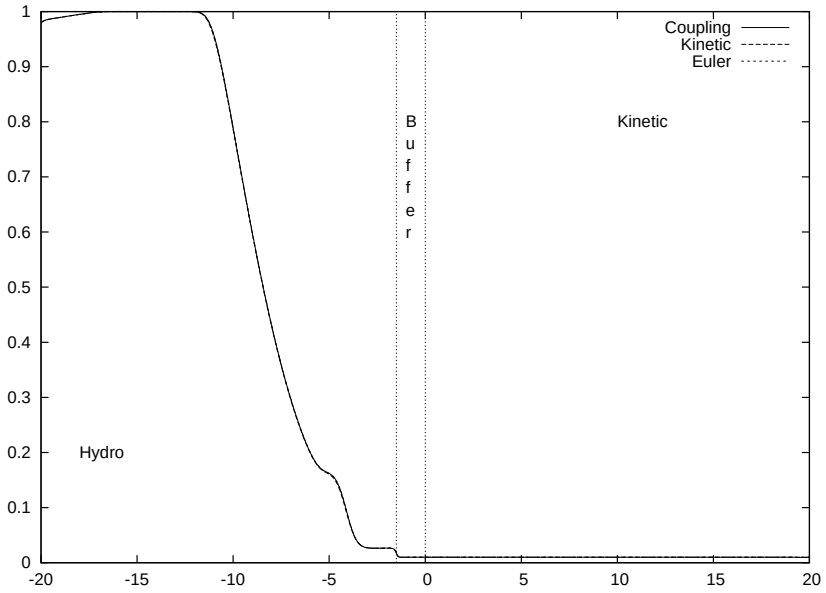

Fig. 2 Density profile for the ions at time $t = 8.4 \times 10^{-3}$. Continuous line coupling method, dashed line Vlasov-BGK-Poisson model, dotted line Euler-Poisson system.

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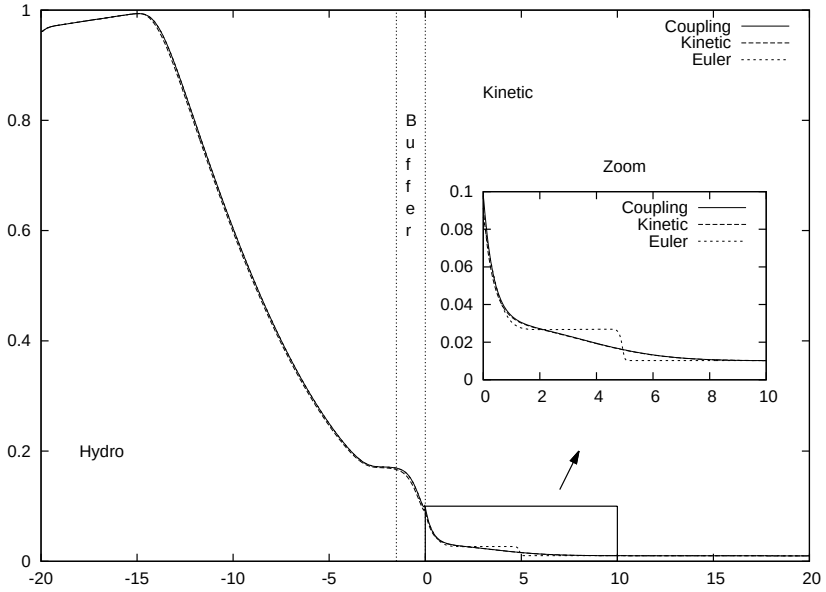

Fig. 3 Density profile for the ions at time $t = 1.32 \times 10^{-2}$. Continuous line coupling method, dashed line Vlasov-BGK-Poisson model, dotted line Euler-Poisson system.