On Iterative Substructuring Methods for Multiscale Problems

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1 Introduction

Model Problem Let $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ be a Lipschitz polytope with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, where $\Gamma_D \cap \Gamma_N = \emptyset$. We are interested in finding $u_h \in V_D^h(\Omega)$ such that

$$\int_\Omega \alpha \nabla u_h \cdot \nabla v_h \, dx \; = \; \langle f, v_h \rangle \qquad \forall u_h \in V_D^h(\Omega). \tag{1}$$

Above, $V_D^h(\Omega)$ denotes the finite element space of continuous and piecewise linear functions with respect to a mesh $\mathscr{T}^h(\Omega)$ that vanish on the Dirichlet boundary Γ_D. The functional $f \in V_D^h(\Omega)^*$ is assumed to be composed of a volume integral over Ω and a surface integral over Γ_N.

The diffusion coefficient $\alpha \in L^\infty(\Omega)$ is assumed to be uniformly positive, i.e., $\text{ess.inf}_{x\in\Omega}\, \alpha(x) > 0$. We allow α to vary by several orders of magnitude in an unstructured way throughout the domain Ω. In particular, we allow α to be discontinuous and exhibit large jumps (high contrast). If the jumps occur at a scale $\eta \ll \text{diam}(\Omega)$, one speaks of a *multiscale problem* (cf. e.g., [1]).

Problem (1) is equivalent to the linear system

$$K_{h,\alpha}\, \underline{u}_h \; = \; \underline{f}_h\,, \tag{2}$$

where the stiffness matrix $K_{h,\alpha}$ and load vector $\underline{f}_h$ are defined with respect to the standard nodal basis of $V_D^h(\Omega)$. For a quasi-uniform mesh, one easily shows that

$$\kappa(K_{h,\alpha}) \leq C\, \frac{\text{ess.sup}_{x\in\Omega}\, \alpha(x)}{\text{ess.inf}_{x\in\Omega}\, \alpha(x)}\, h^{-2}\,.$$

Although in many cases, this might be a pessimistic bound, it is sharp in general. Consequently, an ideal preconditioner for $K_{h,\alpha}$ should be robust in (i) the contrast in α, (ii) the mesh size h, (iii) the scale η at which the coefficient varies, where here we may assume that $h \leq \eta \leq \text{diam}(\Omega)$.

Spectral Properties and the Weighted Poincaré Inequality To get an idea, how difficult it is to precondition System (2), we display the entire *spectrum* of $K_{h,\alpha}$ for the pure Neumann problem ($\Gamma_D = \emptyset$) on the unit square $\Omega = (0,1)^2$ and for three coefficient distributions α (see the top row of Fig. 1). The smallest eigenvalue of $K_{h,\alpha}$ is always zero and not shown in the following plots.

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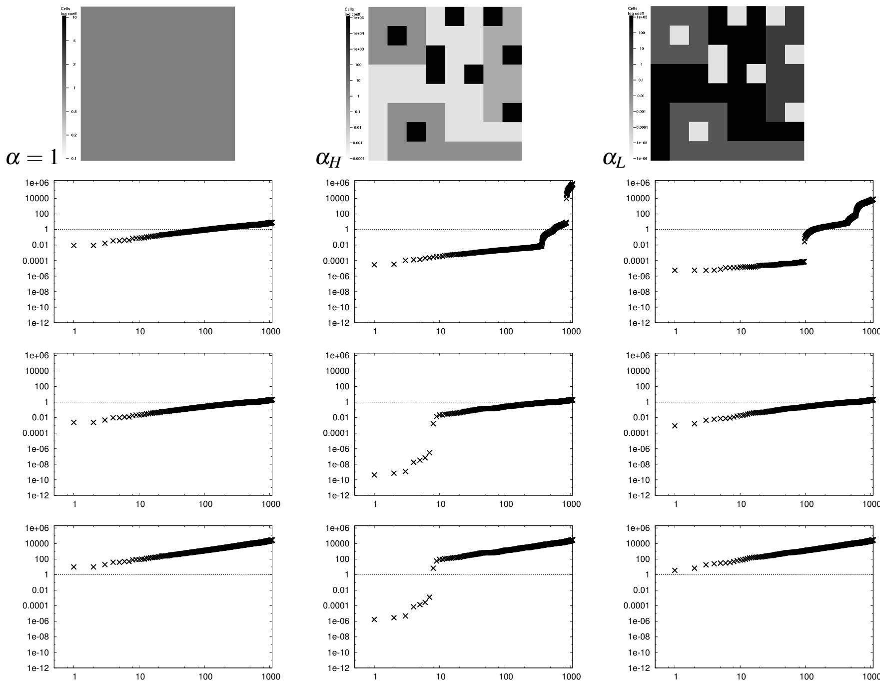

Fig. 1 Top row: three coefficient distributions α. Second row: spectra $\sigma(K_{h,\alpha})$ corresponding to the three distributions. Third row: $\sigma(\mathrm{diag}(K_{h,\alpha})^{-1}K_{h,\alpha})$. Bottom row: $\sigma(M_{h,\alpha}^{-1}K_{h,\alpha})$. In each case structured mesh with mesh size $h = 1/32$. The contrast for $\alpha_H = \alpha_L^{-1}$ is 10^8.

The second row of Fig. 1 displays $\sigma(K_{h,\alpha})$. We see that compared to the reference coefficient $\alpha = 1$, the spectrum is distorted in the two other cases α_H, α_L.

In the third and fourth row, we change the point of view, and display the spectrum of $\mathrm{diag}(K_{h,\alpha})^{-1}K_{h,\alpha}$ and of $M_{h,\alpha}^{-1}K_{h,\alpha}$, where $M_{h,\alpha}$ denotes the weighted mass matrix corresponding to the inner product $(v,\, w)_{L^2(\Omega),\alpha} := \int_\Omega \alpha\, v\, w\, dx$. On a quasi-uniform mesh, one can easily show that $\mathrm{diag}(K_{h,\alpha})$ and $h^{-2}M_{h,\alpha}$ are spectrally equivalent with uniform constants. For this reason, the spectra in the third and fourth row differ mainly by a simple shift. For coefficient α_H, with 8 inclusions of large values (plotted in black), we obtain 7 additional small eigenvalues compared to the reference coefficient. This fact has been theoretically shown by Graham & Hagger [10].

For coefficient α_L, with 8 inclusions of small values (plotted in light grey), the spectra are essentially the same as for the reference coefficient. The theoretical explanation of this fact is the so-called *weighted Poincaré inequality* [17].

Definition 1. Let $\{D_i\}$ be a finite partition of Ω into polytopes, let α be piecewise constant w.r.t. $\{D_i\}$ with value α_i on D_i, and let ℓ^* be an index such that $\alpha_{\ell^*} = \max_i \alpha_i$. Then α is called *quasi-monotone* on Ω iff for each i we can find a path $D_{\ell_1} \cup D_{\ell_2} \cup \ldots \cup D_{\ell_n}$ of subregions connected through proper faces with $\ell_1 = i$, $\ell_n = \ell^*$ such that $\alpha_{\ell_1} \leq \alpha_{\ell_2} \leq \ldots \leq \alpha_{\ell_n}$.

Def. 1 is independent of the choice of ℓ^*: if α attains its maximum in more than one subregion, then α is either not quasi-monotone, or all the maximum subregions are connected. In our example, α_L is quasi-monotone, whereas α_H is not.

Theorem 1. *If α (as in Def. 1) is quasi-monotone on Ω, then there exists a constant $C_{P,\alpha}(\Omega)$ independent of the* values α_i *and of* $\operatorname{diam}(\Omega)$ *such that*

$$\inf_{c\in\mathbb{R}} \|u-c\|_{L^2(\Omega),\alpha} \leq C_{P,\alpha}(\Omega)\, \operatorname{diam}(\Omega)\, |u|_{H^1(\Omega),\alpha} \qquad \forall u \in H^1(\Omega),$$

where $\|v\|^2_{L^2(\Omega),\alpha} := \int_\Omega \alpha\, v^2\, dx$ and $|v|_{H^1(\Omega),\alpha} := \int_\Omega \alpha\, |\nabla v|^2\, dx$.

For the *geometrical* dependence of $C_{P,\alpha}(\Omega)$ on the partition $\{D_i\}$ (in our previous example, the scale η), we refer to [17]. The infimum on the left hand side is attained at the weighted average $c = \overline{u}^{\Omega,\alpha} := \int_\Omega \alpha\, u\, dx / \int_\Omega \alpha\, dx$. Due to the fact that the coefficient α_L in Fig. 1 is *quasi-monotone*, $\lambda_2(M_{h,\alpha}^{-1} K_{h,\alpha}) \geq C_{P,\alpha}(\Omega)^{-2} \operatorname{diam}(\Omega)^{-2}$ and thus bounded from below independently of the contrast in α_L.

Related Preconditioners The simple examples in Fig. 1 show that it is not necessarily contrast alone, which makes preconditioning difficult, but a special *kind* of contrast. The fact that a *small* number of large inclusions leads to essentially well-conditioned problems has, e.g., been exploited in [22]. Overlapping Schwarz theory is given in [11] for coefficients of type α_H, and in [7, 18] for *locally* quasi-monotone coefficients. Robustness theory of FETI methods for locally quasi-monotone coefficients has been developed in [15, 16, 14, 13]. Achieving *robustness* in the general case requires a good coarse space (either for overlapping Schwarz or FETI). Spectral techniques, in particular solving local generalized eigenvalue problems to *compute* coarse basis functions, have come up in [8, 5, 19] (see also the references therein). Very recently, this approach has been even carried over to FETI methods by Spillane and Rixen [20]; see also Axel Klawonn's DD21 talk and proceedings contribution. Although the spectral approaches above guarantee *robust* preconditioners, the dimension of the coarse space may be large, therefore making the preconditioner inefficient. For analyzing the coarse space dimension, tools like the weighted Poincaré inequality are quite useful, cf. [5].

Outline In this paper, we shall

(i) review the available theoretical results of FETI methods for coefficients that are—on each subdomain (or a part of it)—quasi-monotone (i.e., of type α_L),
(ii) present novel theoretical robustness results of FETI methods for coefficients which result from a large number of inclusions with *large* values (i.e., of type α_H far from quasi-monotone). In particular, we allow the inclusions to cut through or touch certain interfaces of the (non-overlapping) domain decomposition.

In both cases, the coarse space is the usual space of constants in each subdomain. After fixing some notation in Sect. 2, we present our review (i) in Sect. 3. Sect. 4 deals with technical tools needed for the novel theory of (ii), which is contained in Sect. 5. In the end, we draw some conclusions.

2 FETI and TFETI

FETI Basics We briefly introduce classical and total FETI; for details see e.g., [21, 13]. The domain Ω is decomposed into non-overlapping subdomains $\{\Omega_i\}_{i=1}^s$, resolved by the fine mesh $\mathscr{T}^h(\Omega)$. The *interface* is defined by $\Gamma := \bigcup_{i\neq j=1}^s (\partial\Omega_i \cap \partial\Omega_j)\setminus\Gamma_D$. Let K_i denote the "Neumann" stiffness matrix corresponding to the local bilinear form $\int_{\Omega_i} \alpha\nabla u\cdot\nabla v\,dx$, and let S_i be the Schur complement of K_i after eliminating the interior degrees of freedom and those corresponding to non-coupling nodes on the Neumann boundary. In the *classical* variant of FETI [6], the corresponding local spaces are chosen to be

$$W_i := \{v \in V^h(\partial\Omega_i \setminus \Gamma_N) : v_{|\Gamma_D} = 0\}.$$

In the case of the *total FETI* (TFETI) method [4], the Dirichlet boundary conditions are not included into K_i, and correspondingly $W_i := V^h(\partial\Omega_i \setminus \Gamma_N)$. We set $W := \prod_{i=1}^s W_i$ and $S := \operatorname{diag}(S_i)_{i=1}^s$. Let R be a block-diagonal full-rank matrix such that $\ker(S) = \operatorname{range}(R)$, and let $B : W \to U$ be a jump operator such that $\ker(B) = \widehat{W}$, where $\widehat{W} \subset W$ is the space of functions being continuous across Γ and fulfilling the homogeneous Dirichlet boundary conditions. The rows of $Bu = 0$ are formed by all (fully redundant) constraints $u_i(x^h) - u_j(x^h) = 0$ for $x^h \in \partial\Omega_i \cap \partial\Omega_j \setminus \Gamma_D$. In TFETI, there are further local constraints of the form $u_i(x^h) = 0$ for $x^h \in \partial\Omega_i \cap \Gamma_D$. Finally, System (2) is reformulated as $\begin{bmatrix} S & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$, where f contains the reduced local load vectors, and further reformulated by

$$\text{find } \widetilde{\lambda} \in \operatorname{range}(P): \quad P^\top F \widetilde{\lambda} \;=\; \widetilde{d} := P^\top B S^\dagger (f - B^\top \lambda_0), \tag{3}$$

where $S^\dagger$ is a pseudo-inverse of S, $F := BS^\dagger B^\top$, $P := I - QG(G^\top QG)^{-1}G^\top$, $G := BR$, $\lambda_0 = QG(G^\top QG)^{-1}R^\top f$, and Q is yet to be specified. The solution u can be recovered easily from $\lambda = \lambda_0 + \widetilde{\lambda}$ by using $S^\dagger$ and $(G^\top QG)^{-1}$.

Scaled Dirichlet Preconditioner For each subdomain index j and each degree of freedom (i.e., node) $x^h \in \partial\Omega_j \cap \Gamma$, we fix a weight $\rho_j(x^h) > 0$ and define

$$\delta_j^\dagger(x^h) := \frac{\rho_j(x^h)^\gamma}{\sum_{k\in\mathscr{N}_{x^h}} \rho_k(x^h)^\gamma} \;\in [0,\,1], \qquad \sum_{j\in\mathscr{N}_{x^h}} \delta_j^\dagger(x^h) = 1.$$

Above, $\mathscr{N}_{x^h}$ is the set of subdomain indices sharing node x^h and $\gamma \in [1/2,\,\infty]$ (the limit $\gamma \to \infty$ has to be carried out properly, cf. [13, Rem. 2.27]). We stress that in the presence of jumps in α, the choice of the weights $\rho_j(x^h)$ (or the scalings $\delta_j^\dagger(x^h)$) is highly important for the robustness of the Dirichlet preconditioner and will be discussed further below. Let us note that for any choice $\rho_j(x^h)$ above and any exponent $\gamma \in [1/2,\infty]$, we have the elementary inequality

$$\rho_i(x^h)\,\delta_j^\dagger(x^h)^2 \;\leq\; \min(\rho_i(x^h),\,\rho_j(x^h)) \qquad \forall i,\,j \in \mathscr{N}_{x^h}. \tag{4}$$

The weighted jump operator B_D is defined similarly to B, but each row of $B_D\, w = 0$ is of the form $\delta_j^\dagger(x^h)\, w_i(x^h) - \delta_i^\dagger(x^h)\, w_j(x^h) = 0$ for $x^h \in \partial\Omega_i \cap \partial\Omega_j \setminus \varGamma_D$. In TFETI, there are further rows of the form $w_i(x^h) = 0$ for $x^h \in \partial\Omega_i \cap \varGamma_D$. The preconditioned FETI system now reads

$$\text{find } \widetilde{\lambda} \in \operatorname{range}(P): \quad PM^{-1}P^\top F \widetilde{\lambda} \;=\; PM^{-1}\widetilde{d}, \tag{5}$$

where $M^{-1} := B_D S B_D^\top$. Since $P^\top F$ is SPD on range(P) up to $\ker(B^\top)$, this system can be solved by CG. Hence, one is interested in a bound on the condition number $\kappa_{\text{FETI}} := \kappa(PM^{-1}P^\top F_{|\operatorname{range}(P)/\ker(B^\top)})$. In the sequel, we set $Q = M^{-1}$. To avoid complications, we exclude the case of TFETI with $\varGamma_D = \partial\Omega$, and the case $\gamma = \infty$; otherwise $GM^{-1}G^\top$ may be singular. As the analysis in [21], [13, Chap. 2] shows, the estimate

$$|P_D\, w|_S^2 \;\le\; \mu\, |w|_S^2 \qquad \forall w \in W^\perp, \tag{6}$$

implies $\kappa_{\text{FETI}} \le 4\mu$. Above, $P_D := B_D^\top B$ is a *projection* (due to the partition of unity property of $\delta_j^\dagger$), $W^\perp = \prod_{i=1}^s W_i^\perp$, and each $W_i^\perp \subset W_i$ is any complementary subspace such that the sum $W_i = \ker(S_i) + W_i^\perp$ is direct. Note that the same estimate implies a bound of the related balancing Neumann-Neumann (BDD) method.

Choice of Weights Table 1 shows several choices for the weights $\rho_j(x^h)$. In each row, we display a *theoretical* choice, which has been used in certain analyses, and then a *practical* choice, which tries to mimic the theoretical one. Choices (a)–(c) in Table 1 are not suitable for coefficients with jumps (see column *problems*). The theoretical choice (d) will be used in the analyses below and leads to "good" condition number bounds under suitable assumptions; however, it is practically infeasible. Under suitable assumptions on the variation of α, the practical choice (d) can be shown to be essentially equivalent to the theoretical one, if one sets $\gamma = \infty$. "Good" means that the bounds are robust with respect to contrast in α. However, they depend on the spatial scale η of the coefficient variation.

$\rho_j(x^h)$	*theoretical*	*practical*	*problems*
(a)	1	1 (multiplicity scaling)	jumps across interfaces
(b)	$\alpha_{\Omega_j}^{\max}$	$\lVert K_j^{\text{diag}} \rVert_{\ell^\infty}$	jumps within subdomains
(c)	$\max\limits_{\tau \subset \Omega_j : x^h \in \overline{\tau}} \alpha_{\lvert\tau}$	$K_j^{\text{diag}}(x^h)$ (stiffness scaling)	oscillating coefficients, unstructured meshes
(d)	$\max\limits_{Y_j^{(k)} : x^h \in \overline{Y}_j^{(k)}} \alpha_{Y_j^{(k)}}^{\max}$	$\begin{cases} 1 & \text{if } K_j^{\text{diag}}(x^h) \simeq \max_{k \in \mathcal{N}_{x^h}} K_k^{\text{diag}}(x^h) \\ 0 & \text{else} \end{cases}$	small geometric scale η

Table 1 Various choices for the weights $\rho_j(x^h)$. Here, K_j^{diag} denotes the diagonal of K_j, $\lVert\cdot\rVert_{\ell^\infty}$ the maximum norm, $K_j^{\text{diag}}(x^h)$ the diagonal entry of K_j corresponding to node x^h, and $\{Y_j^{(k)}\}_k$ is a partition of a neighborhood of $\partial\Omega_j \cap \varGamma$, as coarse as possible, such that α is constant or only mildly varying in each subregion $Y_j^{(k)}$, cf. [13, Sect. 3.3].

Remark 1. A further choice, named *Schur scaling*, has been suggested in [3], see also [2]. There, for each subdomain vertex/edge/face $\mathscr{G}$, the *scalar* values $\delta_j^\dagger(x^h)$ for $x^h \in \mathscr{G}$ are replaced by the *matrix* $(\sum_{k\in\mathscr{N}_\mathscr{G}} S_{k,\mathscr{G}\mathscr{G}})^{-1} S_{j,\mathscr{G}\mathscr{G}}$, where $S_{k,\mathscr{G}\mathscr{G}}$ denotes the restriction of S_k to the nodes on the subdomain vertex/edge/face $\mathscr{G}$. This choice is the only known (practical) candidate that could allow for robustness also with respect to the spatial scale η, but its analysis is still under development, cf. [2]. Nevertheless, it has been successfully analyzed in the context of BDDC methods for the eddy current problem $\mathbf{curl}(\alpha\,\mathbf{curl}\,\mathbf{u}) + \beta\,\mathbf{u} = \mathbf{f}$, where $\alpha, \beta > 0$ are constant in each subdomain [3].

3 Robustness Results for Locally Quasi-monotone Coefficients

In this section, we review robustness results of TFETI, developed originally in [15, 16] and further refined in [13, Chap. 3]. Because of space limitation, we do not list the full set of assumptions, but refer to [13, Sect. 3.3.1, Sect. 3.5]. The essential assumption is that α is piecewise constant with respect to a shape-regular mesh $\mathscr{T}^\eta(\Omega)$, at least in the neighborhood of the interface Γ and the Dirichlet boundary Γ_D, and that this mesh resolves $\Gamma \cup \Gamma_D$. For simplicity of the presentation, we assume further that each subdomain Ω_i is the union of a few elements of a coarse mesh $\mathscr{T}^H(\Omega)$, and that the three meshes $\mathscr{T}^h(\Omega)$, $\mathscr{T}^\eta(\Omega)$, and $\mathscr{T}^H(\Omega)$ are nested, shape-regular, and globally quasi-uniform with mesh parameters $h \le \eta \le H$.

All the following results hold for the TFETI method as defined in Sect. 2 with the theoretical choice (d) for $\rho_j(x^h)$ and with $Q = M^{-1}$, where the regions $Y_j^{(k)}$ are unions of a few elements from $\mathscr{T}^\eta(\Omega)$. The general bound reads

$$\kappa_{\mathrm{FETI}} \;\le\; C\left(\frac{H}{\eta}\right)^{\beta} (1+\log(\eta/h))^2\,, \tag{7}$$

where C is independent of H, η, h, and α. The exponent β is specified below in each particular case.

Definition 2. For each subdomain index i, the *boundary layer* $\Omega_{i,\eta}$ is the union of those elements from $\mathscr{T}^\eta(\Omega)$ that lie in Ω_i and touch $\Gamma \cup \Gamma_D$.

The following theorem is essentially [13, Thm. 3.64] and shows that contrast in the interior of subdomains is taken care of by TFETI (in form of the subdomain solves), except that the geometrical scale shows up in the condition number bound. The original result on classical FETI can be found in [15, Thm. 3.3].

Theorem 2 (Constant Coefficients in the Boundary Layers). *If α is constant in each boundary layer $\Omega_{i,\eta}$, $i = 1,\dots,s$, then (7) holds with $\beta = 2$. The exponent $\beta = 2$ is sharp in general. If the values of α in $\Omega_i \setminus \Omega_{i,\eta}$ do not fall below the constant value in $\Omega_{i,\eta}$ for each $i = 1,\dots,s$, then (7) holds with $\beta = 1$.*

The next theorem (cf. [13, Sect. 3.5.2]) extends the above result to coefficients that are quasi-monotone in each boundary layer.

Theorem 3 (Quasi-monotone coefficients in the Boundary Layers). *If α is quasi-monotone in each boundary layer $\Omega_{i,\eta}$, $i = 1,\dots,s$, then (7) holds with $\beta = 2$ if $d = 2$ and $\beta = 4$ if $d = 3$. Under suitable additional assumptions on α in $\Omega_{i,\eta}$, one can achieve $\beta = 2$ for $d = 3$ as well.*

In many cases, quasi-monotonicity may not hold in each boundary layer, but in a certain sense on a larger domain. The following theorem summarizes essentially [13, Sect. 3.5.3]. We note that the concept of an *artificial coefficient* in the context of FETI goes back to [16].

Theorem 4 (Quasi-monotone Artificial Coefficients). *If for each $i = 1,\dots,s$ there exists an auxiliary domain Λ_i with $\Omega_{i,\eta} \subset \Lambda_i \subset \Omega_i$ and an* artificial coefficient α^{art} *such that*

$$\begin{aligned}
&\alpha^{\mathrm{art}} = \alpha \quad \text{in } \Omega_{i,\eta}\,,\\
&\alpha^{\mathrm{art}} \le \alpha \quad \text{in } \Lambda_i \setminus \Omega_{i,\eta}\,,\\
&\alpha^{\mathrm{art}} \text{ quasi-monotone on } \Lambda_i\,,
\end{aligned}$$

then (7) holds with C independent of α and α^{art}. The exponent β depends on Λ_i and α^{art}. If $\Lambda_i = \Omega_i$ then $\beta \le 2d$. Under additional assumptions on α^{art}, one can achieve, e.g., $\beta \le d+1$.

Remark 2. The proofs of Thm. 3 and Thm. 4 make heavy use of the weighted Poincaré inequality (Thm. 1). We note that Thm. 3 and Thm. 4 can be generalized to so-called type-m quasi-monotonicity (see [17]). Also, all the results of this section can be generalized to (i) coefficients that vary mildly in each element of $\mathscr{T}^{\eta}(\Omega)$ in the neighborhood of $\Gamma \cup \Gamma_D$, (ii) to a certain extent to suitable diagonal choices of the matrix Q, and (iii) under suitable conditions to classical FETI. However, we do not present these results here but refer to [13, Chap. 3] and [15, 16] for the full theory.

4 Technical Tools

In this section, we present two technical tools needed for Sect. 5. The first tools is an extension operator on so-called *quasi-mirrors*.

Definition 3. Let $D_1, D_2 \subset \mathbb{R}^d$ be two disjoint Lipschitz domains sharing a $(d-1)$-dimensional manifold Γ. For $i = 1,2$ let D_{ia} and D_{ib} be open and disjoint Lipschitz domains such that $\overline{D}_i = \overline{D}_{ia} \cup \overline{D}_{ib}$. We say that (D_{2a}, D_{2b}) is a *quasi-mirror* of (D_{1a}, D_{1b}) iff there exists a continuous and piecewise C^1 bijection ϕ with $\|\nabla\phi\|_{L^\infty}$ and $\|\nabla\phi^{-1}\|_{L^\infty}$ bounded, such that D_{ia}, D_{ib}, Γ are mapped to $\hat{D}_{ia}$, $\hat{D}_{ib}$, $\hat{\Gamma}$, respectively, where $\hat{\Gamma}$ lies in the hyperplane $x_d = 0$ and $\hat{D}_{2a}$, $\hat{D}_{2b}$ are the reflections through that hyperplane of $\hat{D}_{1a}$, $\hat{D}_{1b}$, respectively (for an illustration see Fig. 2).

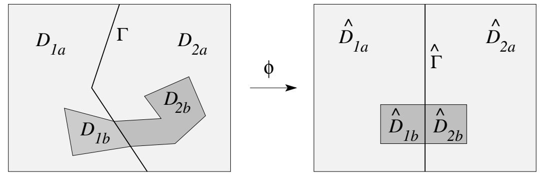

Fig. 2 Illustration of Def. 3: a quasi-mirror in 2D.

Lemma 1. *Let (D_{2a}, D_{2b}) be a quasi-mirror of (D_{1a}, D_{1b}) as in Def. 3. Then there exists a linear operator $E : H^1(D_1) \to H^1(D_2)$ such that for all $v \in H^1(D_1)$, we have $(Ev)_{|\Gamma} = v_{|\Gamma}$ and*

$$|E\,v|_{H^1(D_{2a})} \le C\,|v|_{H^1(D_{1a})}, \qquad |E\,v|_{H^1(D_{2b})} \le C\,|v|_{H^1(D_{1b})},$$
$$\|E\,v\|_{L^2(D_{2a})} \le C\,\|v\|_{L^2(D_{1a})}, \qquad \|E\,v\|_{L^2(D_{2b})} \le C\,\|v\|_{L^2(D_{1b})}.$$

The constant C is dimensionless, but depends on the transformation ϕ from Def. 3.

The proof of the above and the next lemma can be found in [12, Sect. 4]. Our second tool is a special Scott-Zhang quasi-interpolation operator.

Lemma 2. *Let the domain D be composed from two disjoint Lipschitz regions $\overline{D} = \overline{D}_1 \cup \overline{D}_2$ with interface $\Gamma = \partial D_1 \cap \partial D_2$, and let $\Sigma \subset \partial D$ be non-trivial. Let $\mathscr{T}^h(D)$ be a shape-regular mesh resolving Γ and Σ, and let $V^h(D)$ denote the corresponding space of continuous and piecewise linear finite element functions. Then there exists a projection operator $\Pi_h : H^1(D) \to V^h(D)$ such that (i) for any $v \in H^1(D)$ that is piecewise linear on Γ and Σ, $(\Pi_h v)_{\Gamma\cup\Sigma} = v_{|\Gamma\cup\Sigma}$ and (ii) for all $v \in H^1(D)$,*

$$|\Pi_h v|_{H^1(D_i)} \le C\,|v|_{H^1(D_i)}, \qquad \|\Pi_h v\|_{L^2(D_i)} \le C\,\|v\|_{L^2(D_i)}, \qquad \textit{for } i = 1, 2,$$

where the constant C only depends on the shape-regularity of the mesh.

5 Novel Robustness Results for Inclusions

For this section, we adopt again the notations of Sect. 2 and 3. However, we restrict ourselves to coefficients $\alpha \in L^\infty(\Omega)$, given by

$$\alpha(x) \;=\; \begin{cases} \alpha_k & \text{if } x \in D_k \text{ for some } k = 1, \ldots, n_H\,, \\ \alpha_L & \text{else,} \end{cases} \tag{8}$$

where $\alpha_k \ge \alpha_L$ are constants and the regions $\overline{D}_k \subset \overline{\Omega}$ are pairwise disjoint (disconnected) Lipschitz polytopes that are contractible (i.e., topologically isomorphic to the ball). Furthermore, we assume that the subdomains Ω_i as well as the inclusion regions D_k are resolved by a global mesh $\mathscr{T}^\eta(\Omega)$. For the sake of simplicity let $\mathscr{T}^h(\Omega)$ and $\mathscr{T}^\eta(\Omega)$ be nested, shape-regular, and quasi-uniform with mesh sizes h and η, respectively ($h \le \eta$). Our main assumption concerns the location of the inclusion regions D_k relative to the interface.

Assumption A1. Each region D_k, $k = 1,\dots,n_H$, is either
(a) an *interior inclusion*: $D_k \subset\subset \Omega_i$ for some index i,
(b) a *docking inclusion*: there is a unique index i with $D_k \subset \Omega_i$ and $\overline{D}_k \cap \partial\Omega_i \neq \emptyset$, or
(c) a *(proper) face inclusion*: there exists a subdomain face $\mathscr{F}_{ij}$ (shared by only two subdomains Ω_i, Ω_j) such that

- $\overline{D}_k \cap \Gamma \subset\subset \mathscr{F}_{ij}$,
- $\partial(D_k \cap \Omega_i) \cap \mathscr{F}_{ij} = \partial(D_k \cap \Omega_j) \cap \mathscr{F}_{ij}$,
- $\overline{D}_k \cap \Gamma$ is simply connected,
- the neighborhood $\mathscr{U}_k$ constructed from D_k by adding one layer of elements from $\mathscr{T}^\eta(\Omega)$ fulfills $D_k \subset\subset \mathscr{U}_k \subset \overline{\Omega_i \cup \Omega_j}$.

Above, $\subset\subset$ means compactly contained. Note that since the regions $\overline{D}_k$ are disjoint and resolved by $\mathscr{T}^\eta(\Omega)$, in Case (c) above, it follows that $\alpha = \alpha_L$ in $\mathscr{U}_k \setminus D_k$. The second condition in (c) avoids that a part of D_k is only "docking". The third condition ensures that D_k passes through the face $\mathscr{F}_{ij}$ only once.

Theorem 5. *Let the above assumptions, in particular Assumption A1, be fulfilled. For the case of classical FETI, assume that for $d = 3$ the intersection of a subdomain with Γ_D is either empty, or contains at least an edge of $\mathscr{T}^\eta(\Omega)$. For the case of TFETI, assume that none of the docking inclusions in Assumption A1(b) intersects the Dirichlet boundary. Then*

$$\kappa_{\text{FETI}} \leq C(\eta)\,(1 + \log(\eta/h))^2\,,$$

where $C(\eta)$ is independent of h, the number of subdomains, and α_k, α_L.

The dependence of $C(\eta)$ on η can theoretically be made explicit but is ignored here. In general, it is at least $(H/\eta)^2$. Due to space limitations, we can only give a sketch of the proof for the case of classical FETI; the detailed proof can be found in [12]. To get the condition number bound, we show estimate (6). If $\ker(S_i) = \operatorname{span}\{1\}$, we choose $W_i^\perp := \{w \in W_i : \overline{w}^{\partial\Omega_i} = 0\}$, and $W_i^\perp = W_i$ otherwise. Let $w \in W^\perp$ be arbitrary but fixed. To estimate $|P_D w|_S$, we decompose the interface Γ into *globs* g. These are vertices, edges, or faces of the mesh $\mathscr{T}^\eta(\Omega)$, with one exception: for a face inclusion D_k, we combine all vertices/edges/faces of $\mathscr{T}^\eta(\Omega)$ contained in $\overline{D}_k \cap \Gamma$ into a single glob g. Following [13, Lem. 3.21, Lem. 3.27], we get

$$|(P_D w)_i|_{S_i}^2 \;\leq\; C \sum_{\mathsf{g} \subset \partial\Omega_i \cap \Gamma} \underbrace{\sum_{j \in \mathscr{N}_{\mathsf{g}} \setminus \{i\}} (\delta_{j|\mathsf{g}}^\dagger)^2\, |I^h(\vartheta_{\mathsf{g}}(\widetilde{w}_{ii}^{\mathsf{g}} - \widetilde{w}_{ij}^{\mathsf{g}}))|_{H^1(\mathsf{U}_{i,\mathsf{g}}),\alpha}^2}_{=:\Upsilon_{i,\mathsf{g}}}\,, \tag{9}$$

where $\vartheta_{\mathsf{g}} \in V^h(\Omega)$ is a cut-off function (yet to be specified) that equals one on all the nodes on g and vanishes on all other nodes on Γ, I^h is the nodal interpolation operator, and $\mathsf{U}_{i,\mathsf{g}} = \operatorname{supp}(\vartheta_{\mathsf{g}}) \cap \Omega_i$. The (generic) constant C above only depends the shape regularity constant of $\mathscr{T}^\eta(\Omega)$ and is thus uniformly bounded. For $j \in \mathscr{N}_{\mathsf{g}}$, the function $\widetilde{w}_{ij}^{\mathsf{g}} \in V^h(\mathsf{U}_{i,\mathsf{g}})$ is an extension of w_j (yet to be specified) in the sense that $\widetilde{w}_{ij}^{\mathsf{g}}(x^h) = w_j(x^h)$ for all nodes x^h on g. We treat two cases.

Case 1: g is not part of a face inclusion, i.e., for all $k \in \{1,\dots,n_H\}$ with D_k being a face inclusion, $\overline{D}_k \cap \mathrm{g} = \emptyset$. We choose the cut-off function ϑ_{g} like in [21, Sect. 4.6] (where the subdomains there are the elements of $\mathscr{T}^\eta(\Omega)$). Using that

$$(\delta^\dagger_{j|\mathrm{g}})^2 \rho_{i|\mathrm{g}} \le \min(\rho_{i|\mathrm{g}}, \rho_{j|\mathrm{g}}) = \alpha_L \qquad \forall j \in \mathscr{N}_{\mathrm{g}} \setminus \{i\}, \tag{10}$$

and the available techniques from [16, 13], one can show that

$$\Upsilon_{i,\mathrm{g}} \le C \sum_{j \in \mathscr{N}_{\mathrm{g}}} \alpha_L \Big(\omega^2 |\widetilde{w}^{\mathrm{g}}_{ij}|^2_{H^1(\mathsf{U}_{i,\mathrm{g}})} + \frac{\omega}{\eta^2} \|\widetilde{w}^{\mathrm{g}}_{ij}\|^2_{L^2(\mathsf{U}_{i,\mathrm{g}})} \Big), \tag{11}$$

where above and in the following, $\omega := (1 + \log(\eta/h))$.
Case 2: g is part of a face inclusion (see Assumption A1), i.e., there exists k with $\mathrm{g} = \overline{D}_k \cap \Gamma$. Recall that in this case g can be the union of many vertices/edges/faces of $\mathscr{T}^\eta(\Omega)$. We choose a special cut-off function ϑ_{g} supported in $\mathsf{U}_{i,\mathrm{g}} := \mathscr{U}_k \cap \Omega_i$:

- $\vartheta_{\mathrm{g}}(x^h) = 1$ for all nodes $x^h \in \overline{D}_k$,
- $\vartheta_{\mathrm{g}}(x^h) = 0$ for all nodes $x^h \in \partial \mathscr{U}_k \cup (\mathscr{U}_k \cap (\Gamma \setminus \mathrm{g}))$,
- on the elements of the layer, i.e., those elements $T \in \mathscr{T}^\eta(\Omega)$ with $T \subset \mathscr{U}_k \setminus D_k$, we set ϑ_{g} to the sum of local cut-off functions (similar to Case 1).

By construction, $\vartheta_{\mathrm{g}} = 1$ on D_k, where $\alpha = \alpha_k$. On the remainder, $\mathscr{U}_k \setminus D_k$, by the assumptions on the coefficient, $\alpha = \alpha_L$. A careful analysis shows that

$$\Upsilon_{i,\mathrm{g}} \le C \sum_{j \in \mathscr{N}_{\mathrm{g}}} \Big(\omega^2 |\widetilde{w}^{\mathrm{g}}_{ij}|^2_{H^1(\mathsf{U}_{i,\mathrm{g}}),\alpha} + \alpha_L \frac{\omega}{\eta^2} \|\widetilde{w}^{\mathrm{g}}_{ij}\|^2_{L^2(\Omega_i \cap (\mathscr{U}_k \setminus D_k))} \Big). \tag{12}$$

Choice of $\widetilde{w}^{\mathrm{g}}_{ij}$ in Case 1: We set $\widetilde{w}^{\mathrm{g}}_{ij} := E^h_{j,\mathrm{g}} \mathscr{H}^{\alpha,h}_j w_j$, where $\mathscr{H}^{\alpha,h}_j : W_j \to V^h(\Omega_j)$ denotes the discrete extension operator such that $|w_j|_{S_j} = |\mathscr{H}^{\alpha,h}_j w_j|_{H^1(\Omega_j),\alpha}$ and $E^h_{j,\mathrm{g}}$ is a suitable transfer operator (see [13, Sect. 2.5.7] or [16, Lem. 5.5]). This results in the estimates

$$|\widetilde{w}^{\mathrm{g}}_{ij}|_{H^1(\mathsf{U}_{i,\mathrm{g}})} \le C |\mathscr{H}^{\alpha,h}_j w_j|_{H^1(\mathsf{U}'_{j,\mathrm{g}})}, \qquad \|\widetilde{w}^{\mathrm{g}}_{ij}\|_{L^2(\mathsf{U}_{i,\mathrm{g}})} \le C \|\mathscr{H}^{\alpha,h}_j w_j\|_{L^2(\mathsf{U}'_{j,\mathrm{g}})}. \tag{13}$$

where $\mathsf{U}'_{j,\mathrm{g}} \subset \Omega_j$ is an element of $\mathscr{T}^\eta(\Omega)$ with $\mathrm{g} \subset \overline{\mathsf{U}}'_{j,\mathrm{g}}$.

Choice of $\widetilde{w}^{\mathrm{g}}_{ij}$ in Case 2: Recall that in this case we are dealing with a face inclusion such that g is part of the face shared by Ω_i and Ω_j and we choose $\mathsf{U}_{i,\mathrm{g}} = \mathscr{U}_k \cap \Omega_j$. To define the extension $\widetilde{w}^{\mathrm{g}}_{ij} \in V^h(\mathsf{U}_{i,\mathrm{g}})$, we shall combine the technical tools from Sect. 4. Let $\mathsf{U}'_{j,\mathrm{g}} := \mathscr{U}_k \cap \Omega_j$. It can be seen from Assumption A1 that $(\mathsf{U}_{i,\mathrm{g}} \setminus D_k, \mathsf{U}_{i,\mathrm{g}} \cap D_k)$ is a quasi-mirror of $(\mathsf{U}'_{j,\mathrm{g}} \setminus D_k, \mathsf{U}'_{j,\mathrm{g}} \cap D_k)$. We can therefore set

$$\widetilde{w}^{\mathrm{g}}_{ij} := \Pi^{h,\alpha}_{j,\mathrm{g}} \mathscr{E}^{\alpha}_{j,\mathrm{g}} \mathscr{H}^{\alpha,h}_j w_j\,,$$

where $\Pi^{h,\alpha}_{j,\mathrm{g}}$ is the Scott-Zhang interpolator from Lem. 2, $\mathscr{E}^{\alpha}_{j,\mathrm{g}}$ the extension operator from Lem. 1, and $\mathscr{H}^{\alpha,h}_j$ is defined as above. It has now to be argued that the transformation ϕ in Def. 3 can be chosen such that $\mathscr{E}^{\alpha}_{j,\mathrm{g}}\mathscr{H}^{\alpha,h}_j w_j$ is still piecewise linear on the interface $\mathscr{U}'_{j,\mathrm{g}} \cap \partial D_k$. This implies that $\widetilde{w}^{\mathrm{g}}_{ij}$ is indeed an extension of w_j. Due to the properties of the above operators, we obtain the total stability estimates

$$|\widetilde{w}^{\mathrm{g}}_{ij}|_{H^1(\mathsf{U}_{i,\mathrm{g}}),\alpha} \leq C|\mathscr{H}^{\alpha}_j w_j|_{H^1(\mathsf{U}'_{j,\mathrm{g}}),\alpha}\,, \quad \|\widetilde{w}^{\mathrm{g}}_{ij}\|_{L^2(\mathsf{U}_{i,\mathrm{g}})} \leq C\|\mathscr{H}^{\alpha}_j w_j\|_{L^2(\mathsf{U}'_{j,\mathrm{g}})} \tag{14}$$

for all $w_j \in V^h(\partial\Omega_j)$, with C independent of α_L and α_k. Combining the local estimates (11), (12), (13), and (14), using a finite overlap argument, as well as a *conventional* Poincaré or Friedrichs inequality, one arrives at (6) with $\mu = C\,\omega^2$.

6 Conclusions

Section 3 shows robustness of TFETI for (artificial) coefficients that are quasi-monotone in boundary layers. Sect. 5 shows that these conditions are far from necessary for the robustness of FETI or TFETI. Note that the assumptions and robustness properties of Sect. 5 are similar to the theory in [11] for overlapping Schwarz. Actually, several ideas from the latter theory have been reused in the analysis of Sect. 5. However, the robustness for overlapping Schwarz requires a sophisticated coarse space, whereas for FETI/TFETI, the usual coarse space can be used, which simplifies the implementation a lot.

A combination of the two theories (Sect. 3 and Sect. 5) is of course desirable. However, the general case of α remains open. The problematic cases in FETI/TFETI are certainly (a) a multiple number of inclusions on vertices (or edges in 3D), and (b) long channels that traverse through more than one face, or traverse a face more than once; this is seen in numerical examples; see [12, Sect. 6].

Item (a) might be fixed using suitable FETI-DP/BDDC methods, and we hope that novel analysis of Sect. 5 will have a positive impact here (the known theory of FETI-DP/BDDC for multiscale coefficients is yet limited, cf. [13, 14, 9]). Item (b) can only be addressed by a larger coarse space: either by FETI-DP/BDDC with more sophisticated primal DOFs and/or by spectral techniques as suggested in [20]. Robustness in the spatial scale η is achieved neither in Sect. 3 nor Sect. 5. We believe that the only possibility to gain robustness is a more sophisticated weight selection (cf. Rem. 1) and probably again a larger coarse space.

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