A domain decomposition algorithm for contact problems with Coulomb's friction

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1 Introduction

Contact problems of elasticity are used in many fields of science and engineering, especially in structural mechanics, geology and biomecanics. Many numerical procedures solving contact problems have been proposed in the engineering literature. They are based on standard discretization techniques for partial differential equations in combination with a special implementation of non-linear contact conditions (e.g., see [3, 5, 6, 8]).

The use of domain decomposition methods turns out to be one of the most efficient approaches. Recently, Dirichlet-Neumann and FETI type algorithms have been proposed and studied for solving multibody contact problems with Coulomb friction (see for example [7, 1, 2]).

In this paper, the Neumann-Neumann algorithm is extended to two-body contact problems with Coulomb friction. The main difficulty is due to the boundary conditions at the contact interface. They are highly non-linear, both in the normal direction (unilateral contact conditions) and in the tangential one (Coulomb's law). A fixed point procedure is introduced to ensure the continuity of the contact stresses. Numerical results illustrate that an optimal relaxation parameter exists and its value is nearly independent of the friction coefficient and the mesh size.

2 Setting of the problem

Let us consider two plane elastic bodies, occupying bounded domains Ω^α, $\alpha = 1,2$. The boundary $\Gamma^\alpha := \partial\Omega^\alpha$ is assumed to be piecewise continuous, and it is split into three non empty disjoint parts Γ_u^α, Γ_p^α and Γ_c^α such that $\overline{\Gamma_u^\alpha} \cap \overline{\Gamma_c^\alpha} = \emptyset$. Each body Ω^α is fixed on Γ_u^α and subject to surface tractions $\phi^\alpha \in (L^2(\Gamma_p^\alpha))^2$ on Γ_p^α. The body forces are denoted by $f^\alpha \in (L^2(\Omega^\alpha))^2$. In the initial configuration, both bodies have a common contact portion $\Gamma_c := \Gamma_c^1 = \Gamma_c^2$. In other words, we consider the case when the contact zone cannot grow during the deformation process and there is no gap between Ω^1 and Ω^2. Unilateral contact conditions with local Coulomb's friction are prescribed on Γ_c. The problem consists in finding the displacement field $u = (u^1, u^2)$

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(the notation u^α stands for $u|_{\Omega^\alpha}$) and the stress tensor field $\sigma = (\sigma(u^1), \sigma(u^2))$ such that:

$$\left.\begin{array}{lll} div\,\sigma(u^\alpha) + f^\alpha = 0 & \text{in } \Omega^\alpha, \\ \sigma(u^\alpha)n^\alpha = \phi^\alpha & \text{on } \Gamma_p^\alpha, \\ u^\alpha = 0 & \text{on } \Gamma_u^\alpha, \end{array}\right\} \tag{1}$$

$\alpha = 1,2$. The elastic constitutive law, is given by Hooke's law for homogeneous and isotropic material:

$$\sigma_{ij}(u^\alpha) = \mathsf{A}^\alpha_{ijkh} e_{kh}(u^\alpha), \quad e(u^\alpha) = \frac{1}{2}\left(\nabla u^\alpha + (\nabla u^\alpha)^T\right), \tag{2}$$

where $\mathsf{A}^\alpha = (A^\alpha_{ijkh})_{1\le i,j,k,h\le 2} \in (L^\infty(\Omega^\alpha))^{16}$ is the fourth-order elasticity tensor satisfying the usual symmetry and ellipticity conditions and $e(u^\alpha)$ is the respective strain tensor. The summation convention is adopted.

Further the normal and tangential components of the displacement u and the stress vector on Γ_c are defined by

$$\left.\begin{array}{ll} u_N^\alpha = u_i^\alpha n_i^\alpha, & u_{T_i}^\alpha = u_i^\alpha - u_N^\alpha n_i^\alpha, \\ \sigma_N^\alpha = \sigma_{ij}(u^\alpha) n_i^\alpha n_j^\alpha, & \sigma_T{}_i^\alpha = \sigma_{ij}(u^\alpha) n_j^\alpha - \sigma_N^\alpha n_i^\alpha, \end{array}\right\} \tag{3}$$

where n^α denotes the outward normal unit vector to the boundary. On the interface Γ_c, the unilateral contact law conditions are prescribed:

$$\sigma_N := \sigma_N^1 = \sigma_N^2, \qquad \sigma_T := \sigma_T^1 = \sigma_T^2, \tag{4}$$

$$[u_N] \le 0, \quad \sigma_N \le 0, \quad \sigma_N [u_N] = 0, \tag{5}$$

where $[v_N] = v^1 \cdot n^1 + v^2 \cdot n^2$ is the jump across the interface Γ_c of a function v defined on $\Omega^1 \cup \Omega^2$. Coulomb's law of local friction reads as follows

$$\left.\begin{array}{l} |\sigma_T| \le \mathscr{F}|\sigma_N|, \\ |\sigma_T| < \mathscr{F}|\sigma_N| \Longrightarrow [u_T] = 0, \\ |\sigma_T| = \mathscr{F}|\sigma_N| \Longrightarrow \exists \nu \ge 0 \quad [u_T] = -\nu \sigma_T, \end{array}\right\} \tag{6}$$

where $\mathscr{F} \in L^\infty(\Gamma_c)$, $\mathscr{F} \ge 0$ on Γ_c is the coefficient of friction and $[u_T]$ stands for the jump of the tangential displacements.

Weak solutions of the contact problem obeying Coulomb's law of friction can be defined as a fixed point of the mapping $\Phi : \Lambda \mapsto \Lambda$, where $\Lambda = \{\mu \in H^{-1/2}(\Gamma_c), \mu \ge 0\}$ and $\Phi(g) = -\sigma_N(u)$ with $u \in \mathbb{K}$ being the unique solution of the variational inequality:

$$u := u(g) \in \mathbb{K} : \; a(u, v-u) + \langle \mathscr{F} g, |[v_T]| - |[u_T]| \rangle \ge L(v-u), \quad \forall v \in \mathbb{K}. \qquad (\mathscr{P})$$

Here

$$\mathbb{K} = \{v \in \mathbb{V} |\ [v_N] \le 0 \text{ on } \Gamma_c\}, \qquad \mathbb{V} = \mathbb{V}^1 \times \mathbb{V}^2,$$
$$\mathbb{V}^\alpha = \{v^\alpha \in (H^1(\Omega^\alpha))^2 \mid v^\alpha = 0 \ \text{ on } \Gamma_u^\alpha\},\ \alpha = 1,2.$$

The bilinear and linear form $a(\cdot,\cdot)$, $L(\cdot)$ represent the inner energy of the system, and the work of applied forces, respectively:

$$a(v,w) = a^1(v^1,w^1) + a^2(v^2,w^2), \qquad L(v) = L^1(v^1) + L^2(v^2),\ \ v,w \in \mathbb{V},$$

where

$$a^\alpha(v^\alpha,w^\alpha) = \int_{\Omega^\alpha} \mathsf{A}^\alpha_{ijkh} e_{kh}(v^\alpha) e_{ij}(w^\alpha)\,dx,$$
$$L^\alpha(v^\alpha) = \int_{\Omega^\alpha} f^\alpha \cdot v^\alpha\,dx + \int_{\Gamma_p^\alpha} \phi^\alpha \cdot v^\alpha\,ds,$$

$\alpha = 1,2$. The symbol $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $H^{-1/2}(\Gamma_c)$ and $H^{1/2}(\Gamma_c)$ or for the scalar product in $L^2(\Gamma_c)$, if $g \in L^2(\Gamma_c)$.

3 Domain decomposition algorithm for contact problems with given friction

We present the continuous version of the domain decomposition algorithm for solving $(\mathscr{P})$. The mathematical justification of all results presented below can be found in [4]. We introduce the following notation: by $\pi^\alpha : (H^{1/2}(\Gamma_c))^2 \mapsto \mathbb{V}^\alpha$ we denote the extension mapping defined for $\lambda \in (H^{1/2}(\Gamma_c))^2$ by

$$\left.\begin{array}{c} \pi^\alpha\lambda \in \mathbb{V}^\alpha : a^\alpha(\pi^\alpha\lambda, v^\alpha) = 0 \quad \forall v^\alpha \in \mathbb{V}_0^\alpha, \\ \pi^\alpha\lambda = \lambda \ \text{ on } \ \Gamma_c, \end{array}\right\} \tag{7}$$

where

$$\mathbb{V}_0^\alpha = \{v^\alpha \in (H^1(\Omega^\alpha))^2 |\ v^\alpha = 0 \text{ on } \ \Gamma_u^\alpha \cup \Gamma_c\}. \tag{8}$$

Further for $\varphi \in L^2(\Gamma_c)$ given, we define:

$$\mathbb{K}^2(\varphi) = \{v^2 \in \mathbb{V}^2 |\ v^2 \cdot n^2 \le -\varphi \text{ on } \Gamma_c\}$$

and the frictional term $j : \mathbb{V} \mapsto \mathbb{R}$ by

$$j(v) := j(v_1,v_2) = \int_{\Gamma_c} g|[v_T]|\,ds, \quad v = (v_1,v_2) \in \mathbb{V}.$$

The algorithm is based on the following result.

Proposition 1. *A pair $u=(u^1,u^2)\in\mathbb{V}$ is a solution of $(\mathscr{P})$ if and only if $u^1\in\mathbb{V}^1$, $u^2\in\mathbb{V}^2$ solve the following problems:*

$$\left.\begin{aligned}&\text{Find } u^1\in\mathbb{V}^1 \text{ such that}\\ &a^1(u^1,v^1)=L^1(v^1)-a^2(u^2,\pi^2v^1)+L^2(\pi^2v^1)\quad\forall v^1\in\mathbb{V}^1\end{aligned}\right\} \tag{9}$$

and

$$\left.\begin{aligned}&\text{Find } u^2\in\mathbb{K}^2(u^1\cdot v^1) \text{ such that}\\ &a^2(u^2,v^2-u^2)+j(u^1,v^2)-j(u^1,u^2)\geq L^2(v^2-u^2)\quad\forall v^2\in\mathbb{K}^2(u^1\cdot v^1),\end{aligned}\right\} \tag{10}$$

respectively.

Suppose that $\lambda\in(H^{1/2}(\Gamma_c))^2$ is given and u^1, u^2 are the solutions of the following *decoupled* problems:

$$\left.\begin{aligned}&\text{Find } u^1:=u^1(\lambda)\in\mathbb{V}^1 \text{ such that}\\ &a^1(u^1,v^1)=L^1(v^1)\quad\forall v^1\in\mathbb{V}^1_0\\ &u^1=\lambda\quad\text{on }\Gamma_c\end{aligned}\right\} \tag{$\mathscr{P}_1(\lambda)$}$$

and

$$\left.\begin{aligned}&\text{Find } u^2:=u^2(\lambda)\in\mathbb{K}^2(\lambda\cdot n^1) \text{ such that}\\ &a^2(u^2,v^2-u^2)+j(\lambda,v^2)-j(\lambda,u^2)\geq L^2(v^2-u^2)\\ &\qquad\qquad\forall v^2\in\mathbb{K}^2(\lambda\cdot n^1).\end{aligned}\right\} \tag{$\mathscr{P}_2(\lambda)$}$$

If $\lambda\in(H^{1/2}(\Gamma_c))^2$ was chosen in such a way that $\sigma_N^1=\sigma_N^2$ and $\sigma_T^1=\sigma_T^2$ on Γ_c, then the couple $u=(u^1,u^2)\in\mathbb{K}$ would be a solution of $(\mathscr{P})$. To find such λ ensuring continuity of the normal and tangential contact stress across Γ_c, we shall use the following auxiliary Neumann problems defined in Ω^1 and Ω^2:

$$\left.\begin{aligned}&\text{Find } w^1\in\mathbb{V}^1 \text{ such that}\\ &a^1(w^1,v^1)=\tfrac{1}{2}(-a^1(u^1,v^1)+L^1(v^1)-a^2(u^2,\pi^2v^1)+L^2(\pi^2v^1))\\ &\qquad\qquad\forall v^1\in\mathbb{V}^1\end{aligned}\right\} \tag{$\mathscr{P}_3(\lambda)$}$$

and

$$\left.\begin{aligned}&\text{Find } w^2\in\mathbb{V}^2 \text{ such that}\\ &a^2(w^2,v^2)=\tfrac{1}{2}(a^2(u^2,v^2)-L^2(v^2)+a^1(u^1,\pi^1v^2)-L^1(\pi^1v^2))\\ &\qquad\qquad\forall v^2\in\mathbb{V}^2,\end{aligned}\right\} \tag{$\mathscr{P}_4(\lambda)$}$$

where $u^1:=u^1(\lambda)$, $u^2:=u^2(\lambda)$ are the solutions of $(\mathscr{P}_1(\lambda))$, and $(\mathscr{P}_2(\lambda))$, respectively. The algorithm consists of the following five steps:

ALGORITHM (**DD**) Let $\lambda_0 \in (H^{1/2}(\Gamma_c))^2$ and $\theta > 0$ be given. For $k \geq 1$ integer, define u_k^α, w_k^α, $\alpha = 1,2$ and λ_k by:

Step 1. $u_k^1 \in \mathbb{V}^1$ solves $(\mathscr{P}_1(\lambda_{k-1}))$;

Step 2. $u_k^2 \in \mathbb{K}^2(\lambda_{k-1} \cdot n^1)$ solves $(\mathscr{P}_2(\lambda_{k-1}))$;

Step 3. $w_k^1 \in \mathbb{V}^1$ solves $(\mathscr{P}_3(\lambda_{k-1}))$;

Step 4. $w_k^2 \in \mathbb{V}^2$ solves $(\mathscr{P}_4(\lambda_{k-1}))$;

Step 5. $\lambda_k = \lambda_{k-1} + \theta(w_k^1 - w_k^2)$ on Γ_c.

The convergence property of this algorithm follows from the next theorem.

Theorem 1. *There exist:* $0 < \theta^* < 4$ *and functions* $\lambda_* \in (H^{1/2}(\Gamma_c))^2$, $u_*^\alpha, w_*^\alpha \in \mathbb{V}^\alpha$, $\alpha = 1,2$ *such that for any* $\theta \in (0, \theta^*)$ *it holds:*

$$\left.\begin{array}{ll} \lambda_k \to \lambda_* & \text{in } (H^{1/2}(\Gamma_c))^2, \\ u_k^\alpha \to u_*^\alpha & \\ w_k^\alpha \to w_*^\alpha & \text{in } (H^1(\Omega^\alpha))^2,\ \alpha = 1,2, \end{array}\right\} \quad k \to \infty \tag{11}$$

where the sequence $\{(u_k^\alpha, w_k^\alpha, \lambda_k)\}$ *is generated by* ALGORITHM (**DD**). *In addition, the couple* (u_*^1, u_*^2) *solves* $(\mathscr{P})$.

A discrete version of algorithm is obtained by a finite element approximation of *Steps 1-4*. In [4] we used piecewise linear functions on triangulations of Ω^1 and Ω^2. These triangulations are supposed to be compatible on the contact part Γ_c. Using a similar technique as in Theorem 1, one can prove the convergence property of the discrete version with θ^* independent of the mesh norm.

4 Numerical experiments

In this section, we shall test the performance of variants of ALGORITHM (**DD**) for solving contact problems with Coulomb friction. For this reason, we combine ALGORITHM (**DD**) with the method of succesive approximations that enables us to compute fixed points of the mapping Φ. To get an efficient algorithm, we perform only one iteration of ALGORITHM (**DD**) in each step of the method of succesive approximations. In other words, we update the slip bound g in each *Step 2* using the result of the previous iteration, i.e., $g = -\sigma_N(u_{k-1}^2)$ (and $g = 0$, if $k = 1$). This algorithm will be called ALGORITHM I in this numerical part.

Note that *Step 2* in ALGORITHM I treats simultaneously both, the non-penetration and the friction conditions. A natural idea occurs, namely to split these conditions between *Steps 1* and *2*. This modification of ALGORITHM (**DD**) will be called ALGORITHM II.

In both, ALGORITHM I and II, one can perform splitting of the Gauss-Seidel type so that computatios of the normal and tangential contact stresses are decoupled by performing one Gauss-Seidel iteration; see [4] for more details. In the respective columns of the tables below we show the results without (column *without*) and with the Gauss-Seidel splitting in *Step 1*, *2*, and in both these steps.

Example 1. Let us consider two plane elastic bodies

$$\Omega^1 = (0,3)\times(1,2) \quad \text{and} \quad \Omega^2 = (0,3)\times(0,1)$$

made of an isotropic, homogeneous material characterized by the Young modulus 2.1×10^{11} and the Poisson ratio 0.277 (steel). The decompositions of $\partial\Omega^{\alpha}$, $\alpha = 1,2$ are as follows:

$$\Gamma_u^1 = \{0\}\times(1,2),\ \Gamma_c^1 = (0,3)\times\{1\},\ \Gamma_p^1 = \partial\Omega^1 \setminus \overline{\Gamma_u^1 \cup \Gamma_c^1},$$
$$\Gamma_u^2 = \{0\}\times(0,1),\ \Gamma_c^2 = (0,3)\times\{1\},\ \Gamma_p^2 = \partial\Omega^2 \setminus \overline{\Gamma_u^2 \cup \Gamma_c^2}.$$

The volume forces $f^{\alpha} = 0$ in Ω^{α}, $\alpha = 1,2$ while the following surface tractions of density $\phi^1 = (\phi_1^1, \phi_2^1)$ act on Γ_p^1:

$$\phi_1^1(s,2) = 0,\ \phi_2^1(s,2) = \phi_{2,L}^1 + \phi_{2,R}^1 s, \quad s\in(0,3),$$
$$\phi_1^1(3,s) = \phi_{1,B}^1(2-s) + \phi_{1,U}^1(s-1), \quad s\in(1,2),$$
$$\phi_2^1(3,s) = \phi_{2,B}^1(2-s) + \phi_{2,U}^1(s-1), \quad s\in(1,2),$$

where $\phi_{2,L}^1 = -6\times 10^7$, $\phi_{2,R}^1 = -1/3\times 10^7$, $\phi_{1,B}^1 = 2\times 10^7$, $\phi_{1,U}^1 = 2\times 10^7$, $\phi_{2,B}^1 = 4\times 10^7$, and $\phi_{2,U}^1 = 2\times 10^7$. The coefficient of friction is $\mathscr{F} = 0.3$.

We compare performance of ALGORITHMS I and II with different splittings of Gauss-Seidel type for various values of θ and degrees of freedom n (twice the number of nodes) and m (the number of the contact nodes). In the tables we report the computational time in seconds, the number $\#iter$ of the (outer) iterations, and the total number of actions $n_{\mathbf{A}}$ of the inverses to the stiffness matrices. Further we quote the total efficiency of the method assessed by the ratio $eff := n_{\mathbf{A}}/(2m)$ which gives a comparison of our algorithms with the realization of "similar linear problems" by the standard conjugate gradient method. It is well-known that the number of conjugate gradient iterations, i.e. the number of matrix-vector multiplications, is bounded by the size of the problem. Therefore, one can say that our algorithms exhibit the complexity comparable with the conjugate gradient method when *eff* is less than two. All computations are performed in Matlab 8.2 on Intel(R)Core(TM)2 Duo CPU, 2 GHz with 3 GB RAM. We set the relative terminating precision on the computed contact stresses to $tol = 10^{-4}$. The inner problems in *Step 1* and *2* are solved by optimization algorithms based on the conjugate gradient method with the adaptive precision control respecting the accuracy achieved in the outer loop; see [4] for more details.

Table 1 Characteristics of ALGORITHM I without and with splitting.

$n\|m$	*without* $\#iter\|n_{\mathbf{A}}$ $[time\|eff]$	*in Step 1* $\#iter\|n_{\mathbf{A}}$ $[time\|eff]$	*in Step 2* $\#iter\|n_{\mathbf{A}}$ $[time\|eff]$	*in Step 1+2* $\#iter\|n_{\mathbf{A}}$ $[time\|eff]$
504|18	60|667 [0.80|18.53]	61|1075 [0.98|29.86]	59|742 [0.67|20.61]	60|1146 [0.70|31.83]
6072|66	61|1044 [8.19|7.91]	61|1492 [8.35|11.30]	61|824 [4.63|6.24]	60|1236 [6.91|9.36]
17784|114	62|1313 [31.73|5.76]	63|1816 [43.71|7.96]	61|855 [33.24|3.75]	63|1365 [32.89|5.99]
35640|162	61|1839 [126.94|5.68]	62|1819 [133.30|5.61]	61|892 [59.59|2.75]	62|1377 [91.82|4.25]
59640|210	60|1583 [238.32|3.77]	61|2336 [341.33|5.56]	61|876 [127.42|2.09]	61|1377 [196.11|3.28]
89784|258	60|1627 [405.31|3.15]	59|2333 [585.25|4.52]	60|864 [216.09|1.67]	61|1421 [359.08|2.75]

Table 2 Characteristics of ALGORITHM II without and with splitting.

$n\|m$	*without* $\#iter\|n_{\mathbf{A}}$ $[time\|eff]$	*in Step 1* $\#iter\|n_{\mathbf{A}}$ $[time\|eff]$	*in Step 2* $\#iter\|n_{\mathbf{A}}$ $[time\|eff]$	*in Step 1+2* $\#iter\|n_{\mathbf{A}}$ $[time\|eff]$
504|18	37|530 [0.19|14.72]	36|520 [0.16|14.44]	37|714 [0.19|19.83]	38|770 [0.19|21.39]
6072|66	36|987 [5.76|7.48]	37|586 [3.29|4.44]	37|964 [5.35|7.30]	38|829 [4.59|6.28]
17784|114	36|1417 [34.32|6.21]	38|626 [15.16|2.75]	37|1347 [32.81|5.91]	35|794 [19.00|3.48]
35640|162	37|1864 [119.50|5.75]	36|608 [38.74|1.88]	36|1399 [89.79|4.32]	36|863 [54.83|2.66]
59640|210	37|2132 [290.71|5.08]	37|624 [93.40|1.49]	37|1401 [191.30|3.34]	35|851 [115.64|2.03]
89784|258	37|2532 [631.80|4.91]	37|619 [154.52|1.20]	37|1806 [451.65|3.50]	36|877 [225.59|1.70]

Figure 1 illustrates the sensitivity of the different variants of our algorithms with respect to θ. From these results one may conclude at least two facts: *(i)* ALGORITHM II without splitting is more stable than ALGORITHM I in sense that it converges for larger values of θ; *(ii)* splitting used *Step 2* of ALGORITHM II leads to the convergent process for all $\theta \in (0,1]$.

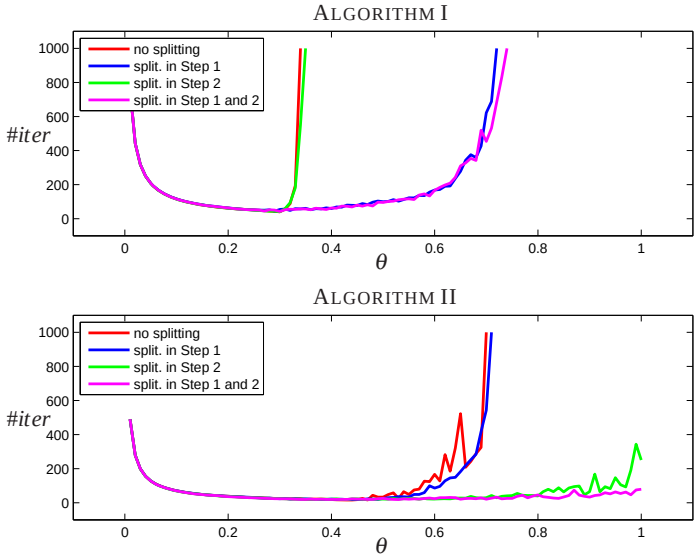

Fig. 1 For each θ we display the number of iterations *#iter* satisfying the terminating precision as above ($n = 1872$, $m = 36$).

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