Optimized interface preconditioners for the FETI method

Martin J. Gander and Hui Zhang

1 Motivation

In the past two decades, the FETI method introduced in [10] and its variants have become a class of popular methods for the parallel solution of large-scale finite element problems, see e.g. [11], [9], [14], [15], [8]. A key ingredient in this class of methods is a good preconditioner for the dual Schur complement system whose operator is a weighted sum of subdomain Neumann to Dirichlet (*NtD*) maps. One choice is the so-called Dirichlet preconditioner, which is the primal Schur complement, i.e. a weighted sum of subdomain Dirichlet to Neumann (*DtN*) maps. The Dirichlet preconditioner is quasi-optimal in the sense that together with an appropriate coarse space, it leads to a polylogarithmic condition number in H/h, see e.g. [14]. However, in terms of total CPU time, often a cheaper alternative called the lumped preconditioner performs better [11, 8].

We show here that the lumped preconditioner can be further improved by introducing parameters into the tangential interface operator and optimizing them to get condition numbers as small as possible while keeping the cost of the preconditioner low. Since these preconditioners, like the lumped preconditioner, only involve computations along the interface, and no computations in the interior of subdomains, we call them *interface preconditioners*.

We consider the model problem

$$\begin{cases} -u_{xx} - u_{yy} = f, & (x,y) \in \mathbb{R}^2 \\ \lim_{(x,y) \to \infty} u = 0, \end{cases}$$

which can be decomposed into two non-overlapping subproblems as follows:

$$\begin{cases} -(u_1)_{xx} - (u_1)_{yy} = f, & (x,y) \in (-\infty, 0) \times \mathbb{R}, \\ \quad\quad\quad\quad (u_1)_x = \lambda, & (x,y) \in \{0\} \times \mathbb{R}, \\ \lim_{(x,y) \to \infty} u_1 = 0, \end{cases} \tag{1}$$

$$\begin{cases} -(u_2)_{xx} - (u_2)_{yy} = f, & (x,y) \in (0, \infty) \times \mathbb{R}, \\ \quad\quad\quad\quad (u_2)_x = \lambda, & (x,y) \in \{0\} \times \mathbb{R}, \\ \lim_{(x,y) \to \infty} u_2 = 0, \end{cases} \tag{2}$$

$$\frac{1}{2} u_1 - \frac{1}{2} u_2 = 0, \quad (x,y) \in \{0\} \times \mathbb{R}. \tag{3}$$

University of Geneva, 2-4 rue du Liévre, Case postale 64, {martin.gander, hui.zhang}@unige.ch

The FETI method takes the common Neumann trace λ as unknown and the equation to be solved for λ is defined by (3). To analyze the operator of the equation for λ, we let $f = 0$ and do a Fourier transform in $y \in \mathbb{R}$ for (1), (2) and (3) to obtain

$$\begin{cases} -(\hat{u}_1)_{xx} - k^2 \hat{u}_1 = 0, & x \in (-\infty, 0), \\ (\hat{u}_1)_x = \hat{\lambda}, & x = 0, \\ \lim_{x \to -\infty} \hat{u}_1(x,k) = 0, \end{cases} \tag{4}$$

$$\begin{cases} -(\hat{u}_2)_{xx} - k^2 \hat{u}_2 = 0, & x \in (0, \infty), \\ (\hat{u}_2)_x = \hat{\lambda}, & x = 0, \\ \lim_{x \to \infty} \hat{u}_2(x,k) = 0, \end{cases} \tag{5}$$

and

$$\frac{1}{2}\hat{u}_1 - \frac{1}{2}\hat{u}_2 = 0, \quad x = 0. \tag{6}$$

The subdomain solutions $\hat{u}_i, i = 1, 2$ can be obtained from (4) and (5), and substituting them into the left hand side of (6) yields the equation for $\hat{\lambda}$,

$$\hat{F}\hat{\lambda} := \frac{1}{\sqrt{k^2}}\hat{\lambda},$$

where $\hat{F}$ is the symbol of the averaged NtD operator F. Similarly, one can obtain the symbol of the Dirichlet preconditioner (i.e. the averaged DtN operator): it is exactly $\hat{F}^{-1}$, which means that the Dirichlet preconditioner is an exact preconditioner for our symmetric partition into two subdomains. However, using the Dirichlet preconditioner requires to solve the Dirichlet boundary value problems on the subdomains, which are in addition to the Neumann boundary value problems involved in F.

As a cheaper alternative, Farhat and Roux introduced the lumped preconditioner for F, see e.g. [11], which corresponds to the submatrix of the assembled matrix for the original problem restricted to the interface. Here we explain it as an operator at the continuous level, that is $P_L^{-1} := -\partial_{yy} + p$ acting on the interface, where $p = O(h^{-2})$. To see this, let us consider a 5-point stencil discretization of the minus Laplacian (see the left part of the following illustration)

$$\frac{1}{h^2} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}, \qquad \frac{1}{h^2} \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}.$$

Assuming the interface is along the vertical direction, the lumped preconditioner corresponds to a 3-point stencil along interface, shown on the right part of the above illustration. So the symbol of the preconditioned operator will be

$$\hat{P}_L^{-1}\hat{F} := \frac{k^2 + p}{\sqrt{k^2}}.$$

Note that practically $|k|$ varies in $[k_{\min}, k_{\max}] := [\frac{\pi}{H}, \frac{\pi}{h}]$, where H is the domain size and h is the mesh size, both along the y direction, and that usually we have $\sqrt{p} \in$

$[k_{\min}, k_{\max}]$. In this case, the spectra of $\hat{P}_L^{-1}\hat{F}$ are bounded by

$$\sigma(\hat{P}_L^{-1}\hat{F}) \subset [2\sqrt{p}, \max\{k_{\min} + \frac{p}{k_{\min}}, k_{\max} + \frac{p}{k_{\max}}\}].$$

If we fix H and let $h \to 0$, we will find that the condition number of $P_L^{-1}F$ is $O(h^{-1})$. So the drawback of the lumped preconditioner is that the condition number deteriorates at the same rate as the unpreconditioned method as the mesh size tends to zero. This is conforming to the result in [9] where it was also pointed out that the lumped preconditioner has a favorable spectral distribution for the Conjugate Gradient method. However, in our special cases of numerical experiments, we have not found this superiority, see Sec. 3 (in which we did use the original form of the lumped preconditioner).

Since the symbol of F is $\hat{F} = \frac{1}{\sqrt{k^2}}$, it is clear that an exact preconditioner for F is the square-root of the Laplacian operator on the interface. We already know that the Dirichlet preconditioner implements the square-root through subdomain solves. There are also other ways to approximate the square-root or its inverse, the latter is useful for the primal Schur complement methods. Some are based on the idea of FFT and its extensions, see e.g. [7, 4, 13]. Two multilevel methods are proposed in [5]. In [16], the Green's function is used for approximating the inverse square-root in general geometry. In the more recent approach [3, 2], a Krylov subspace method is adopted for the approximate application of the inverse square-root. In the context of integral equation methods for scattering problems, Padé approximation is adopted for preconditioning, see e.g. [6]. Our work is more related to that of [1][1], in which the ideas of using quadratic approximations and minimizing the condition number were first presented. The first difference of our work from that of [1] lies in the problems studied: the positive definite Helmholtz equation is considered in [1] while we study the Poisson equation here. The second difference is that we propose two new approaches, in addition to the quadratic approximation.

2 Optimized Interface Preconditioners

In this section we will introduce some approximations of the square-root of the interface Laplacian, which define our new preconditioners. Parameters involved in these approximations will be optimized so that condition numbers of the corresponding preconditioned operators are as small as possible.

We first consider the preconditioner whose symbol is of the form $\hat{P}^{-1} := k^2 + p$, the same as that of the lumped preconditioner. We now optimize however the parameter $p \geq 0$ by solving the minimization problem

$$\min_{p \geq 0} \text{cond}(P^{-1}F) = \min_{p \geq 0} \frac{\max_{k \in [k_{\min}, k_{\max}]}(k^2 + p)/\sqrt{k^2}}{\min_{k \in [k_{\min}, k_{\max}]}(k^2 + p)/\sqrt{k^2}}. \tag{7}$$

---

[1] We only discovered this reference when we already finished our present investigation.

Theorem 1. *The solution of problem (7) is given by $p^* = k_{\min}k_{\max}$. In particular, if $k_{\min} = O(1)$ and $k_{\max} = O(h^{-1})$, we have $\mathrm{cond}(P^{-1}F) = O(h^{-1/2})$ when $p = p^*$.*

Remark 1. It is also possible to include the first-order derivative into the preconditioner. But in that case, symmetry is destroyed, and minimizing the condition number is then not necessarily the relevant goal.

In the second approach, the symbol of the preconditioner is chosen to be of the form

$$\hat{P}^{-1} = \frac{p_0 + p_2 k^2 + k^4}{q + k^2}, \tag{8}$$

and we optimize p_0, p_2, q by solving the minimization problem

$$\min_{p_0, p_2, q \geq 0} \mathrm{cond}(P^{-1}F) = \min_{p_0, p_2, q \geq 0} \frac{\max_{k \in [k_{\min}, k_{\max}]} \rho(k)}{\min_{k \in [k_{\min}, k_{\max}]} \rho(k)},$$

where $\rho(k)$ is the symbol of the preconditioned operator, i.e.

$$\rho(k) := \frac{p_0 + p_2 k^2 + k^4}{(q + k^2)\sqrt{k^2}}.$$

Theorem 2. *Assume $k_{\max} = Ch^{-1}$ and let $p_0 = p_2 = k_{\max}^{4/3}\left(2k_{\min} + \frac{2}{k_{\min}}\right)^{2/3}$, $q = \left(k_{\min} + \frac{1}{k_{\min}}\right)^{4/3}(k_{\max}/2)^{2/3}$. Then we have $\mathrm{cond}(P^{-1}F) = O(h^{-1/3})$.*

Remark 2. We found numerically that the smaller $k_{\min}$ is, the smaller h needs to be before the asymptotics set in. We also observed that there exist better choices of parameters in the pre-asymptotic regime, but a formula still needs to be found.

Remark 3. There are many possible ways to implement (8) in the physical domain. We found that a good way in practice is formally given by

$$P^{-1} = (-\partial_{yy} + r_0)(-\partial_{yy} + q)^{-1}(-\partial_{yy} + r_2),$$

where r_0, r_2 are related to p_0, p_2 of (8) such that $k^4 + p_2 k^2 + p_0 = (k^2 + r_0)(k^2 + r_2)$.

Now we propose a third approach for approximating the square root. Suppose we have a good preconditioner $\tilde{A}$ (e.g. Jacobi) for an operator A such that

(i) $\|S\| < 1$ where $S := I - \tilde{A}^{-1}A$ is the iteration operator,
(ii) $\tilde{A}^{1/2}$ is cheap to apply,
(iii) A commutes with $\tilde{A}$ (this can be omitted in practice).

Then, using Newton's binomial series, we have

$$A^{1/2} = \tilde{A}^{1/2}(I - S)^{1/2} = \tilde{A}^{1/2} \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i}(-S)^i.$$
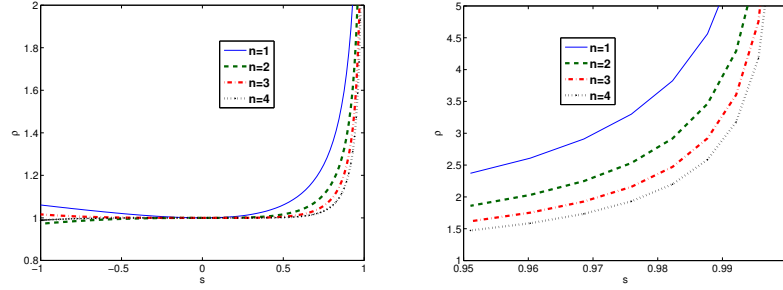
Fig. 1 Values of the symbol (10) with p_i the binomial coefficient in (9).

A preconditioner for $A^{-1/2}$ can be obtained by truncating the infinite series,

$$P^{-1} = \tilde{A}^{1/2} \sum_{i=0}^{n} \binom{\frac{1}{2}}{i} (-S)^i, \tag{9}$$

so n iterations of S are needed in one application of P^{-1}. We can also consider the more general polynomial

$$P^{-1} = \tilde{A}^{1/2} \sum_{i=0}^{n} p_i (-S)^i.$$

The right preconditioned operator is then

$$A^{-1/2} P^{-1} = T \sum_{i=0}^{n} p_i (-S)^i, \quad T := A^{-1/2} \tilde{A}^{1/2}.$$

Assume the symbol of S to be $s \in [s_{\min}, s_{\max}] \subset (-1, 1)$ and the symbol of T to be $\hat{T} = \frac{1}{\sqrt{1-s}}$, which can be obtained for example by Fourier analysis. Hence, the symbol of the preconditioned operator is

$$\rho(s) := \hat{A}^{-1/2} \hat{P}^{-1} = \frac{1}{\sqrt{1-s}} \sum_{i=0}^{n} p_i (-s)^i. \tag{10}$$

In the case of truncation of the binomial series and $n = 1, \ldots, 4$, the symbols are plotted in Fig.1. This clearly shows that the symbol value tends to infinity as s goes to one. In fact, we have $\rho = O(t^{-1/2})$ for $t := 1 - s \to 0$.

For example, when S is Jacobi for the 1d discrete Laplacian, we have $s \in \{\cos(\frac{j\pi}{N}), j = 1, \ldots, N-1\}$, where the mesh size is $h = 1/N$ and we assumed Dirichlet conditions on the boundary. In this case, we have $\max_{[-1, s_{\max}]} \rho(s) = O(h^{-1})$ no matter how many orders are kept in the truncation! To make things worse, the discrete points in s are more clustered near $s = \pm 1$ than elsewhere.

Remark 4. Since the direct truncation of the binomial series is really good away from low frequencies (small j), it is natural to approximate the low frequency part by a coarse grid or multigrid. We will however not investigate this further here.

The idea to improve is optimizing the parameters $\{p_i\}$ such that the corresponding condition number is minimized. We begin with the approximation of order $n = 1$.

Theorem 3. *Let $s \in [-1, s_{\max}]$ with $0 < s_{\max} < 1$, $n = 1$ and assume $p_1 = 1$ is used for the preconditioned operator (10). If the operator is positive definite, then the condition number of the preconditioned operator is minimized if and only if*

$$p_0 = \frac{2s_{\max} + \sqrt{2 - 2s_{\max}}}{2 - \sqrt{2 - 2s_{\max}}},$$

in which case $\operatorname{cond}(P^{-1}A^{-1/2}) = O(t^{-1/4})$ *as* $t := 1 - s_{\max} \to 0$.

We also tried the approximation of order $n = 2$. The scaling of the condition number when s goes to one is *not improved for the exponent* but is improved for the constant. We do however not have closed formulas for the optimized parameters when $n \geq 2$.

3 Numerical experiments

All the numerical experiments are coded in FreeFem++ [12] with P1 elements. We solve homogeneous equations on $\Omega = (0,1)^2$ with the zero solution. We take a random initial guess for the CG iterations which stop when the relative preconditioned residual norms are less than 10^{-15}. It is worthwhile to note that the proposed preconditioners involve only *integer-order* differential operators easily implementable as matrices from standard discretization (FFT is unnecessary). So in methodology, they are applicable to general geometry though the optimal parameters could change.

First, we solve the Laplace equation in two equal subdomains. The maximum errors of the iterates to the exact zero solution are illustrated in Fig. 2 against the iteration numbers, from which we can see that with the optimized interface preconditioners the iterations converge faster than without or with the lumped preconditioner, and the optimized rational preconditioners eventually outperform the others in terms of iteration numbers as the mesh size h becomes small.

Next, we consider a diffusion problem with smoothly varying coefficient,

$$\begin{aligned} -\nabla \cdot (a(x,y)\nabla u) &= 0, \quad (x,y) \in (0,1)^2, \\ u &= 0, \quad \text{if } xy(1-x)(1-y) = 0, \end{aligned} \tag{11}$$

where $a = v\,x^2y^2 + 0.1$. The coefficient a is continuous but varies along the interface. To study the effect of the variation, we take the constant v to be 1, 10, 100, and 1000. We use a fifth-order quadrature rule to ensure accurate numerical integration in the discretization. The results using two subdomains are shown in Fig. 3, and clearly show the robustness of the optimized interface preconditioners, except for the one based on the Jacobi preconditioner.
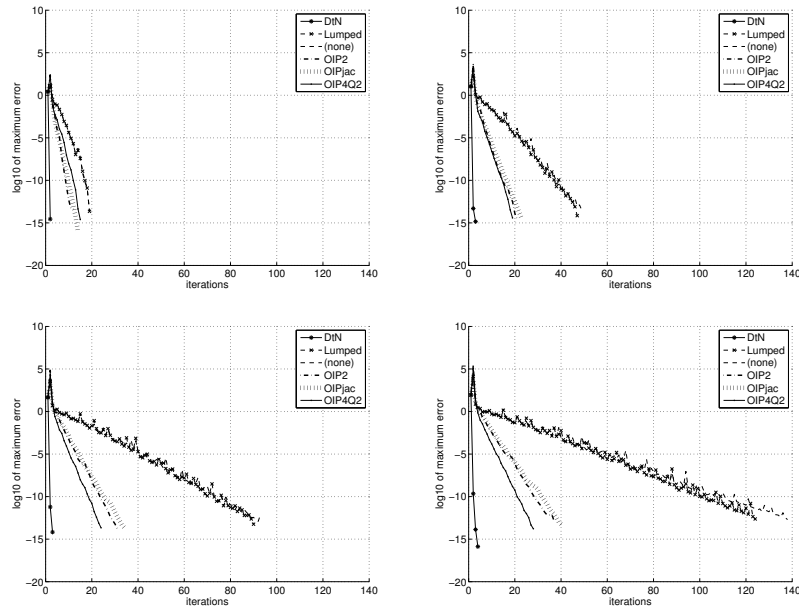
Fig. 2 Maximum errors between iterates and the FEM solutions for the Laplacian in the unit square for $h = 1/16, 1/64, 1/256, 1/512$.
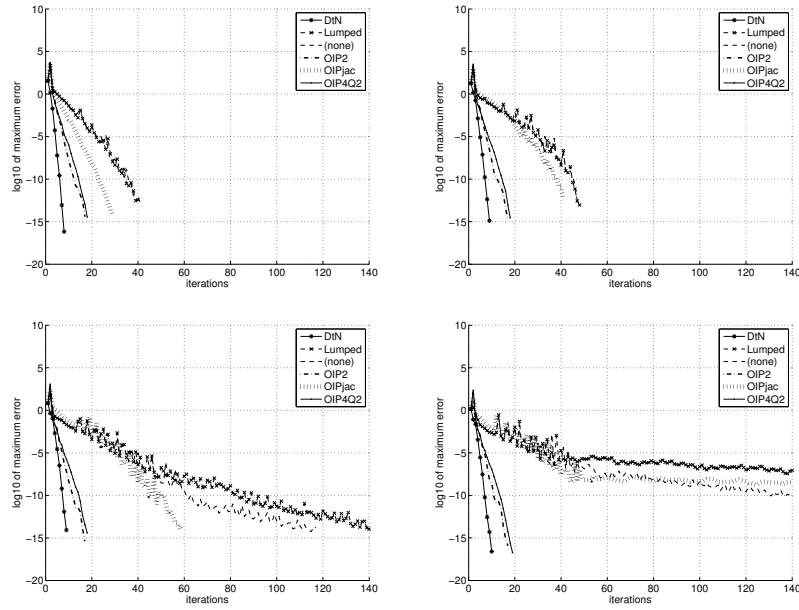

Fig. 3 Maximum errors between iterates and the FEM solutions for (11) with $h = 1/32$ for $\nu = 1, 10, 100, 1000$.

Remark 5. For the quadratic and the quartic/quadratic approximation, we adapted the interface Laplacian to $\partial_y(a(x,y)\partial_y)$ and at the same time multiplied the optimized parameters with $a(x,y)$. For the Jacobi induced preconditioner, we still use the interface Laplacian operator, which is better than using the diffusion operator.

References

1. Achdou, Y., Nataf, F.: Preconditioners for the mortar method based on local approximations of the Steklov-Poincaré operator. Mathematical Models and Methods in Applied Sciences **5**(7), 967–997 (1995)
2. Arioli, M., Kourounis, D., Loghin, D.: Discrete fractional Sobolev norms for domain decomposition preconditioning. IMA J. Numer. Anal. **33**(1), 318–342 (2013)
3. Arioli, M., Loghin, D.: Discrete interpolation norms with applications. SIAM J. Numer. Anal. **47**(4), 2924–2951 (2009)
4. Bjorstad, P.E., Widlund, O.B.: Iterative methods for the solution of elliptic problems on regions partitioned into substructures. SIAM J. Numer. Anal. **23**(6), 1097–1120 (1986)
5. Bramble, J.H., Pasciak, J.E., Xu, J.: A multilevel preconditioner for domain decomposition boundary systems. In: Proceedings of the 10th international conference on computing methods in applied sciences and engineering on Computing methods in applied sciences and engineering, Paris, pp. 107–118. Nova Science Publishers (1991)
6. Darbas, M., Darrigrand, E., Lafranche, Y.: Combining analytic preconditioner and fast multipole method for the 3-D Helmholtz equation. Journal of Computational Physics **236**, 289 – 316 (2013)
7. Dryja, M.: A capacitance matrix method for Dirichlet problem on polygon region. Numerische Mathematik **39**, 51–64 (1982)
8. Farhat, C., Lesoinne, M., Pierson, K.: A scalable dual-primal domain decomposition method. Numerical Linear Algebra with Applications **7**(7-8), 687–714 (2000)
9. Farhat, C., Mandel, J., Roux, F.X.: Optimal convergence properties of the FETI domain decomposition method. Comput. Methods Appl. Mech. Engrg. **115**, 365–385 (1994)
10. Farhat, C., Roux, F.X.: A method of finite element tearing and interconnecting and its parallel solution algorithm. Int. J. Numer. Meth. Engng. **32**(6), 1205–1227 (1991)
11. Farhat, C., Roux, F.X.: Implicit parallel processing in structural mechanics. Computational Mechanics Advances **II**(1), 1–124 (1994)
12. Hecht, F., Pironneau, O., Morice, J., Hyaric, A., Ohtsuka, K.: Freefem++. Universite Pierre et Marie Curie (2012). URL http://www.freefem.org/ff++
13. Krautle, S.: A domain decomposition method using efficient interface-acting preconditioners. Math. Comp. **74**, 1231–1256 (2004)
14. Mandel, J., Tezaur, R.: Convergence of a substructuring method with Lagrange multipliers. Numerische Mathematik **73**(4), 473–487 (1996)
15. Rixen, D.J., Farhat, C.: A simple and efficient extension of a class of substructure based preconditioners to heterogeneous structural mechanics problems. Int. J. Numer. Meth. Engng **44**(4), 489–516 (1999)
16. Xu, J., Zhang, S.: Preconditioning the Poincaré-Steklov operator by using Green's function. Math. Comput. **66**(217), 125–138 (1997)