A finite volume Ventcell-Schwarz algorithm for advection-diffusion equations

Laurence Halpern[1] and Florence Hubert[2]

1 Introduction

Consider a two-dimensional domain Ω, and the boundary value problem

$$\mathscr{L}u := -\mathrm{div}(\nu(x)\nabla u) + \mathrm{div}(\mathbf{b}(x)u) + \eta(x)u = f, \tag{1}$$

with homogeneous boundary condition $u = 0$ on the boundary $\partial\Omega$. The Ventcell-Schwarz iterative method has been introduced in [9] for the resolution of (1) in parallel. A nonoverlapping decomposition of Ω into two subdomains Ω_j is given, with common boundary Γ. The algorithm defines a sequence of solutions u_j^n of equation (1) in Ω_j, related by two transmission conditions, for $(i,j) = (1,2)$ or $(2,1)$:

$$(\nu\partial_{n_j} - \frac{1}{2}\mathbf{b}\cdot\mathbf{n}_j + \Lambda)\,u_j^n = (-\nu\partial_{n_i} + \frac{1}{2}\mathbf{b}\cdot\mathbf{n}_i + \Lambda)\,u_i^{n-1} \text{ on } \Gamma.$$

The boundary operator Λ involves second order derivatives along the boundary. In the case where Γ is a vertical line, it can be written as $\Lambda\phi = p\phi - q\partial_y(\nu\partial_y\phi)$, with two real parameters p and q to be chosen adequately. By Lax-Milgram theorem, if $\nu \geq \nu_0 > 0$ and $\eta + \frac{1}{2}\mathrm{div}(\mathbf{b}) > 0$, the well-posedness of the boundary value problem is ensured as soon as p and q are positive. If $q = 0$, Λ reduces to Robin operator, first used in [10]. Numerical evidences with a finite element scheme were given in [9] that these transmission conditions outperform significantly the Robin-Schwarz algorithm . Further analysis has been conducted in [5] in a model case, where the coefficients p and q were obtained by optimization of the convergence factor of the algorithm, defined for two half planes, in the Fourier variables. Asymptotic values in terms of the discretization parameters were given (see Section 4).

The discrete counterpart of the algorithm in the Robin case $q = 0$ has been analyzed first in [1] and extended in [3] and [2] in the finite volume framework. For an analysis in the finite element context see [6]. The study of the Ventcell case $(p, q > 0)$ is, as far as we know, new. The scheme is fully described for the first time in this paper, and simulations are presented. The error analysis and the proofs of well-posedness and convergence will appear in an extended paper [7].

The first step, in section 2, is to write a finite volume scheme for the discretization of the subdomain problem. We use a two point flux approximation for the diffusive flux and a family of discrete convective fluxes as in [4], specially designed to handle the boundary condition. The discretisation of the boundary operator appearing in (1)

[1] LAGA, UMR 7539 CNRS, Université PARIS 13, 93430 VILLETANEUSE, FRANCE, e-mail: halpern@math.univ-paris13.fr ·[2] Université Aix-Marseille, LATP, 39 rue F. Joliot Curie 13 453 Marseille Cedex 13, FRANCE e-mail: florence.hubert@univ-amu.fr

is performed. Non conforming meshes on the interface are considered as they can be useful for local refinement, see [8] for large scale computations.

The discrete Schwarz algorithm is described in section 3. In opposition to the Robin case, the convective flux on the interface has to be modified to get the convergence towards the approximation of (1) on Ω.

Finally, numerical examples illustrate the properties of the scheme , among which the improvement of the algorithm over the Robin algorithm.

2 Finite volume discretization for Ventcell transmission condition

We first introduce the necessary tools for finite volume design in the case of elliptic equation with mixed boundary conditions, Dirichlet on $\Gamma_D \subset \partial\Omega$ and possibly Ventcell (2) on $\Gamma \subset \partial\Omega$ (see [3] for the standard part of the notations).

Admissible Meshes Let Ω be an open polygonal set, $\mathfrak{M}$ a family of polygonal control volumes such that $\bar{\Omega} = \cup_{\mathrm{K}\in\mathfrak{M}}\mathrm{K}$, with $\mathrm{K}\cap\mathrm{L} = \emptyset$ if $\mathrm{K} \neq \mathrm{L}$. $\mathfrak{M}$ is an *admissible finite volume mesh* if there exists a family of points $(x_\mathrm{K})_{\mathrm{K}\in\mathfrak{M}}$ that satisfies $(x_\mathrm{K}, x_\mathrm{L}) \perp \sigma$ if $\sigma = \partial\mathrm{K}\cap\partial\mathrm{L}$. If all control volumes K are triangles, the family of circumcenters of the triangles satisfies this orthogonality condition. The set of all edges σ of control volumes is denoted by $\mathscr{E}$. It is divided into three sets: the edges located inside the domain Ω, $\mathscr{E}_{int} = \{\sigma \in \mathscr{E} / \sigma = \partial\mathrm{K}\cap\partial\mathrm{L}\}$, the edges $\mathscr{E}_D$ located on an external Dirichlet boundary Γ_D, and the edges $\mathscr{E}_\Gamma$ located on Γ. Finally, for any K in $\mathfrak{M}$, $\mathscr{E}_\mathrm{K}$ stands for the edges of its boundary $\partial\mathrm{K}$.

For any $\sigma \in \mathscr{E}_\mathrm{K}$, $\mathbf{n}_{\mathrm{K}\sigma}$ is the outward-pointing unit vector orthogonal to σ, $d_{\mathrm{K},\sigma} > 0$ the distance from x_K to σ, $d_\sigma = d_{\mathrm{K},\sigma}$ if $\sigma \in \mathscr{E}_D \cup \mathscr{E}_\Gamma$ and $d_\sigma = d_{\mathrm{K},\sigma} + d_{\mathrm{L},\sigma}$ is the distance between x_K and x_L if $\sigma = \partial\mathrm{K}\cap\partial\mathrm{L} \in \mathscr{E}_{int}$.

Let $|\mathscr{E}_\Gamma|$ be the cardinality of $\mathscr{E}_\Gamma$, the edges of $\mathscr{E}_\Gamma$ are reordered as $\{\sigma_i\}$, with $\sigma_i \cap \sigma_{i+1}$ reduced to a single point denoted by $x_{i+\frac{1}{2}}$. The control volume associated to σ_i is denoted by K_i.

For each $\mathrm{K} \in \mathfrak{M}$ or $\sigma \subset \Gamma$, $|\mathrm{K}|$ denotes the area of K, and $|\sigma|$ is the length of σ.

The complete admissible finite volume mesh for the boundary value problem is $\mathscr{T} = \mathfrak{M} \cup \mathscr{E}_\Gamma$. Figure 1 summarizes these notations.

Composite meshes The subdomains Ω_j are endowed with admissible meshes $\mathscr{T}_j = \mathfrak{M}_j \cup \mathscr{E}^j_\Gamma$, with two different meshes on Γ. The meshes $\mathscr{T}_1$ and $\mathscr{T}_2$ are said to be compatible if they coincide on Γ or equivalently if $\mathscr{E}^1_\Gamma = \mathscr{E}^2_\Gamma$. We then define $\mathscr{E}_\Gamma = \mathscr{E}^1_\Gamma = \mathscr{E}^2_\Gamma$. Any non compatible couple of meshes $(\mathscr{T}_1, \mathscr{T}_2)$ is made compatible by redefining the edges on Γ: in the example of Grid # 2 in Fig. 2, $\#\mathscr{E}_\mathrm{K} = 5$ for any control volume $\mathrm{K} \in \mathfrak{M}_1$ touching Γ. An edge of $\mathscr{E}_\Gamma$ is $\partial\mathrm{K}_1 \cap \partial\mathrm{K}_2$ with $\mathrm{K}_i \in \mathscr{T}_i$.

Finally a *composite mesh* associated to $\Omega = \Omega_1 \cup \Omega_2$ is a quadruplet $\mathscr{T} = (\mathfrak{M}, \mathfrak{M}_1, \mathfrak{M}_2, \mathscr{E}_\Gamma)$ such that each mesh $\mathfrak{M}_j$ is an admissible mesh for Ω_j, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are compatible, and $\mathfrak{M} = \{K \in \mathfrak{M}_1 \cup \mathfrak{M}_2\}$.

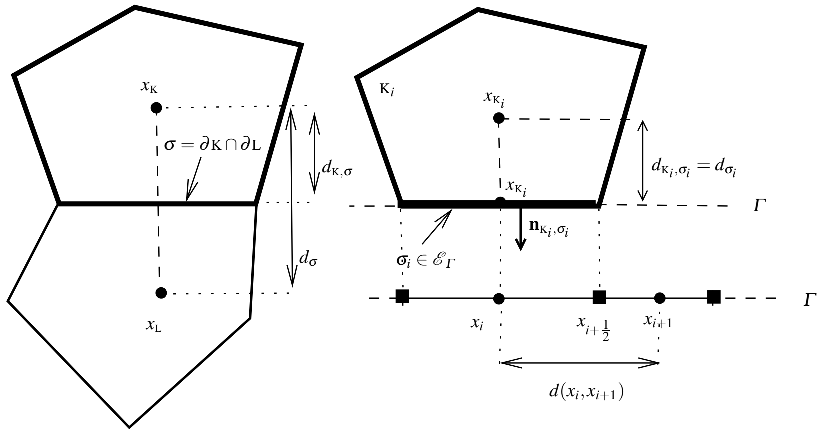

Fig. 1 Notations for an admissible mesh

A two-points flux approximation for Ventcell boundary conditions On each subdomain Ω_j, we approximate the problem $\mathscr{L}u_j = f$ with homogeneous Dirichlet boundary condition on $\Gamma_D^j = \partial\Omega_j \cap \partial\Omega$, and Ventcell boundary condition on $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$:

$$(\nu\partial_{n_j} - \frac{1}{2}\mathbf{b}\cdot\mathbf{n}_j + \Lambda)\,u_j = g_j. \tag{2}$$

For sake of clarity, the dependency on the index of the subdomain Ω_j will be omitted in this paragraph.

We introduce two sets $\mathbf{u}^{\mathfrak{M}} = (u_{\mathrm{K}})_{\mathrm{K}\in\mathfrak{M}}$ and $\mathbf{u}^{\mathscr{E}_\Gamma} = (u_\sigma)_{\sigma\in\mathscr{E}_\Gamma}$ of unknowns, one for the control volumes, one for the edges of the boundary $\mathscr{E}_\Gamma$. We define $\mathbf{u}^{\mathscr{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathscr{E}_\Gamma})$. The discrete volume equations will be obtained, first by integrating the volume equation on a control volume K, second by integrating the boundary condition on the boundary control cell σ_i.

Equation on $\mathrm{K}\in\mathfrak{M}$

Integrating the equation (1) on the control volume K, we get:

$$\sum_{\sigma\in\mathscr{E}_{\mathrm{K}}}\left(-\int_\sigma \nu\nabla u\cdot\mathbf{n}_{\mathrm{K}\sigma}\,ds + \int_\sigma \mathbf{b}\cdot\mathbf{n}_{\mathrm{K}\sigma}u\,ds\right) + \int_{\mathrm{K}} \eta u\,dx = \int_{\mathrm{K}} f(x)dx.$$

The volume term $\int_{\mathrm{K}} \eta u\,dx$ can be approximated by $\eta_{\mathrm{K}}u_{\mathrm{K}}$ with $\eta_{\mathrm{K}} = \frac{1}{|\mathrm{K}|}\int_{\mathrm{K}}\eta$. The total flux in K is the sum on the edges of K of the diffusive fluxes$-\int_\sigma \nu\nabla u\cdot\mathbf{n}_{\mathrm{K}\sigma}\,ds$ and the convective fluxes $\int_\sigma \mathbf{b}\cdot\mathbf{n}_{\mathrm{K}\sigma}u\,ds$, that can be approximated respectively by the discrete fluxes $F^d_{\mathrm{K},\sigma}$, $F^c_{\mathrm{K},\sigma}$ to be defined below. Defining the total discrete flux on the edge σ as $F_{\mathrm{K},\sigma} = F^d_{\mathrm{K},\sigma} + F^c_{\mathrm{K},\sigma}$, the equation on $\mathrm{K}\in\mathfrak{M}$ can be approximated by

$$\forall \mathrm{K}\in\mathfrak{M}, \quad \sum_{\sigma\in\mathscr{E}_{\mathrm{K}}} F_{\mathrm{K},\sigma} + |\mathrm{K}|\eta_{\mathrm{K}}u_{\mathrm{K}} = \int_{\mathrm{K}} f(x)\,dx. \tag{3}$$

We use the classical diffusive discrete flux

$$F^d_{\mathrm{K},\sigma} = |\sigma|\nu_\sigma \frac{u_\mathrm{K} - \bar{u}_\sigma}{d_\sigma} \text{ with } \bar{u}_\sigma = \begin{cases} u_\mathrm{L} & \text{if } \sigma = \partial\mathrm{K} \cap \partial\mathrm{L} \in \mathscr{E}_{int}, \\ 0 & \text{if } \sigma \in \mathscr{E}_D, \\ u_\sigma & \text{if } \sigma \in \mathscr{E}_\Gamma, \end{cases} \tag{4}$$

with $\nu_\sigma = \frac{1}{|\sigma|}\int_\sigma \nu(s)\,ds$ or $\nu_\sigma = \nu(x_\sigma)$, ($x_\sigma$ center of σ) in the case of regular ν.

We introduce a general discrete convection flux in the form

$$F^c_{\mathrm{K},\sigma} = \frac{1}{2}|\sigma| b_{\mathrm{K}\sigma}(u_\mathrm{K} + \bar{u}_\sigma) + \frac{|\sigma|\nu_\sigma}{d_\sigma} B_\sigma \left(\frac{d_\sigma b_{\mathrm{K}\sigma}}{\nu_\sigma}\right)(u_\mathrm{K} - \bar{u}_\sigma), \tag{5}$$

where $b_{\mathrm{K}\sigma} = \frac{1}{|\sigma|}\int_\sigma \mathbf{b}\cdot\mathbf{n}_{\mathrm{K}\sigma}$, and for all edge σ, B_σ is an even Lipschitz continuous function such that

$$B_\sigma(0) = 0, \quad B_\sigma(s) + 1 > \underline{c} > 0 \text{ for } s \neq 0. \tag{6}$$

This frame, introduced in [4], includes the centered scheme $B_\sigma(s) := B^c(s) = 0$, the upwind scheme $B_\sigma(s) := B^{up}(s) = \frac{1}{2}|s|$, and the Scharfetter-Gummel scheme $B_\sigma(s) := B^{SG}(s) = \frac{1}{2}(\frac{s}{e^s - 1} - \frac{s}{e^{-s}-1}) - 1$. Each of these approximations can be seen as a stabilization of the centered scheme. We will take advantage of this flexibility in the convergence analysis of the algorithm (see Theorem 2).

Equation for $\sigma \in \mathscr{E}_\Gamma$. Integrate the Ventcell boundary condition (2) on the edge $\sigma_i \in \mathscr{E}_\Gamma$ to obtain

$$\int_{\sigma_i} \nu\nabla u\cdot\mathbf{n}_{\mathrm{K}_i\sigma_i}\,ds - \frac{1}{2}\int_{\sigma_i}\mathbf{b}\cdot\mathbf{n}_{\mathrm{K}_i\sigma_i} u\,ds + p\int_{\sigma_i} u\,ds + q\left[-\nu\partial_y u\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \int_{\sigma_i} g(s)\,ds.$$

Define the discrete 1D flux $F_{i+\frac{1}{2}}$ as an approximation of $-\nu\frac{\partial u}{\partial y}(x_{i+\frac{1}{2}})$, given by

$$F_{i+\frac{1}{2}} = -\nu(x_{i+\frac{1}{2}})\frac{u_{\sigma_{i+1}} - u_{\sigma_i}}{d(x_{i+1}, x_i)} \quad \text{for } i = 0, \cdots, |\mathscr{E}_\Gamma|, \tag{7}$$

with the convention $u_{\sigma_0} = 0$ and $u_{\sigma_{|\mathscr{E}_\Gamma|+1}} = 0$. We obtain for all $\sigma \in \mathscr{E}_\Gamma$ the equation

$$-F_{\mathrm{K},\sigma} + \frac{1}{2}b_{\mathrm{K}\sigma}m_\sigma u_\sigma + (\Lambda^{\mathscr{E}_\Gamma}\mathbf{u}^{\mathscr{E}_\Gamma})_\sigma = \int_\sigma g(s)\,ds, \tag{8}$$

where the discrete boundary operator $\Lambda^{\mathscr{E}_\Gamma}$ is defined by

$$(\Lambda^{\mathscr{E}_\Gamma}\mathbf{u}^{\mathscr{E}_\Gamma})_\sigma = p|\sigma|u_\sigma - q(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}), \text{ for } \sigma = \sigma_i. \tag{9}$$

Properties of the scheme

By construction $\Lambda^{\mathscr{E}_\Gamma}$ is a symmetric and positive definite matrix. Therefore classical a priori estimates together with assumptions (6), induce the well-posedness of the scheme (3)-(8), see [7]. Furthermore, the scheme is of order 1.

3 A discrete Schwarz algorithm for Ventcell transmission conditions

Discrete Schwarz algorithm Given a composite mesh $\mathscr{T} = (\mathfrak{M}, \mathfrak{M}_1, \mathfrak{M}_2, \mathscr{E}_\Gamma)$, the discrete Schwarz algorithm consists, with suitable initial data, in finding for all $n \geq 1$, the solutions $\mathbf{u}^{\mathscr{T}_j,n} = (\mathbf{u}^{\mathfrak{M}_j,n}, \mathbf{u}^{\mathscr{E}_\Gamma^j,n})$ of the linear system

$$\forall \mathrm{K} \in \mathfrak{M}_j, \quad \sum_{\sigma \in \mathscr{E}_\mathrm{K}} (F_{\mathrm{K},\sigma})_j^n + |\mathrm{K}| \eta_\mathrm{K} (u_\mathrm{K})_j^n = \int_\mathrm{K} f(x)\,dx, \tag{10-a}$$

$$\forall \sigma \in \mathscr{E}_\Gamma, \quad \begin{aligned} &-(F_{\mathrm{K},\sigma})_j^n + \frac{1}{2}|\sigma| b_{\mathrm{K}_j,\sigma}(u_\sigma)_j^n + (\Lambda^{\mathscr{E}_\Gamma} u^{\mathscr{E}_\Gamma^j,n})_\sigma \\ &= (F_{\mathrm{K},\sigma})_i^{n-1} - \frac{1}{2}|\sigma| b_{\mathrm{K}_i,\sigma}(u_\sigma)_i^{n-1} + (\Lambda^{\mathscr{E}_\Gamma} u^{\mathscr{E}_\Gamma^i,n-1})_\sigma. \end{aligned} \tag{10-b}$$

Limit of the discrete Schwarz algorithm Assume that the algorithm (10) converges as n tends to infinity. The limit $\mathbf{u}^{\mathscr{T}_j,\infty} = (\mathbf{u}^{\mathfrak{M}_j,\infty}, \mathbf{u}^{\mathscr{E}_\Gamma^j,\infty})$ is solution of the scheme

$$\forall \mathrm{K} \in \mathfrak{M}_j, \quad \sum_{\sigma \in \mathscr{E}_\mathrm{K}} (F_{\mathrm{K},\sigma})_j^\infty + |\mathrm{K}| \eta_\mathrm{K} (u_\mathrm{K})_j^\infty = \int_\mathrm{K} f(x)\,dx, \tag{11-a}$$

$$\forall \sigma \in \mathscr{E}_\Gamma, \quad \begin{aligned} &-(F_{\mathrm{K},\sigma})_j^\infty + \frac{1}{2}|\sigma| b_{\mathrm{K}_j,\sigma}(u_\sigma)_j^\infty + (\Lambda^{\mathscr{E}_\Gamma^j} \mathbf{u}^{\mathscr{E}_\Gamma,\infty})_\sigma \\ &= (F_{\mathrm{K},\sigma})_i^\infty - \frac{1}{2}|\sigma| b_{\mathrm{K}_i,\sigma}(u_\sigma)_i^\infty + (\Lambda^{\mathscr{E}_\Gamma} \mathbf{u}^{\mathscr{E}_\Gamma^i,\infty})_\sigma. \end{aligned} \tag{11-b}$$

The expected limit However, we expect the convergence towards the classical two point flux finite volume scheme, associated to the mesh $\mathfrak{M}$ for the problem (1) on Ω, which consists in finding $\mathbf{u}^\mathfrak{M} = (u_\mathrm{K})_{\mathrm{K} \in \mathfrak{M}}$ solution of the discrete problem

$$\forall \mathrm{K} \in \mathfrak{M}, \quad \sum_{\sigma \in \mathscr{E}_\mathrm{K}} F_{\mathrm{K},\sigma} + |\mathrm{K}| \eta_\mathrm{K} u_\mathrm{K} = \int_\mathrm{K} f(x)\,dx. \tag{12}$$

If the composite mesh $\mathfrak{M}$ is non admissible in the neighborhood of Γ (Figure 2 right), the solution $u^\mathfrak{M}$ still approximates the solution u of (1), but with an error of order $\mathrm{size}(\mathfrak{M})^{\frac{1}{2}}$ only (See [3]).

The solutions of the schemes (12) and (11) can coincide only when the fluxes in (11) are modified, as stated in the next theorem.

Theorem 1. *Let* $\mathbf{u}^{\mathfrak{M}}$ *be the solution of* (12)*, with a convective flux in* (5) *defined by a function* B_σ*, satisfying*

$$B_\sigma(0)=0, \qquad B_\sigma(s)>-1+\frac{1}{2}|s|. \tag{13}$$

Define for $\sigma\in\mathscr{E}$ *the functions* $\bar{B}_\sigma$ *by*

$$\bar{B}_\sigma(s)=\begin{cases}B_\sigma(s) & \text{if } \sigma\notin\mathscr{E}_\Gamma,\\ \frac{1}{2}(1-B_\sigma(2s))\pm\frac{1}{2}\sqrt{(1-s+B_\sigma(2s))(1+s+B_\sigma(2s))} & \text{if } \sigma\in\mathscr{E}_\Gamma.\end{cases} \tag{14}$$

Then, for this modified choice of fluxes $\bar{B}_\sigma$*, there exists* $\mathbf{u}^{\mathscr{T}_j,\infty}=(\mathbf{u}^{\mathfrak{M}_j,\infty},\mathbf{u}^{\mathscr{E}_\Gamma,\infty})$ *for* $j=1,2$*, solution of* (11)*, and* $u_K^{\mathfrak{M}}=u_K^{\mathfrak{M}_j}$ *for* $K\in\mathfrak{M}_j$*.*

Proof. Let $u_{\mathrm{K}}^{\mathfrak{M}_j,\infty}=u_{\mathrm{K}}$ for all $\mathrm{K}\in\mathfrak{M}_j$. First for K such that $\mathscr{E}_{\mathrm{K}}\cap\mathscr{E}_\Gamma=\emptyset$, equation (11-a) is nothing but equation (12). However, the construction of the edge unknowns $\mathbf{u}^{\mathscr{E}_\Gamma^j}$ requires some care.

For $\sigma\in\mathscr{E}_\Gamma$, equation (11-b) written for $(j,i)=(1,2)$ and $(2,1)$ yields

$$\Lambda^{\mathscr{E}_\Gamma}u^{\mathscr{E}_\Gamma^1,\infty}=\Lambda^{\mathscr{E}_\Gamma}u^{\mathscr{E}_\Gamma^2,\infty}.$$

Thus, using the invertibility of $\Lambda^{\mathscr{E}_\Gamma}$, we obtain that $u^{\mathscr{E}_\Gamma^1,\infty}=u^{\mathscr{E}_\Gamma^2,\infty}=u^{\mathscr{E}_\Gamma,\infty}$ and $(F_{\mathrm{K},\sigma})_1^\infty=-(F_{\mathrm{K},\sigma})_2^\infty$. Finally equation (11-a) coincides with equation (12) if

$$F_{\mathrm{K},\sigma}=(F_{\mathrm{K},\sigma})_1^\infty. \tag{15}$$

Define $d_{\mathrm{K}_1\sigma}$, $d_{\mathrm{K}_2\sigma}$ and s by $s=\frac{b_{\mathrm{K}_1\sigma}d_{\mathrm{K}_1\sigma}}{\nu_{\mathrm{K}_1\sigma}}=-\frac{b_{\mathrm{K}_2\sigma}d_{\mathrm{K}_2\sigma}}{\nu_{\mathrm{K}_2\sigma}}=\frac{b_{\mathrm{K}_1\sigma}d_{\mathrm{K}_1\mathrm{K}_2}}{2\nu_\sigma}$. We then have for j=1,2

$$(F_{\mathrm{K},\sigma})_j^\infty=\frac{|\sigma|\nu_{\mathrm{K}_j\sigma}}{d_{\mathrm{K}_j\sigma}}(u_{\mathrm{K}_j}^\infty-u_\sigma^\infty)(1+\bar{B}_\sigma(s))+\frac{1}{2}|\sigma|b_{\mathrm{K}_j\sigma}(u_{\mathrm{K}_j}^\infty+u_\sigma^\infty).$$

Identifying $(F_{\mathrm{K},\sigma})_1^\infty$ to $-(F_{\mathrm{K},\sigma})_2^\infty$ defines u_σ^∞, then (15) is equivalent to

$$B_\sigma(2s)=\bar{B}_\sigma(s)+\frac{1}{4}s^2(1+\bar{B}_\sigma(s))^{-1}. \tag{16}$$

Hence, to express $\bar{B}_\sigma(s)$ in terms of $B_\sigma(s)$, we solve the equation $X^2+(1-B_\sigma(2s))X+\left(\frac{1}{4}s^2-B_\sigma(2s)\right)=0$, Under condition (13), there exists a unique solution satisfying $\bar{B}_\sigma(0)=0$, which is given in (14).

In this case, any solution of (11) is a solution of (12), which has a unique solution. □

Remark 1. Assumption (13) is satisfied by the upwind scheme, the Scharfetter-Gummel scheme and the centered scheme if $|s|<1$. In the case of the Scharfetter-Gummel scheme, $\bar{B}_\sigma=B_\sigma$.

Convergence of the Schwarz algorithm

Theorem 2. *Let $\mathscr{T}_j$ be two compatible meshes of Ω_j, $j=1,2$ and $\mathscr{T}$ the associated composite mesh. With the assumptions in Theorem 1, the solution $(\mathbf{u}^{\mathfrak{M}_{j},n})_{j=1,2}$ of the discrete Schwarz algorithm* (10) *converges to $\mathbf{u}^{\mathfrak{M}}$ solution of* (12) *as n tends to infinity.*

Hint on the proof. The proof is too long to be developed here, and will appear in [7]. By Theorem 1 the convergence of the Schwarz algorithm is equivalent to the convergence to 0 of the solution $\mathbf{u}^{\mathscr{T}_j,n}=(\mathbf{u}^{\mathfrak{M}_j,n},\mathbf{u}^{\mathscr{E}^j_\Gamma,n})$ of (10) when f is identically zero. That convergence is then obtained by an extension of P.L. Lions trick in [10], using the fact that $\Lambda^{\mathscr{E}_\Gamma}$ is a symmetric positive definite matrix.

4 Numerical experiments

The domain $\Omega=]-1,1[\times]0,1[$ is split into $\Omega_1=]-1,0[\times]0,1[$ and $\Omega_2=]0,1[\times]0,1[$ with an interface Γ at $x=0$. We compare the convergence behaviour of the optimized Schwarz algorithm for Robin and Ventcell transmission conditions. Define the mesh size on the interface, $h=\min(\max(|\sigma|,\sigma\in\mathscr{E}_j),\ j=1,2)$. Asymptotically optimal parameters (for small h) are taken from [5]. They have been determined to produce the smallest convergence factor over all frequencies supported by the grid.

$$\text{Robin}: \quad p^\star=\frac{h^{-\frac{1}{2}}}{2}\sqrt{2\pi\nu(b_x^2+4\nu\eta)^{\frac{1}{2}}}, \qquad q^\star=0.$$

$$\text{Ventcell}: \quad p^\star=\frac{h^{-\frac{1}{4}}}{2}\sqrt[4]{\frac{\nu\pi(b_x^2+4\nu\eta)^{\frac{3}{2}}}{2}}, \qquad q^*=\frac{h^{\frac{3}{4}}}{2}\sqrt[4]{\frac{8\nu}{\pi^3}(b_x^2+4\nu\eta)^{-\frac{1}{2}}}.$$

The corresponding theoretical convergence factor of the algorithm (i.e. the factor of reduction of the L^2 norm of the error in one iteration) is

$$\text{Robin}: 1-\mathscr{O}(h^{\frac{1}{2}}), \quad \text{Ventcell}: 1-\mathscr{O}(h^{\frac{1}{4}}),$$

showing an improvement from Robin to Ventcell, since it is less dependent of the size of the mesh.

We choose $\nu=0.1$, $\mathbf{b}=(1,1)^t$, $\eta=1$. The source f is such that the exact solution of (1) is $u(x,y)=\sin(3\pi x)\sin(3\pi y)$. The Scharfetter-Gummel scheme is used for all edges. The algorithm is initialized with random data $(\mathbf{u}^{\mathfrak{M}_j,0})_{j=1,2}$. We illustrate our results on two families of grids presented in Figure 2, one is conforming (Grid # 1), the other non conforming (Grid # 2) at the interface Γ. We draw the convergence history for increasing mesh refinement, given by $i=3,4,5,6$. We stopped the algorithm as soon as $\left(\sum_{j=1,2}\|\mathbf{u}^{\mathfrak{M}_j,n+1}-\mathbf{u}^{\mathfrak{M}_j,n}\|^2_{L^2(\Omega_j)}\right)^{\frac{1}{2}}\leq 10^{-7}$. We can see the drastic improvement obtained by using the second order transmission condition, for which the convergence lines seem almost independent of h. The numerical con-

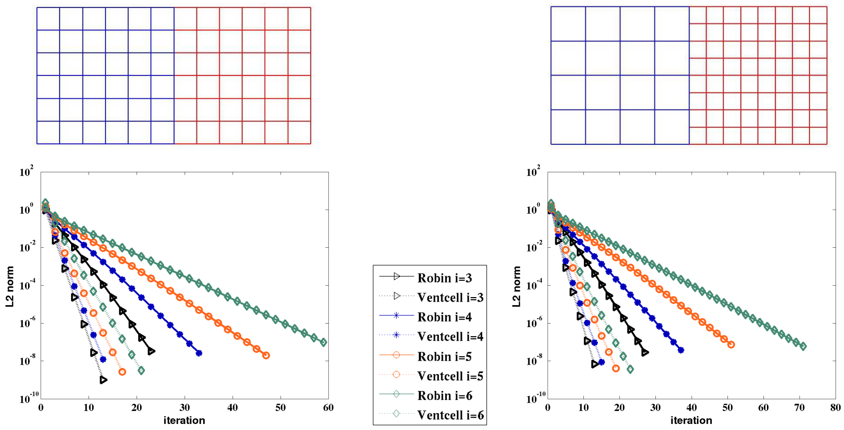

Fig. 2 (Left) A $6*2^i \times 6*2^i$ square grid on both Ω_1 and Ω_2. (Right) A $4*2^i \times 4*2^i$ grid on Ω_1 and a $8*2^i \times 8*2^i$ grid on Ω_2. Robin vs Ventcell. L^2 norm error w.r.t. iterations for increasing mesh refinements.

vergence factor behaves in $1-\mathscr{O}(h^\alpha)$ with $\alpha = 0.43$ for Robin-Grid # 1, $\alpha = 0.44$ for Robin-Grid # 2, $\alpha = 0.17$ for Ventcell-Grid # 1, $\alpha = 0.19$ for Ventcell-Grid # 2.

References

1. Achdou, Y., Japhet, C., Nataf, F., Maday, Y.: A new cement to glue non-conforming grids with Robin interface conditions: The finite volume case. Numer. Math. **92**(4), 593–620 (2002)
2. Boyer, F., Hubert, F., Krell, S.: Non-overlapping Schwarz algorithm for solving 2D m-DDFV schemes. IMA J. Numer. Anal. **30**, 1062–1100 (2009)
3. Cautrès, R., Herbin, R., Hubert, F.: The Lions domain decomposition algorithm on non-matching cell-centred finite volume meshes. IMA J. Numer. Anal. **24**(3), 465–490 (2004)
4. Chainais-Hillairet, C., Droniou, J.: Finite volume schemes for non-coercive elliptic problems with Neumann boundary conditions. IMA Journal of Numerical Analysis **31**(1), 61–85 (2011)
5. Dubois, O.: Optimized Schwarz methods for the advection-diffusion equation and for problems with discontinuous coefficients. Ph.D. thesis, McGill University, Canada (2007)
6. Gander, M.J., Japhet, C., Maday, Y., Nataf, F.: A new cement to glue nonconforming grids with Robin interface conditions: the finite element case. Lect. Notes Comp. Sci. Eng. **40**, 259–266 (2005)
7. Halpern, L., Hubert., F.: Optimized Schwarz algorithms in the classical finite volume framework. http://www.math.univ-paris13.fr/ halpern/Publis/HH.pdf (2013)
8. Halpern, L., Japhet., C., Szeftel, J.: Optimized Schwarz waveform relaxation and discontinuous Galerkin time stepping for heterogeneous problem. SIAM Journal on Numerical Analysis **50**(5), 2588–2611 (2012)
9. Japhet, C.: Méthode de décomposition de domaine et conditions aux limites artificielles en mécanique des fluides : méthode Optimisée d'Ordre 2. Ph.D. thesis, Université Paris 13, France (1998)
10. Lions, P.L.: On the Schwarz alternating method. III. A variant for nonoverlapping subdomains. Third International Symposium on Domain Decomposition Methods for Partial Differential Equations (Houston, TX, 1989), pp. 202–223.