# DG discretization of optimized Schwarz methods for Maxwell's equations

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## 1 Introduction

In the last decades, Discontinuous Galerkin (DG) methods have seen rapid growth and are widely used in various application domains (see [13] for an historical introduction). This is due to their main advantage of combining the best of finite element and finite volume methods. For the time-harmonic Maxwell equations, once the problem is discretized with a DG method, finding robust solvers is a difficult task since one has to deal with indefinite problems. From the pioneering work of Després [5] where the first provably convergent domain decomposition (DD) algorithm for the Helmholtz equation was proposed and then extended to Maxwell's equations in [6], other studies followed. Preliminary attempts to obtain better algorithms for this kind of equations were given in [3, 4, 12], where the first ideas of optimized Schwarz methods can be found. Then, the advantage of the optimization process was used for the second order Maxwell system in [1]. Later on, an entire hierarchy of optimized transmission conditions for the first order Maxwell's equations was proposed in [9, 11] . For the second order or curl-curl Maxwell's equations second order optimized transmission conditions can be found in [14, 15, 16, 17]. We study here optimized Schwarz DD methods for the time-harmonic Maxwell equations discretized by a DG method. Due to the particularity of the latter, DG discretization applied to more sophisticated Schwarz methods is not straightforward. In this work we show a strategy of discretization and prove the equivalence between multi-domain and single-domain solutions. The proposed discrete framework is then illustrated by some numerical results in the two-dimensional case.

We consider time-harmonic Maxwell's equations in a homogeneous medium written as a first order system (see [10] for more details)

$$G_0\mathbf{W} + G_x\partial_x\mathbf{W} + G_y\partial_y\mathbf{W} + G_z\partial_z\mathbf{W} = 0, \tag{1}$$

where

$$\mathbf{W} = \begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}, G_0 = \begin{pmatrix}(\sigma + i\omega)\mathbb{I}_{3\times3} & 0_{3\times3}\\ 0_{3\times3} & i\omega\mathbb{I}_{3\times3}\end{pmatrix}$$

with $\mathbf{E}$, $\mathbf{H}$ the complex-valued electric and magnetic fields, $\omega$ the angular frequency of the time-harmonic wave, $\sigma$ the electric conductivity. For a general vector $\mathbf{n} =$

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$\left(n_x\ n_y\ n_z\right)$, we also define the matrices

$$G_{\mathbf{n}} = \begin{pmatrix} 0_{3\times 3} & N_{\mathbf{n}} \\ N_{\mathbf{n}}^T & 0_{3\times 3} \end{pmatrix} \text{ and } N_{\mathbf{n}} = \begin{pmatrix} 0 & n_z & -n_y \\ -n_z & 0 & n_x \\ n_y & -n_x & 0 \end{pmatrix}.$$

Then, for $l \in \{x,y,z\}$, we have that $N_l = N_{\mathbf{e}_l}$ and $G_l = G_{\mathbf{e}_l}$, where $\mathbf{e}_l$, $l = 1,2,3$ are the canonical basis vectors. Our goal is to solve the boundary-value problem

$$\begin{aligned} G_0\mathbf{W} + G_x\partial_x\mathbf{W} + G_y\partial_y\mathbf{W} + G_z\partial_z\mathbf{W} &= 0 \text{ in } \Omega, \\ (M_{\Gamma_m} - G_{\mathbf{n}})\mathbf{W} = 0 \text{ on } \Gamma_m \ \text{ and } \ (M_{\Gamma_a} - G_{\mathbf{n}})(\mathbf{W} - \mathbf{W}_{\text{inc}}) &= 0 \text{ on } \Gamma_a, \end{aligned} \tag{2}$$

where $\mathbf{W}_{\text{inc}}$ is a given *incident field*, while $M_{\Gamma_m}$ and $M_{\Gamma_a}$ are trace operators defined on the *metallic* and *absorbing* boundaries $\Gamma_m$ and $\Gamma_a$ (see [10] for more details)

$$M_{\Gamma_m} = \begin{pmatrix} 0_{3\times 3} & N_{\mathbf{n}} \\ -N_{\mathbf{n}}^T & 0_{3\times 3} \end{pmatrix} \text{ and } M_{\Gamma_a} = |G_{\mathbf{n}}| = \begin{pmatrix} N_{\mathbf{n}}N_{\mathbf{n}}^T & 0_{3\times 3} \\ 0_{3\times 3} & N_{\mathbf{n}}^T N_{\mathbf{n}} \end{pmatrix}.$$

The matrices $G_{\mathbf{n}}^+$ and $G_{\mathbf{n}}^-$ are the positive and negative parts of $G_{\mathbf{n}}$ based on its diagonalization and we have that $|G_{\mathbf{n}}| = G_{\mathbf{n}}^+ - G_{\mathbf{n}}^-$.

## 2 Continuous classical and optimized Schwarz algorithms

We now decompose the domain $\Omega$ into two non-overlapping subdomains $\Omega_1$ and $\Omega_2$, and denote by $\Sigma$ the interface between $\Omega_1$ and $\Omega_2$, by $\mathbf{W}_j$ the restriction of $\mathbf{W}$ to $\Omega_j$ and by $\mathbf{n}$ the unit outward normal vector to $\Sigma$ directed from $\Omega_1$ to $\Omega_2$. Schwarz algorithms consist in computing iteratively $\mathbf{W}_j^{n+1}$ from $\mathbf{W}_j^n$, for $j = 1,2$

$$\begin{aligned} G_0\mathbf{W}_1^{n+1} + G_x\partial_x\mathbf{W}_1^{n+1} + G_y\partial_y\mathbf{W}_1^{n+1} + G_z\partial_z\mathbf{W}_1^{n+1} &= 0, \text{ in } \Omega_1, \\ (G_{\mathbf{n}}^- + S_1 G_{\mathbf{n}}^+)\mathbf{W}_1^{n+1} &= (G_{\mathbf{n}}^- + S_1 G_{\mathbf{n}}^+)\mathbf{W}_2^n, \text{ on } \Sigma, \\ G_0\mathbf{W}_2^{n+1} + G_x\partial_x\mathbf{W}_2^{n+1} + G_y\partial_y\mathbf{W}_2^{n+1} + G_z\partial_z\mathbf{W}_2^{n+1} &= 0, \text{ in } \Omega_2, \\ (G_{\mathbf{n}}^+ + S_2 G_{\mathbf{n}}^-)\mathbf{W}_2^{n+1} &= (G_{\mathbf{n}}^+ + S_2 G_{\mathbf{n}}^-)\mathbf{W}_1^n, \text{ on } \Sigma, \end{aligned} \tag{3}$$

where $S_1$ and $S_2$ are differential operators. When $S_1 = S_2 = 0_{6\times 6}$, the interface conditions become the positive and negative flux operators $G_{\mathbf{n}}^+$ and $G_{\mathbf{n}}^-$, and the *classical Schwarz algorithm* is obtained. Applying $G_{\mathbf{n}}^+$ (respectively $G_{\mathbf{n}}^-$) to a vector $\mathbf{W}$ means to select the characteristic variables associated to out-going (respectively in-coming) waves, which is very natural considering the hyperbolic nature of the problem, see [9] (section 3.1). We note that

| Algorithm | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\mathscr{F}(\tilde{S}_j)$ | 0 | $-\frac{s-i\omega}{s+i\omega}$ | $-\frac{k^2+i\omega\sigma}{k^2-2\omega^2+i\omega\sigma+2i\omega s}$ | $-\frac{s_j-i\omega}{s_j+i\omega}$ | $-\frac{k^2+i\omega\sigma}{k^2-2\omega^2+i\omega\sigma+2i\omega s_j}$ |

**Table 1** Five different choices for the symbols of the operators in the transmission conditions (6) leading to five different optimized Schwarz algorithms

$$\begin{aligned} G_{\mathbf{n}}^- &= \begin{pmatrix} -N_{\mathbf{n}}N_{\mathbf{n}}^T & N_{\mathbf{n}} \\ N_{\mathbf{n}}^T & -N_{\mathbf{n}}^T N_{\mathbf{n}} \end{pmatrix} = \begin{pmatrix} \mathbb{I}_{3\times3} \\ -N_{\mathbf{n}}^T \end{pmatrix} \left( -N_{\mathbf{n}}N_{\mathbf{n}}^T \; N_{\mathbf{n}} \right), \\ G_{\mathbf{n}}^+ &= \begin{pmatrix} N_{\mathbf{n}}N_{\mathbf{n}}^T & N_{\mathbf{n}} \\ N_{\mathbf{n}}^T & N_{\mathbf{n}}^T N_{\mathbf{n}} \end{pmatrix} = \begin{pmatrix} \mathbb{I}_{3\times3} \\ N_{\mathbf{n}}^T \end{pmatrix} \left( N_{\mathbf{n}}N_{\mathbf{n}}^T \; N_{\mathbf{n}} \right). \end{aligned} \tag{4}$$

Thus the classical transmission conditions are equivalent to impedance conditions,

$$\begin{aligned} G_{\mathbf{n}}^-\mathbf{W}_1^{n+1} &= G_{\mathbf{n}}^-\mathbf{W}_2^n \Leftrightarrow \mathscr{B}_{\mathbf{n}}(\mathbf{E}_1^{n+1},\mathbf{H}_1^{n+1}) = \mathscr{B}_{\mathbf{n}}(\mathbf{E}_2^n,\mathbf{H}_2^n), \\ G_{\mathbf{n}}^+\mathbf{W}_2^{n+1} &= G_{\mathbf{n}}^+\mathbf{W}_1^n \Leftrightarrow \mathscr{B}_{-\mathbf{n}}(\mathbf{E}_2^{n+1},\mathbf{H}_2^{n+1}) = \mathscr{B}_{-\mathbf{n}}(\mathbf{E}_1^n,\mathbf{H}_1^n). \end{aligned} \tag{5}$$

with $\mathscr{B}_{\mathbf{n}}(\mathbf{E},\mathbf{H}) = N_{\mathbf{n}}^T\mathbf{E} - N_{\mathbf{n}}^T N_{\mathbf{n}}\mathbf{H}$. For $\Omega_2$ we have used the fact that $G_{\mathbf{n}}^+ = -G_{-\mathbf{n}}^-$. The classical Schwarz algorithm is adopted in [10] together with low order DG methods in the 3D case. Along the lines of (5), we have the equivalences

$$\begin{aligned} &(G_{\mathbf{n}}^- + S_1G_{\mathbf{n}}^+)\mathbf{W}_1^{n+1} = (G_{\mathbf{n}}^- + S_1G_{\mathbf{n}}^+)\mathbf{W}_2^n \\ &\quad \Leftrightarrow (\mathscr{B}_{\mathbf{n}} + \tilde{S}_1\mathscr{B}_{-\mathbf{n}})(\mathbf{E}_1^{n+1},\mathbf{H}_1^{n+1}) = (\mathscr{B}_{\mathbf{n}} + \tilde{S}_1\mathscr{B}_{-\mathbf{n}})(\mathbf{E}_2^n,\mathbf{H}_2^n), \\ &(G_{\mathbf{n}}^+ + S_2G_{\mathbf{n}}^-)\mathbf{W}_2^{n+1} = (G_{\mathbf{n}}^+ + S_2G_{\mathbf{n}}^-)\mathbf{W}_1^n \\ &\quad \Leftrightarrow (\mathscr{B}_{-\mathbf{n}} + \tilde{S}_2\mathscr{B}_{\mathbf{n}})(\mathbf{E}_2^{n+1},\mathbf{H}_2^{n+1}) = (\mathscr{B}_{-\mathbf{n}} + \tilde{S}_2\mathscr{B}_{\mathbf{n}})(\mathbf{E}_1^n,\mathbf{H}_1^n), \end{aligned} \tag{6}$$

where $\tilde{S}_1$ and $\tilde{S}_2$ denote differential operators which are approximations of the transparent operators. From these transparent operators we can obtain a hierarchy of optimized algorithms with appropriate choices for $\tilde{S}_1$ and $\tilde{S}_2$ [11]. The operators $S_1$ and $S_2$ are eventually defined to guarantee the equivalences in (6).

If we consider the TM formulation of Maxwell's equations, that is with $\mathbf{E} = (0\ 0\ E_z)^T$ and $\mathbf{H} = (H_x\ H_y\ 0)^T$, then $\mathbf{W} = (E_z\ H_x\ H_y)^T$, $N_{\mathbf{n}} = (n_y\ -n_x)^T$, and

$$G_0 = \begin{pmatrix} \sigma + i\omega & 0_{1\times2} \\ 0_{2\times1} & i\omega\mathbb{I}_{2\times2} \end{pmatrix}, \quad G_x = \begin{pmatrix} 0 & N_{\mathbf{e}_x} \\ N_{\mathbf{e}_x}^T & 0 \end{pmatrix} \quad \text{and} \quad G_y = \begin{pmatrix} 0 & N_{\mathbf{e}_y} \\ N_{\mathbf{e}_y}^T & 0 \end{pmatrix}.$$

We give in Table 1 the symbols $\mathscr{F}(\tilde{S}_j)$ of $\tilde{S}_j$ in the 2d case for conductive media for five different Schwarz algorithms, where the parameters $s = p(1+i)$, $s_1 = p_1(1+i)$ and $s_2 = p_2(1+i)$ are solutions of some min-max problems, as explained in [11] (section 5, table 5.1). Note that the Fourier symbols of the operators in algorithms 1, 2 and 4 are constants, therefore they have the same expression as in the physical space. In this case (6) can be written in the 2d situation considered here as

$$\begin{aligned} E_1^{n+1} - N_{\mathbf{n}}\mathbf{H}_1^{n+1} + \tilde{S}_1(E_1^{n+1} + N_{\mathbf{n}}\mathbf{H}_1^{n+1}) &= E_2^n - N_{\mathbf{n}}\mathbf{H}_2^n + \tilde{S}_1(E_2^n + N_{\mathbf{n}}\mathbf{H}_2^n), \\ E_2^{n+1} + N_{\mathbf{n}}\mathbf{H}_2^{n+1} + \tilde{S}_2(E_2^{n+1} - N_{\mathbf{n}}\mathbf{H}_2^{n+1}) &= E_1^n + N_{\mathbf{n}}\mathbf{H}_1^n + \tilde{S}_2(E_1^n - N_{\mathbf{n}}\mathbf{H}_1^n). \end{aligned} \tag{7}$$

This is not the case for algorithms 3 and 5 which involved second order transmission conditions. Here, the $\tilde{S}_j$ are operators whose Fourier symbols have the form

$$\mathscr{F}(\tilde{S}_j) = \frac{q_j(k)}{r_j(k)} \text{ with } q_j(k) = -(k^2 + i\omega\sigma) \text{ and } r_j(k) = k^2 - 2\omega^2 + i\omega\sigma + 2i\omega s_j.$$

where the Fourier variable $k$ corresponds to a transform with respect to the tangential direction $\tau$ along the interface, assuming a two-subdomain decomposition with a straight interface. In that case, $\mathscr{F}^{-1}(q_j)$ and $\mathscr{F}^{-1}(r_j)$ are partial differential operators in the $\tau$ variable,

$$\mathscr{F}^{-1}(q_j) = \partial_{\tau\tau} - i\omega\sigma,\ \mathscr{F}^{-1}(r_j) = -\partial_{\tau\tau} - 2\omega^2 + i\omega\sigma + 2i\omega s_j,\ s_j \in \mathbb{C},$$

and (7) can be re-written as

$$\begin{aligned}
\mathscr{F}^{-1}\left(r_1(E_1^{n+1} - N_{\mathbf{n}}\mathbf{H}_1^{n+1})\right) &+ \mathscr{F}^{-1}\left(q_1(E_1^{n+1} + N_{\mathbf{n}}\mathbf{H}_1^{n+1})\right) \\
&= \mathscr{F}^{-1}\left(r_1(E_2^{n} - N_{\mathbf{n}}\mathbf{H}_2^{n})\right) + \mathscr{F}^{-1}\left(q_1(E_2^{n} + N_{\mathbf{n}}\mathbf{H}_2^{n})\right), \\
\mathscr{F}^{-1}\left(r_2(E_2^{n+1} + N_{\mathbf{n}}\mathbf{H}_2^{n+1})\right) &+ \mathscr{F}^{-1}\left(q_2(E_2^{n+1} - N_{\mathbf{n}}\mathbf{H}_2^{n+1})\right) \\
&= \mathscr{F}^{-1}\left(r_2(E_1^{n} + N_{\mathbf{n}}\mathbf{H}_1^{n})\right) + \mathscr{F}^{-1}\left(q_2(E_1^{n} - N_{\mathbf{n}}\mathbf{H}_1^{n})\right).
\end{aligned}$$

## 3 Discontinuous Galerkin approximation

Let $\mathscr{T}_h$ be a discretization of $\Omega$ and $\Gamma^0$, $\Gamma^m$ and $\Gamma^a$ be the sets of purely internal, metallic and absorbing faces of $\mathscr{T}_h$. We denote by $K$ an element of $\mathscr{T}_h$ and by $F = K \cap \tilde{K}$ the face shared by two neighboring elements $K$ and $\tilde{K}$. On this face $F$, we define the *average* by $\{\mathbf{W}\} = \frac{1}{2}(\mathbf{W}_K + \mathbf{W}_{\tilde{K}})$ and the *tangential trace jump* by $[[\mathbf{W}]] = G_{\mathbf{n}_K}\mathbf{W}_K + G_{\mathbf{n}_{\tilde{K}}}\mathbf{W}_{\tilde{K}}$. For two vector functions $\mathbf{U}$ and $\mathbf{V}$ in $(L^2(D))^6$, we denote $(\mathbf{U},\mathbf{V})_D = \int_D \mathbf{U}\cdot\overline{\mathbf{V}}\,\mathrm{d}x$, if $D$ is a domain of $\mathbb{R}^3$ and $\langle \mathbf{U},\mathbf{V}\rangle_F = \int_F \mathbf{U}\cdot\overline{\mathbf{V}}\,\mathrm{d}s$ if $F$ is a face of $\mathbb{R}^2$. For sake of simplicity, we will skip some subscripts, that is $(\cdot,\cdot) = (\cdot,\cdot)_{\mathscr{T}_h} = \sum_{K\in\mathscr{T}_h}(\cdot,\cdot)_K$. On the boundaries we define

$$M_{F,K} = \begin{cases} \begin{pmatrix} \eta_F N_{\mathbf{n}_K} N_{\mathbf{n}_K}^T & N_{\mathbf{n}_K} \\ -N_{\mathbf{n}_K}^T & 0_{3\times 3} \end{pmatrix} \text{ with } \eta_F \neq 0, \text{ if } F \text{ belongs to } \Gamma^m, \\ |G_{\mathbf{n}_K}| \text{ if } F \text{ belongs to } \Gamma^a. \end{cases}$$

Using these notations, the weak formulation of the problem is

$$\begin{aligned}
(G_0\mathbf{W},\mathbf{V}) + \left(\sum_{l\in\{x,y,z\}} G_l\partial_l\mathbf{W},\mathbf{V}\right) - \sum_{F\in\Gamma^0}\langle[[\mathbf{W}]],\{\mathbf{V}\}\rangle_F + \sum_{F\in\Gamma^0}\left\langle\frac{1}{2}[[\mathbf{W}]],\{\mathbf{V}\}\right\rangle_F \\
+ \sum_{F\in\Gamma^m\cup\Gamma^a}\left\langle\frac{1}{2}(M_{F,K} - G_{\mathbf{n}_K})\mathbf{W},\mathbf{V}\right\rangle_F = \sum_{F\in\Gamma^a}\left\langle\frac{1}{2}(M_{F,K} - G_{\mathbf{n}_K})\mathbf{W}_{\mathrm{inc}},\mathbf{V}\right\rangle_F.
\end{aligned}$$

Note that we have implicitly adopted an upwind scheme for the calculation of the boundary integral over an internal face $F \in \Gamma^0$. An alternative choice is that of a centered scheme. Both of these options are discussed and compared in [8]. Let $\mathbb{P}_p(D)$ denote the space of polynomial functions of degree at most $p$ on a domain $D$. For any element $K \in \mathscr{T}_h$, let $\mathbf{D}^p(K) \equiv (\mathbb{P}_p(K))^6$. The vectors $\mathbf{W}$ and $\mathbf{V}$ will be taken in the space $\mathbf{D}_h^p = \left\{\mathbf{V} \in (L^2(\Omega))^6 \mid \mathbf{V}|_K \in \mathbf{D}^p(K),\ \forall K \in \mathscr{T}_h\right\}$.

For the discretization of optimized transmission conditions, let $\Gamma_\Sigma$ be the set of faces on $\Sigma$, $\Gamma_0^j$ be the set of interior faces of $\Omega_j$ and $\Gamma_b^j$ be the set of faces of $\Omega_j$ lying on $\partial\Omega$. Then the weak form in the two-subdomain case can be written as

$$\begin{aligned}
&\mathscr{L}(\mathbf{W}_1,\mathbf{V}_1)+\sum_{\Gamma_0^1}\diamond+\sum_{\Gamma_b^1}\diamond+\sum_{F\in\Gamma_\Sigma}\left\langle\frac{1}{2}\left(|G_{\mathbf{n}_K}|-G_{\mathbf{n}_K}\right)(\mathbf{W}_1-\mathbf{W}_2),\mathbf{V}_1\right\rangle_F=0,\\
&\mathscr{L}(\mathbf{W}_2,\mathbf{V}_2)+\sum_{\Gamma_0^2}\diamond+\sum_{\Gamma_b^2}\diamond+\sum_{F\in\Gamma_\Sigma}\left\langle\frac{1}{2}\left(|G_{\mathbf{n}_{\tilde{K}}}|-G_{\mathbf{n}_{\tilde{K}}}\right)(\mathbf{W}_2-\mathbf{W}_1),\mathbf{V}_2\right\rangle_F=0,
\end{aligned}\tag{8}$$

where $\mathscr{L}(\mathbf{W}_j,\mathbf{V}_j) \equiv (G_0\mathbf{W}_j,\mathbf{V}_j)+(\sum_l G_l\partial_l\mathbf{W}_j,\mathbf{V}_j)$ and, for simplicity, we have replaced some terms on the faces that are not important for the presentation by a $\diamond$. For any face $F = K\cap\tilde{K}$ on $\Sigma$, if $\mathbf{n}$ denotes the normal on $\Sigma$ directed from $\Omega_1$ towards $\Omega_2$, and $K$ and $\tilde{K}$ are elements of $\Omega_1$ and $\Omega_2$, we have $\mathbf{n}_K = \mathbf{n} = -\mathbf{n}_{\tilde{K}}$. In order to simplify the notation, we make use of $G_{\mathbf{n}}^- = \frac{1}{2}(G_{\mathbf{n}}-|G_{\mathbf{n}}|)$ and $G_{\mathbf{n}}^+ = \frac{1}{2}(G_{\mathbf{n}}+|G_{\mathbf{n}}|)$. Then, starting from initial guesses $\mathbf{W}_1^0$ and $\mathbf{W}_2^0$, the classical Schwarz algorithm computes the iterates $\mathbf{W}_j^{n+1}$ from $\mathbf{W}_j^n$ by solving on $\Omega_1$ and $\Omega_2$ the subproblems

$$\begin{aligned}
&\mathscr{L}(\mathbf{W}_1^{n+1},\mathbf{V}_1)+\sum_{\Gamma_0^1}\diamond+\sum_{\Gamma_b^1}\diamond-\sum_{F\in\Gamma_\Sigma}\left\langle G_{\mathbf{n}}^-(\mathbf{W}_1^{n+1}-\mathbf{W}_2^n),\mathbf{V}_1\right\rangle_F=0,\\
&\mathscr{L}(\mathbf{W}_2^{n+1},\mathbf{V}_2)+\sum_{\Gamma_0^2}\diamond+\sum_{\Gamma_b^2}\diamond+\sum_{F\in\Gamma_\Sigma}\left\langle G_{\mathbf{n}}^+(\mathbf{W}_2^{n+1}-\mathbf{W}_1^n),\mathbf{V}_2\right\rangle_F=0.
\end{aligned}\tag{9}$$

In order to introduce optimized transmission conditions (3) into the DG discretization, we first want to show explicitly what transmission conditions the classical relaxation in (9) corresponds to. To do so, the subdomain problems solved in (9) are not allowed to depend on variables of the other subdomain anymore, since the coupling will be performed with the transmission conditions, and we thus need to introduce additional unknowns, namely $\mathbf{W}_{2,\Omega_1}^{n+1}$ on $\Omega_1$ and $\mathbf{W}_{1,\Omega_2}^{n+1}$ on $\Omega_2$, in order to write the classical Schwarz iteration with local variables only, *i.e.*

$$\begin{aligned}
&\mathscr{L}(\mathbf{W}_1^{n+1},\mathbf{V}_1)+\sum_{\Gamma_0^1}\diamond+\sum_{\Gamma_b^1}\diamond-\sum_{F\in\Gamma_\Sigma}\left\langle G_{\mathbf{n}}^-(\mathbf{W}_1^{n+1}-\mathbf{W}_{2,\Omega_1}^{n+1}),\mathbf{V}_1\right\rangle_F=0,\\
&\mathscr{L}(\mathbf{W}_2^{n+1},\mathbf{V}_2)+\sum_{\Gamma_0^2}\diamond+\sum_{\Gamma_b^2}\diamond+\sum_{F\in\Gamma_\Sigma}\left\langle G_{\mathbf{n}}^+(\mathbf{W}_2^{n+1}-\mathbf{W}_{1,\Omega_2}^{n+1}),\mathbf{V}_2\right\rangle_F=0.
\end{aligned}\tag{10}$$

Comparing with the classical Schwarz algorithm (9), we see that in order to obtain the same algorithm, the transmission conditions for (10) need to be chosen as $G_{\mathbf{n}}^-\mathbf{W}_{2,\Omega_1}^{n+1} = G_{\mathbf{n}}^-\mathbf{W}_2^n$ and $G_{\mathbf{n}}^+\mathbf{W}_{1,\Omega_2}^{n+1} = G_{\mathbf{n}}^+\mathbf{W}_1^n$, which implies that at the limit, when the algorithm converges, we must verify the coupling conditions

$$G_{\mathbf{n}}^{-}\mathbf{W}_{2,\Omega_1} = G_{\mathbf{n}}^{-}\mathbf{W}_2, \qquad G_{\mathbf{n}}^{+}\mathbf{W}_{1,\Omega_2} = G_{\mathbf{n}}^{+}\mathbf{W}_1, \tag{11}$$

where we dropped the iteration index to denote the limit quantities. The Schwarz algorithm (10) can however also be used with optimized transmission conditions (3), which have to be the DG discretization of the strong relations

$$\begin{aligned} G_{\mathbf{n}}^{-}\mathbf{W}_{2,\Omega_1}^{n+1} + S_1 G_{\mathbf{n}}^{+}\mathbf{W}_1^{n+1} &= G_{\mathbf{n}}^{-}\mathbf{W}_2^{n} + S_1 G_{\mathbf{n}}^{+}\mathbf{W}_{1,\Omega_2}^{n}, \\ G_{\mathbf{n}}^{+}\mathbf{W}_{1,\Omega_2}^{n+1} + S_2 G_{\mathbf{n}}^{-}\mathbf{W}_2^{n+1} &= G_{\mathbf{n}}^{+}\mathbf{W}_1^{n} + S_2 G_{\mathbf{n}}^{-}\mathbf{W}_{2,\Omega_1}^{n}. \end{aligned} \tag{12}$$

Then, we want to show the equivalence between (11) and the DG discretization we adopt for the transmission conditions (12) at convergence in a 2d case. First, from (4) note that relation (11) is equivalent to

$$\begin{aligned} N_{\mathbf{n}}N_{\mathbf{n}}^{T}\mathbf{E}_{2,\Omega_1} - N_{\mathbf{n}}\mathbf{H}_{2,\Omega_1} &= N_{\mathbf{n}}N_{\mathbf{n}}^{T}\mathbf{E}_2 - N_{\mathbf{n}}\mathbf{H}_2, \\ N_{\mathbf{n}}N_{\mathbf{n}}^{T}\mathbf{E}_{1,\Omega_2} + N_{\mathbf{n}}\mathbf{H}_{1,\Omega_2} &= N_{\mathbf{n}}N_{\mathbf{n}}^{T}\mathbf{E}_1 + N_{\mathbf{n}}\mathbf{H}_1. \end{aligned} \tag{13}$$

We translate these relations using auxiliary variables $\Lambda_{2,\Omega_1} := E_{2,\Omega_1} - N_{\mathbf{n}}\mathbf{H}_{2,\Omega_1}$, $\Lambda_2 := E_2 - N_{\mathbf{n}}\mathbf{H}_2$, $\Lambda_{1,\Omega_2} := E_{1,\Omega_2} + N_{\mathbf{n}}\mathbf{H}_{1,\Omega_2}$ and $\Lambda_1 := E_1 + N_{\mathbf{n}}\mathbf{H}_1$ belonging to the trace space $M_h^p = \left\{\eta \in L^2(\Sigma) \mid \eta|_F \in \mathbb{P}_p(F), \forall F \in \Sigma\right\}$. Then (13) becomes

$$\Lambda_{2,\Omega_1} = \Lambda_2 \text{ and } \Lambda_{1,\Omega_2} = \Lambda_1. \tag{14}$$

From (12) and (14), we have to find for optimized transmission conditions a suitable DG discretization of the relations

$$\Lambda_{2,\Omega_1} + \tilde{S}_1\Lambda_1 = \Lambda_2 + \tilde{S}_1\Lambda_{1,\Omega_2} \text{ and } \Lambda_{1,\Omega_2} + \tilde{S}_2\Lambda_2 = \Lambda_1 + \tilde{S}_2\Lambda_{2,\Omega_1}. \tag{15}$$

We focus on the case of second order transmission conditions and (15) becomes

$$\begin{aligned} (-\partial_\tau^2 + i\omega\sigma - 2\omega^2 + 2i\omega s_1)(\Lambda_{2,\Omega_1} - \Lambda_2) + (-\partial_\tau^2 + i\omega\sigma)(\Lambda_{1,\Omega_2} - \Lambda_1) = 0, \\ (-\partial_\tau^2 + i\omega\sigma - 2\omega^2 + 2i\omega s_2)(\Lambda_{1,\Omega_2} - \Lambda_1) + (-\partial_\tau^2 + i\omega\sigma)(\Lambda_{2,\Omega_1} - \Lambda_2) = 0. \end{aligned} \tag{16}$$

Let $(\eta_j)_j$ be a basis of $M_h^p$. We define the discrete matrices $M_\Sigma$ and $K_\Sigma$ by

$$\begin{aligned} (M_\Sigma)_{i,j} &= \sum_{F\in\Sigma} \langle \eta_i, \eta_j \rangle_F, \\ (K_\Sigma)_{i,j} &= \sum_{F\in\Sigma} \langle \partial_\tau\eta_i, \partial_\tau\eta_j \rangle_F + \sum_{n\in\Sigma^0} \alpha_n h^{-1} [[[[\eta_i]]]]_n [[[[\eta_j]]]]_n \\ &\quad - \sum_{n\in\Sigma^0} \{\{\partial_\tau\eta_i\}\}_n [[[[\eta_j]]]]_n - [[[[\eta_i]]]]_n \{\{\partial_\tau\eta_j\}\}_n, \end{aligned}$$

where positiveness is guaranteed for sufficiently large $\alpha_n$, $\Sigma^0$ denotes the set of interior nodes of $\Sigma$, $[[[[\cdot]]]]_n$ and $\{\{\cdot\}\}_n$ denotes the jump and the average at a node $n$ between values of the neighboring segments. The matrix $K_\Sigma$ comes from the discretization of $-\partial_\tau^2$ using a symmetric interior penalty approach [2]. If we denote by $A_\Sigma = (K_\Sigma + i\omega\sigma M_\Sigma)$, the DG discretization of (16) we consider is

$$\begin{pmatrix} A_\Sigma - 2(\omega^2 - i\omega s_1)M_\Sigma & A_\Sigma \\ A_\Sigma & A_\Sigma - 2(\omega^2 - i\omega s_2)M_\Sigma \end{pmatrix} \begin{pmatrix} \Lambda_{2,\Omega_1} - \Lambda_2 \\ \Lambda_{1,\Omega_2} - \Lambda_1 \end{pmatrix} = 0. \qquad (17)$$

**Theorem 1.** *If $s_1$ and $s_2$ are defined as given in [11] (section 5, table 5.1) then relations (14) and (17) are equivalent.*

The proof is based on the invertibility of the matrix of (17) and can be found in [7].

## 4 Numerical results

In order to illustrate numerically the proposed discrete versions of the optimized Schwarz algorithms, we consider the propagation of a plane wave in a homogeneous conductive medium with $\Omega = [0,1]^2$ and $\sigma = 0.5$. We use DG with several orders of polynomial interpolation, denoted by DG-$\mathbb{P}_k$ with $k = 1,2,3,4$, and impose on $\partial\Omega = \Gamma_a$ an incident wave $\mathbf{W}_{\text{inc}} = (\ \frac{k_y}{\omega}\ \frac{-k_x}{\omega}\ 1\ )^T e^{-i\mathbf{k}\cdot\mathbf{x}}$, and $\mathbf{k} = (\ k_x\ k_y\ )^T = (\ \omega\sqrt{1 - i\frac{\sigma}{\omega}}\ 0\ )^T$. The domain $\Omega$ is decomposed into two subdomains $\Omega_1 = [0,0.5] \times [0,1]$ and $\Omega_2 = [0.5,1] \times [0,1]$. The aim is to retrieve numerically the asymptotic behavior of the convergence factors of the optimized Schwarz methods. It has been proved that these factors behave like $1 - O(h^{\alpha_i})$, $i = 2,3,4,5$. We show here that numerically they behave like $1 - O(h^{\beta_i})$, $i = 2,3,4,5$, with $\beta_i \approx \alpha_i$. The performance of these algorithms is summarized in Figure 1.

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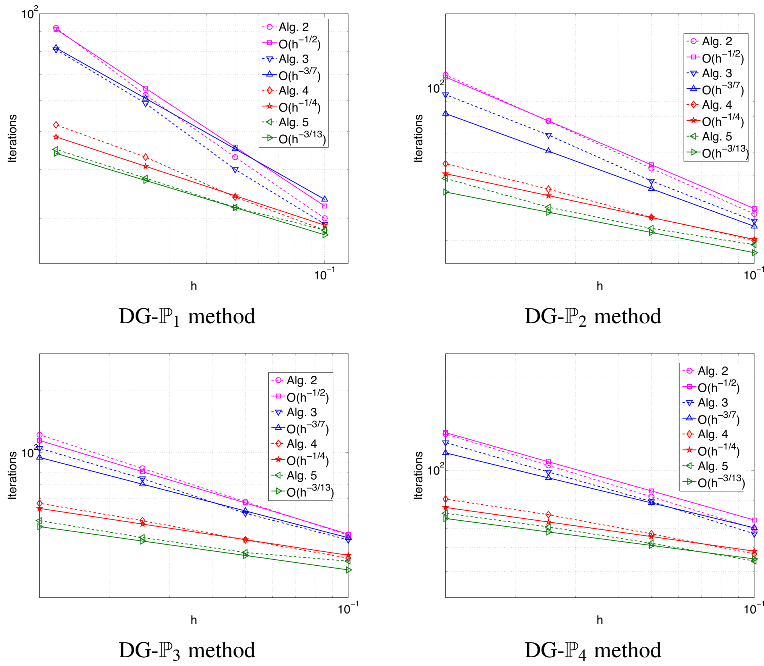


**Fig. 1** Wave propagation in a homogeneous medium. Iteration count vs. $h$.

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