Higher order optimized Schwarz algorithms in the framework of DDFV schemes

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Framework

- An anisotropic diffusion problem:

$$
\begin{aligned}
&-\mathrm{div}(A(x)\nabla u) + \eta u = f \text{ in } \Omega = \cup_i \Omega_i,\\
&u = 0 \text{ on } \partial\Omega
\end{aligned}
$$

  with $A(x) = A_i \in \mathcal{M}_2(\mathbb{R})$ for $x \in \Omega_i$.

- The Schwarz algorithms with Ventcell BC at interface $\Gamma_{ij} = \partial\overline{\Omega}_i \cap \overline{\Omega}_j$

$$
\begin{aligned}
&-\mathrm{div}(A_i \nabla u_i^k) + \eta u_i^k = f \text{ on } \Omega_i,\\
&u_i^k = 0 \text{ on } \partial\Omega \cap \partial\Omega_i,\\
&A_i \nabla u_i^k \cdot \vec{\mathbf{n}}_{ij} + \Lambda(u_i^k) = -A_j \nabla u_j^{k-1} \cdot \vec{\mathbf{n}}_{ji} + \Lambda(u_j^{k-1}) \text{ on } \Gamma_{ij}
\end{aligned}
$$

  with

$$
\Lambda(\phi) = p\phi - q\partial_y(A_{yy}\partial_y \phi)
$$

Goal

- Develop a **discrete** Schwarz algorithm with Ventcell BC at interface.
- Use the Discrete Duality Finite Volume (**DDFV**) discretisation.
  Hermeline 00', Domelevo, Omnès 05', Andreianov, Boyer, Hubert 04'

Approximation of the problem

$$-\mathrm{div}(A(x)\nabla u) = f$$

The finite volume strategy:

- Consider $\mathcal{T} = \cup \kappa$ a partition of Ω.
  Associate a point $x_\mathcal{K}$ and an unknown $u_\mathcal{K}$ to each $\kappa \in \mathcal{T}$.
- Integrate on any control volume κ the equation:

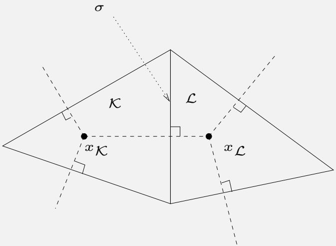
$$-\int_\mathcal{K} \mathrm{div}(A(x)\nabla u)\, dx = -\sum_{\sigma \in \partial\mathcal{K}} \int_\sigma A(x)\nabla u \cdot \vec{\mathbf{n}} = \int_\mathcal{K} f(x)\, dx$$

- Approximate the normal fluxes $\int_\sigma A(x)\nabla u \cdot \vec{\mathbf{n}}$ in a consistant and conservative way.

- In the classical case $A = Id$, $\nabla u \cdot \vec{\mathbf{n}}$ can be approximated by a VF4/TPFA scheme (Two Point Flux Approximation)

  For $\sigma = \kappa|\mathcal{L}$ $\qquad \nabla u \cdot \vec{\mathbf{n}} \sim \dfrac{u_\mathcal{L} - u_\mathcal{K}}{d(x_\mathcal{K}, x_\mathcal{L})}$

  for "admissible" meshes ($x_\mathcal{K} x_\mathcal{L} \,/\!\!/\, \vec{\mathbf{n}}$).



APPROXIMATION OF THE PROBLEM

$$-\mathrm{div}(A(x)\nabla u) = f$$

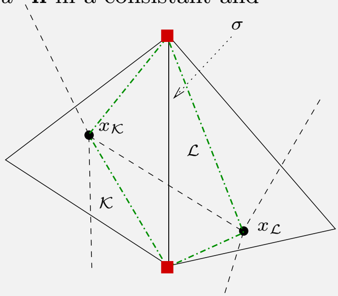
THE FINITE VOLUME STRATEGY:

- Consider $\mathcal{T} = \cup \kappa$ a partition of Ω.
  Associate a point $x_{\mathcal{K}}$ and an unknown $u_{\mathcal{K}}$ to each $\kappa \in \mathcal{T}$.
- Integrate on any control volume κ the equation:

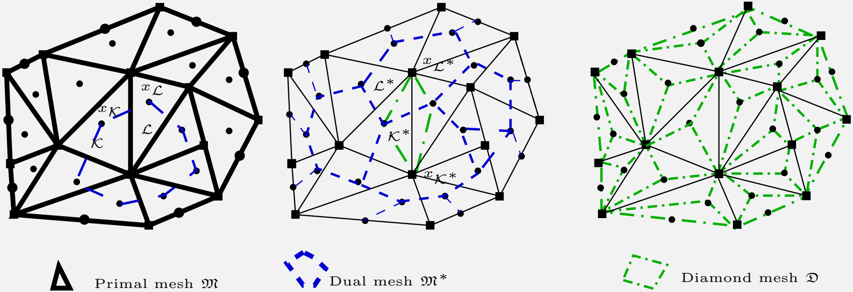
$$-\int_{\mathcal{K}} \mathrm{div}(A(x)\nabla u)\, dx = -\sum_{\sigma \in \partial \mathcal{K}} \int_{\sigma} A(x)\nabla u \cdot \vec{\mathbf{n}} = \int_{\mathcal{K}} f(x)\, dx$$

- Approximate the normal fluxes $\int_\sigma A(x)\nabla u \cdot \vec{\mathbf{n}}$ in a consistant and conservative way.

- For general anisotropy, it is impossible to construct $(x_{\mathcal{K}})$ such that $\vec{\mathbf{n}}^t A // \vec{x_{\mathcal{K}} x_{\mathcal{L}}}$
  $\Rightarrow$ **New unknowns have to be added to reconstruct a whole discrete gradient.**



- Description of the DDFV scheme
- Properties of the scheme
- The associated Schwarz algorithm
- Convergence of the Schwarz algorithm
- Numerical experimentations



Primal cells

$(\mathcal{K}, x_\mathcal{K})_{\mathcal{K}\in\mathfrak{M}}$

Dual cells

$(\mathcal{K}^*, x_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*}$

Diamonds

$(\mathcal{D}, x_\mathcal{D})_{\mathcal{D}\in\mathfrak{D}}$

$\rightsquigarrow u^{\mathfrak{M}} = (u_\mathcal{K})_{\mathcal{K}\in\mathfrak{M}}$ $\quad u^{\mathfrak{M}^*} = (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*}$

$\rightsquigarrow u_\mathcal{T} = (u^{\mathfrak{M}}, u^{\mathfrak{M}^*})$,

$\nabla^\mathfrak{D} u_\mathcal{T}$ discrete gradient

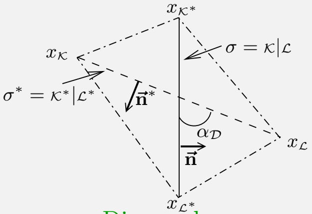
(Hermeline '00), (Domelevo-Omnès '05), (Andreianov-Boyer-Hubert '07)

Discrete gradient for a vector in $\mathbb{R}^{\mathcal{T}}$

$$\nabla^{\mathfrak{D}} : \mathbb{R}^{\mathcal{T}} \longrightarrow \left(\mathbb{R}^2\right)^{\mathfrak{D}}$$

$$\text{where } \begin{cases} \nabla_{\mathcal{D}} u_{\mathcal{T}}.(x_{\mathcal{L}} - x_{\mathcal{K}}) = u_{\mathcal{L}} - u_{\mathcal{K}}, \\ \nabla_{\mathcal{D}} u_{\mathcal{T}}.(x_{\mathcal{L}^*} - x_{\mathcal{K}^*}) = u_{\mathcal{L}^*} - u_{\mathcal{K}^*}. \end{cases}$$

$$\nabla_{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{2m_{\mathcal{D}}}\Big((u_{\mathcal{L}} - u_{\mathcal{K}}) m_{\sigma} \vec{\mathbf{n}} + (u_{\mathcal{L}^*} - u_{\mathcal{K}^*}) m_{\sigma^*} \vec{\mathbf{n}}^*\Big).$$



Diamond

Discrete divergence $\text{div}^{\mathcal{T}} : \left(\mathbb{R}^2\right)^{\mathfrak{D}} \longrightarrow \mathbb{R}^{\mathcal{T}}$

By mimicking the following continuous equality :

$$\int_{\mathcal{K}} \text{div}\xi = \sum_{\sigma \subset \partial\mathcal{K}} \int_{\sigma} \xi \cdot \vec{\mathbf{n}}.$$

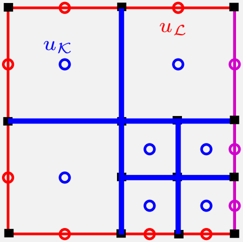
$$\mathcal{K} \in \mathfrak{M}, \quad \text{div}^{\mathcal{K}} \xi^{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}}} \sum_{\sigma \subset \partial\mathcal{K}} m_{\sigma} \xi^{\mathcal{D}} . \vec{\mathbf{n}}.$$

$$\mathcal{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*, \quad \text{div}^{\mathcal{K}^*} \xi^{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}^*}} \sum_{\sigma^* \subset \partial\mathcal{K}^*} m_{\sigma^*} \xi^{\mathcal{D}} . \vec{\mathbf{n}}^*.$$

Stokes formula (Discrete Duality) $-\displaystyle\int_{\Omega} \text{div}^{\mathcal{T}}(\xi^{\mathfrak{D}}) u_{\mathcal{T}} = \int_{\Omega} \xi^{\mathfrak{D}} \cdot \nabla^{\mathfrak{D}} u_{\mathcal{T}}$

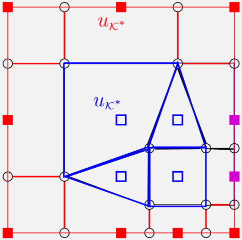
Interface unknowns on Γ

Primal unknowns

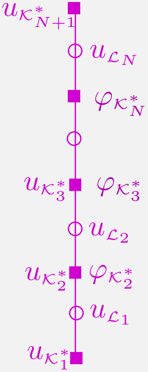


- $\forall \mathcal{K} \in \mathfrak{M}, u_{\mathcal{K}}$
- $\forall \mathcal{L} \in \partial\mathfrak{M}_D, u_{\mathcal{L}}$

Dual unknowns



- $\forall \mathcal{K}^* \in \mathfrak{M}^*, u_{\mathcal{K}^*}$
- $\forall \mathcal{K}^* \in \partial\mathfrak{M}^*_D, u_{\mathcal{K}^*}$



- $\forall \mathcal{L} \in \partial\mathfrak{M}_\Gamma, u_{\mathcal{L}}$
- $\forall \mathcal{K}^* \in \partial\mathfrak{M}^*_\Gamma, u_{\mathcal{K}^*}$
- $\forall \mathcal{K}^* \in \partial\mathfrak{M}^*_\Gamma, \varphi_{\mathcal{K}^*}$

▶ One equation per unknowns

▶ The DDFV scheme with **mixed Dirichlet/Ventcell BC**.

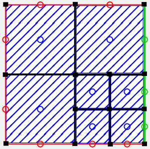
$$\begin{aligned} (1) \quad & -\mathrm{div}(A \cdot \nabla u) + \eta u = f, \quad \text{in } \Omega, \\ (2) \quad & u = 0, \quad \text{on } \partial\Omega \setminus \Gamma, \\ (3) \quad & A\nabla u \cdot \vec{\mathbf{n}} + \Lambda(u) = g, \ \text{ on } \Gamma. \end{aligned}$$

- On the primal mesh :
  - Integrate the equation (1) on interior primal cell $\mathcal{K} \in \mathfrak{M}$,

    $$\Longrightarrow -\mathrm{div}^{\mathcal{K}}\left(A^{\mathfrak{D}} \nabla^{\mathfrak{D}} u_{\mathcal{T}}\right) + \eta_{\mathcal{K}} u_{\mathcal{K}} = f_{\mathcal{K}}$$

  - Impose the Dirichlet boundary condition (2) on $\mathcal{K} \in \partial\mathfrak{M}_D$,

    $$\Longrightarrow u_{\mathcal{K}} = 0$$



▶ The DDFV scheme with **mixed Dirichlet/Ventcell BC**.

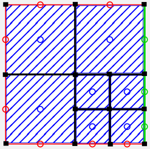
$$\begin{aligned} (1)\quad & -\text{div}(A\cdot\nabla u)+\eta u = f, \quad \text{in } \Omega,\\ (2)\quad & u = 0, \quad \text{on } \partial\Omega\setminus\Gamma,\\ (3)\quad & A\nabla u\cdot\vec{\mathbf{n}}+\Lambda(u)=g, \ \text{on } \Gamma. \end{aligned}$$

- On the primal mesh :
  - Integrate the equation (1) on interior primal cell $\mathcal{K}\in\mathfrak{M}$,

    $$\Longrightarrow -\text{div}^{\mathcal{K}}\left(A^{\mathfrak{D}}\nabla^{\mathfrak{D}}u_{\mathcal{T}}\right)+\eta_{\mathcal{K}}u_{\mathcal{K}}=f_{\mathcal{K}}$$

  - Impose the Dirichlet boundary condition (2) on $\mathcal{K}\in\partial\mathfrak{M}_D$,

    $$\Longrightarrow u_{\mathcal{K}}=0$$

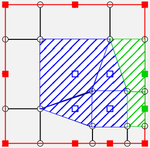


- On the dual mesh :
  - Integrate the equation (1) on interior dual cell $\mathcal{K}^*\in\mathfrak{M}^*$,

    $$\Longrightarrow -\text{div}^{\mathcal{K}^*}\left(A^{\mathfrak{D}}\nabla^{\mathfrak{D}}u_{\mathcal{T}}\right)+\eta_{\mathcal{K}^*}u_{\mathcal{K}^*}=f_{\mathcal{K}^*}$$

  - Impose the Dirichlet boundary condition (2) on $\mathcal{K}^*\in\partial\mathfrak{M}^*_D$,

    $$\Longrightarrow u_{\mathcal{K}^*}=0$$



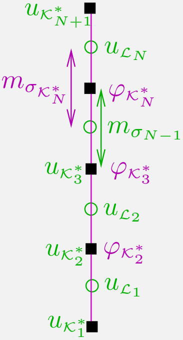
Particular treatment for the interface

- On $\mathcal{L}_s \in \partial\mathfrak{M}_\Gamma, s = 1, \cdots, N$,

$$A_\mathcal{D}\nabla_\mathcal{D} u_\mathcal{T} \cdot \vec{\mathbf{n}} + \Lambda_{\mathcal{L}_s}(u_{\partial\mathfrak{M}_\Gamma}) = g_{\mathcal{L}_s}$$

with

$$\Lambda_{\mathcal{L}_s}(u_{\partial\mathfrak{M}_\Gamma}) = p u_{\mathcal{L}_s} - A_{yy}\frac{q}{m_{\sigma_s}}\left(\frac{u_{\mathcal{L}_{s+1}} - u_{\mathcal{L}_s}}{m_{\sigma_{\mathcal{K}^*_{s+1}}}} - \frac{u_{\mathcal{L}_s} - u_{\mathcal{L}_{s-1}}}{m_{\sigma_{\mathcal{K}^*_s}}}\right)$$

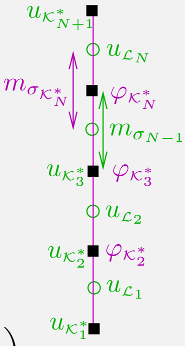


▶ On $\mathcal{L}_s \in \partial\mathfrak{M}_\Gamma, s = 1, \cdots, N$,

$$A_\mathcal{D}\nabla_\mathcal{D}u_\mathcal{T}\cdot\vec{\mathbf{n}} + \Lambda_{\mathcal{L}_s}(u_{\partial\mathfrak{M}_\Gamma}) = g_{\mathcal{L}_s}$$

with

$$\Lambda_{\mathcal{L}_s}(u_{\partial\mathfrak{M}_\Gamma}) = pu_{\mathcal{L}_s} - A_{yy}\frac{q}{m_{\sigma_s}}\left(\frac{u_{\mathcal{L}_{s+1}} - u_{\mathcal{L}_s}}{m_{\sigma_{\mathcal{K}^*_{s+1}}}} - \frac{u_{\mathcal{L}_s} - u_{\mathcal{L}_{s-1}}}{m_{\sigma_{\mathcal{K}^*_s}}}\right)$$



▶ On $\mathcal{K}^*_s \in \partial\mathfrak{M}^*_\Gamma, s = 2, \cdots, N$

$$\varphi_{\mathcal{K}^*_s} + \Lambda_{\mathcal{K}^*_s}(u_{\partial\mathfrak{M}^*_\Gamma}) = g_{\mathcal{K}^*_s}$$

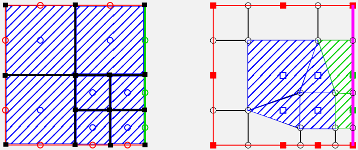
with

$$\Lambda_{\mathcal{K}^*_s}(u_{\partial\mathfrak{M}^*_\Gamma}) = pu_{\mathcal{K}^*_s} - A_{yy}\frac{q}{m_{\sigma_{\mathcal{K}^*_s}}}\left(\frac{u_{\mathcal{K}^*_{s+1}} - u_{\mathcal{K}^*_s}}{m_{\sigma_s}} - \frac{u_{\mathcal{K}^*_s} - u_{\mathcal{K}^*_{s-1}}}{m_{\sigma_{s-1}}}\right)$$

▶ Integrate (1) on boundary dual cell $\mathcal{K}^* \in \partial\mathfrak{M}^*_\Gamma$

$$-\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}}\frac{m_{\sigma^*}}{m_{\mathcal{K}^*}}A_\mathcal{D}\nabla_\mathcal{D}u_\mathcal{T}\cdot\vec{\mathbf{n}}^* - \sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}\\ \mathcal{D}\cap\Gamma\neq\emptyset}}\frac{m_{\sigma_{\mathcal{K}^*}}}{m_{\mathcal{K}^*}}\varphi_{\mathcal{K}^*} + \eta_{\mathcal{K}^*}u_{\mathcal{K}^*} = f_{\mathcal{K}^*}$$

- DDFV scheme



$$
\begin{cases}
u_{\mathcal{K}} = 0, \quad \forall\, \mathcal{K} \in \partial\mathfrak{M}_D, \qquad u_{\mathcal{K}^*} = 0, \quad \forall\, \mathcal{K}^* \in \partial\mathfrak{M}_D^*, \\
-\mathrm{div}^{\mathcal{K}}\left(A^{\mathfrak{D}}\nabla^{\mathfrak{D}} u_{\mathcal{T}}\right) + \eta_{\mathcal{K}} u_{\mathcal{K}} = f_{\mathcal{K}}, \quad \forall\, \mathcal{K} \in \mathfrak{M}, \\
-\mathrm{div}^{\mathcal{K}^*}\left(A^{\mathfrak{D}}\nabla^{\mathfrak{D}} u_{\mathcal{T}}\right) + \eta_{\mathcal{K}^*} u_{\mathcal{K}^*} = f_{\mathcal{K}^*}, \quad \forall\, \mathcal{K}^* \in \mathfrak{M}^*, \\
-\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}} \frac{m_{\sigma^*}}{m_{\mathcal{K}^*}} A_{\mathcal{D}}\nabla_{\mathcal{D}} u_{\mathcal{T}} \cdot \vec{\mathbf{n}}^* - \sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*} \\ \mathcal{D}\cap\Gamma\neq\emptyset}} \frac{m_{\sigma_{\mathcal{K}^*}}}{m_{\mathcal{K}^*}} \varphi_{\mathcal{K}^*} + \eta_{\mathcal{K}^*} u_{\mathcal{K}^*} = f_{\mathcal{K}^*}, \quad \forall\, \mathcal{K}^* \in \partial\mathfrak{M}_\Gamma^*, \\
A_{\mathcal{D}}\nabla_{\mathcal{D}} u_{\mathcal{T}} \cdot \vec{\mathbf{n}} + \Lambda_{\mathcal{L}}(u_{\partial\mathfrak{M}_\Gamma}) = g_{\mathcal{L}}, \quad \forall\, \mathcal{L} \in \partial\mathfrak{M}_\Gamma, \\
\varphi_{\mathcal{K}^*} + \Lambda_{\mathcal{K}^*}(u_{\partial\mathfrak{M}_\Gamma^*}) = g_{\mathcal{K}^*}, \quad \forall\, \mathcal{K}^* \in \partial\mathfrak{M}_\Gamma^*.
\end{cases}
$$

Compact way

$$
\mathcal{L}^{\mathcal{T}}_{\Omega,\Gamma}(u_{\mathcal{T}}, \varphi_{\mathcal{T}}, f^{\mathcal{T}}, g^{\mathcal{T}}) = 0. \tag{4}
$$

- The scheme (4) possesses a unique solution $U^{\mathcal{T}} = (u_{\mathcal{T}}, \varphi_{\mathcal{T}}) \in \mathbb{R}^{\mathcal{T}} \times \Phi_\Gamma^{\mathcal{T}}$.

ENERGY ESTIMATE

- By linearity, it is suffisant to prove

$$\mathcal{L}^{\mathcal{T}}_{\Omega,\Gamma}(u_{\mathcal{T}},\varphi_{\mathcal{T}},0,0)=0 \Longrightarrow u_{\mathcal{T}}=0,\varphi_{\mathcal{T}}=0$$

- Multiplying by $u_{\mathcal{T}}$, suming and using discrete Stokes formula lead to

$$2\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} A_{\mathcal{D}}\nabla_{\mathcal{D}}u_{\mathcal{T}}\cdot\nabla_{\mathcal{D}}u_{\mathcal{T}} - \sum_{\mathcal{L}\in\partial\mathfrak{M}_\Gamma} m_{\sigma_{\mathcal{L}}} \underbrace{A_{\mathcal{D}}\nabla_{\mathcal{D}}u_{\mathcal{T}}\cdot\vec{\mathbf{n}}}_{-\Lambda_{\mathcal{L}}(u_{\partial\mathfrak{M}_\Gamma})} u_{\mathcal{L}}$$

$$-\sum_{\mathcal{K}^*\in\partial\mathfrak{M}^*_\Gamma} m_{\sigma_{\mathcal{K}^*}} \underbrace{\varphi_{\mathcal{K}^*}}_{-\Lambda_{\mathcal{K}^*}(u_{\partial\mathfrak{M}^*_\Gamma})} u_{\mathcal{K}^*} + \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}}\eta_{\mathcal{K}}u_{\mathcal{K}}^2 + \sum_{\mathcal{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*_\Gamma} m_{\mathcal{K}^*}\eta_{\mathcal{K}^*}u_{\mathcal{K}^*}^2 = 0$$

- Ventcell boundary conditions

$$2\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} A_{\mathcal{D}}\nabla_{\mathcal{D}}u_{\mathcal{T}}\cdot\nabla_{\mathcal{D}}u_{\mathcal{T}} + (\Lambda^{\partial\mathfrak{M}_\Gamma}(u_{\partial\mathfrak{M}_\Gamma}),u_{\partial\mathfrak{M}_\Gamma})$$

$$+(\Lambda^{\partial\mathfrak{M}^*_\Gamma}(u_{\partial\mathfrak{M}^*_\Gamma}),u_{\partial\mathfrak{M}^*_\Gamma}) + \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}}\eta_{\mathcal{K}}u_{\mathcal{K}}^2 + \sum_{\mathcal{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*_\Gamma} m_{\mathcal{K}^*}\eta_{\mathcal{K}^*}u_{\mathcal{K}^*}^2 = 0$$

We have obtained

$$2\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} A_{\mathcal{D}} \nabla_{\mathcal{D}} u_{\mathcal{T}} \cdot \nabla_{\mathcal{D}} u_{\mathcal{T}} + (\Lambda^{\partial\mathfrak{M}_\Gamma}(u_{\partial\mathfrak{M}_\Gamma}), u_{\partial\mathfrak{M}_\Gamma})$$
$$+(\Lambda^{\partial\mathfrak{M}^*_\Gamma}(u_{\partial\mathfrak{M}^*_\Gamma}), u_{\partial\mathfrak{M}^*_\Gamma}) + \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} \eta_{\mathcal{K}} u_{\mathcal{K}}^2 + \sum_{\mathcal{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*_\Gamma} m_{\mathcal{K}^*} \eta_{\mathcal{K}^*} u_{\mathcal{K}^*}^2 = 0$$

- The operators $\Lambda^{\partial\mathfrak{M}_\Gamma}$ and $\Lambda^{\partial\mathfrak{M}^*_\Gamma}$ are symmetric positive definite.
- The operators $\Lambda^{\partial\mathfrak{M}_\Gamma\,-1}$ and $\Lambda^{\partial\mathfrak{M}^*_\Gamma\,-1}$ are symmetric positive definite and induce a norm.

Since A is symmetric positive definite and η bounded by below, we get

$$\|\nabla^{\mathfrak{D}} u_{\mathcal{T}}\|_2 = 0 \text{ and } \|u_{\mathcal{T}}\|_2 = 0$$

We deduce

$$u_{\mathcal{T}} = 0$$

Ventcell boundary condition implies

$$\varphi_{\mathcal{T}} = 0.$$

DDFV Schwarz algorithm

- Choose $g^0_{\mathcal{T}_i} \in \Phi^{\mathcal{T}_i}_{\Gamma}$.
- $\forall\ n \geq 0$
  - Calculate
    $$\mathcal{L}^{\mathcal{T}_i}_{\Omega_i,\Gamma}(u^{n+1}_{\mathcal{T}_i}, \varphi^{n+1}_{\mathcal{T}_i}, f_{\mathcal{T}_i}, g^n_{\mathcal{T}_j}) = 0.$$
  - Calculate $g^{n+1}_{\mathcal{T}_i}$ by
    $$\forall\ \mathcal{K}^* \in \partial\mathfrak{M}^*_\Gamma, \quad g^{n+1}_{i,\mathcal{K}^*} = -\varphi^{n+1}_{i,\mathcal{K}^*} + \Lambda_{\mathcal{K}^*}(u^{n+1}_{\mathcal{T}_i})$$
    $$\forall\ \mathcal{L} \in \partial\mathfrak{M}_\Gamma, \quad g^{n+1}_{i,\mathcal{L}} = -A_{\mathcal{D}}\nabla_{\mathcal{D}} u^{n+1}_{\mathcal{T}_i} \cdot \vec{\mathbf{n}} + \Lambda_{\mathcal{L}}(u^{n+1}_{\mathcal{T}_i})$$

Convergence of the algorithm

Theorem

*The solution of the DDFV Schwarz algorithm **converges** when $n \to \infty$ to the solution of the classical DDFV scheme on Ω.*

STEP 1: DEFINE THE ERROS

- Construct $(u^{\infty}_{\mathcal{T}_i}, \varphi^{\infty}_{\mathcal{T}_i})$ from the solution $u_{\mathcal{T}}$ of the DDFV scheme on Ω s. t.

$$\mathcal{L}^{\mathcal{T}_i}_{\Omega_i,\Gamma}(u^{\infty}_{\mathcal{T}_i}, \varphi^{\infty}_{\mathcal{T}_i}, f_{\mathcal{T}_i}, g^{\infty}_{\mathcal{T}_j}) = 0.$$

- Observe that the errors $e^{n+1}_{\mathcal{T}_i} = u^{n+1}_{\mathcal{T}_i} - u^{\infty}_{\mathcal{T}_i}$, $\Phi^{n+1}_{\mathcal{T}_i} = \varphi^{n+1}_{\mathcal{T}_i} - \varphi^{\infty}_{\mathcal{T}_i}$ satisfy

$$\mathcal{L}^{\mathcal{T}_i}_{\Omega_i,\Gamma}(e^{n+1}_{\mathcal{T}_i}, \Phi^{n+1}_{\mathcal{T}_i}, 0, G^{n}_{\mathcal{T}_j}) = 0.$$

with

$$\forall\ \mathcal{K}^* \in \partial\mathfrak{M}^*_{\Gamma}, \quad G^n_{j,\mathcal{K}^*} = -\Phi^n_{i,\mathcal{K}^*} + \Lambda_{\mathcal{K}^*}(e^n_{\mathcal{T}_j})$$
$$\forall\ \mathcal{L} \in \partial\mathfrak{M}_{\Gamma}, \quad G^n_{j,\mathcal{L}} = -A_{\mathcal{D}}\nabla_{\mathcal{D}} e^n_{\mathcal{T}_j} \cdot \vec{\mathbf{n}} + \Lambda_{\mathcal{L}}(e^n_{\mathcal{T}_j})$$

STEP 2: ENERGY ESTIMATE

- Multiplying by $e^{n+1}_{\mathcal{T}_j}$, suming and using discrete Stokes formula lead to

$$2\sum_{\mathcal{D}\in\mathfrak{D}_i} m_{\mathcal{D}} A_{\mathcal{D}} \nabla_{\mathcal{D}} e^{n+1}_{\mathcal{T}_i} \cdot \nabla_{\mathcal{D}} e^{n+1}_{\mathcal{T}_i} - \sum_{\mathcal{L}\in\partial\mathfrak{M}_{i,\Gamma}} m_{\sigma_{\mathcal{L}}} A_{\mathcal{D}} \nabla_{\mathcal{D}} e^{n+1}_{\mathcal{T}_i} \cdot \vec{\mathbf{n}} e^{n+1}_{i,\mathcal{L}}$$
$$- \sum_{\mathcal{K}^*\in\partial\mathfrak{M}^*_{i,\Gamma}} m_{\sigma_{\mathcal{K}^*}} \Phi^{n+1}_{i,\mathcal{K}^*} e^{n+1}_{i,\mathcal{K}^*} + \sum_{\mathcal{K}\in\mathfrak{M}_i} m_{\mathcal{K}} \eta_{\mathcal{K}} (e^{n+1}_{i,\mathcal{K}})^2 + \sum_{\mathcal{K}^*\in\mathfrak{M}^*_i\cup\partial\mathfrak{M}^*_{i,\Gamma}} m_{\mathcal{K}^*} \eta_{\mathcal{K}^*} (e^{n+1}_{i,\mathcal{K}^*})^2 = 0$$

Step 3: Adapt the Lions's trick at the discrete level

- Use the scalar product defined by $(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}$:

$$-\sum_{\mathcal{L}\in\partial\mathfrak{M}_{i,\Gamma}} m_{\sigma_\mathcal{L}} A_\mathcal{D}\nabla_\mathcal{D} e_{\mathcal{T}_i}^{n+1}\cdot\vec{\mathbf{n}} e_{i,\mathcal{L}}^{n+1} = \left(A^\mathfrak{D}\nabla^\mathfrak{D} e_{\mathcal{T}_i}^{n+1}\cdot\vec{\mathbf{n}}, \Lambda^{\partial\mathfrak{M}_\Gamma}(e_{\partial\mathfrak{M}_{i,\Gamma}}^{n+1})\right)_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}}$$

- Use the formula $-ab = \frac{1}{4}\left((a-b)^2-(a+b)^2\right)$:

$$-\sum_{\mathcal{L}\in\partial\mathfrak{M}_{i,\Gamma}} m_{\sigma_\mathcal{L}} A_\mathcal{D}\nabla_\mathcal{D} e_{\mathcal{T}_i}^{n+1}\cdot\vec{\mathbf{n}} e_{i,\mathcal{L}}^{n+1} = \frac{1}{4}\left\|-A^\mathfrak{D}\nabla^\mathfrak{D} e_{\mathcal{T}_i}^{n+1}\cdot\vec{\mathbf{n}}+\Lambda^{\partial\mathfrak{M}_\Gamma}(e_{\partial\mathfrak{M}_{i,\Gamma}}^{n+1})\right\|^2_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}} - \frac{1}{4}\Bigg\|\underbrace{A^\mathfrak{D}\nabla^\mathfrak{D} e_{\mathcal{T}_i}^{n+1}\cdot\vec{\mathbf{n}}+\Lambda^{\partial\mathfrak{M}_\Gamma}(e_{\partial\mathfrak{M}_{i,\Gamma}}^{n+1})}_{=G^n_{j,\partial\mathfrak{M}_{j,\Gamma}}}\Bigg\|^2_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}}$$

- Use the Ventcell BC:

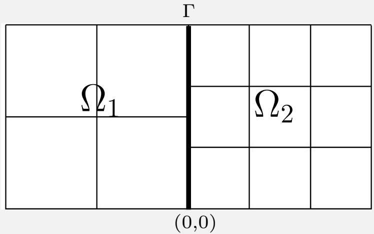
$$-\sum_{\mathcal{L}\in\partial\mathfrak{M}_{i,\Gamma}} m_{\sigma_\mathcal{L}} A_\mathcal{D}\nabla_\mathcal{D} e_{\mathcal{T}_i}^{n+1}\cdot\vec{\mathbf{n}} e_{i,\mathcal{L}}^{n+1} = \frac{1}{4}\left\|-A^\mathfrak{D}\nabla^\mathfrak{D} e_{\mathcal{T}_i}^{n+1}\cdot\vec{\mathbf{n}}+\Lambda^{\partial\mathfrak{M}_\Gamma}(e_{\partial\mathfrak{M}_{i,\Gamma}}^{n+1})\right\|^2_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}} - \frac{1}{4}\left\|-A^\mathfrak{D}\nabla^\mathfrak{D} e_{\mathcal{T}_j}^{n}\cdot\vec{\mathbf{n}}+\Lambda^{\partial\mathfrak{M}_\Gamma}(e_{\partial\mathfrak{M}_{j,\Gamma}}^{n})\right\|^2_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}}$$

Step 4: Conclusion

- Summing over $n = 0, \cdots, n_{max} - 1$ and $i = 1, 2$, we get

$$
\begin{aligned}
& 2 \sum_{n=0}^{n_{max}-1} \sum_{i=1,2} \sum_{\mathfrak{D} \in \mathfrak{D}_i} m_{\mathcal{D}} A_{\mathcal{D}} \nabla_{\mathcal{D}} e_{\mathcal{T}_i}^{n+1} \cdot \nabla_{\mathcal{D}} e_{\mathcal{T}_i}^{n+1} \\
& + \sum_{n=0}^{n_{max}-1} \sum_{i=1,2} \sum_{\mathcal{K} \in \mathfrak{M}_i} m_{\mathcal{K}} \eta_{\mathcal{K}} (e_{i,\mathcal{K}}^{n+1})^2 + \sum_{n=0}^{n_{max}-1} \sum_{i=1,2} \sum_{\mathcal{K}^* \in \mathfrak{M}_i^* \cup \partial\mathfrak{M}_{i,\Gamma}^*} m_{\mathcal{K}^*} \eta_{\mathcal{K}^*} (e_{i,\mathcal{K}^*}^{n+1})^2 \\
& + \frac{1}{4} \sum_{i=1,2} \left\| -A^{\mathfrak{D}} \nabla^{\mathfrak{D}} e_{\mathcal{T}_i}^{n_{max}} \cdot \vec{\mathbf{n}} + \Lambda^{\partial\mathfrak{M}_\Gamma} (e_{\partial\mathfrak{M}_{i,\Gamma}}^{n_{max}}) \right\|_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}}^2 \\
& + \sum_{i=1,2} \frac{1}{4} \left\| -\Phi_{\mathcal{T}_i}^{n_{max}} + \Lambda^{\partial\mathfrak{M}_\Gamma^*} (e_{\partial\mathfrak{M}_{i,\Gamma}^*}^{n_{max}}) \right\|_{(\Lambda^{\partial\mathfrak{M}_\Gamma^*})^{-1}}^2 \\
& = \sum_{i=1,2} \frac{1}{4} \left\| -A^{\mathfrak{D}} \nabla^{\mathfrak{D}} e_{\mathcal{T}_i}^{0} \cdot \vec{\mathbf{n}} + \Lambda^{\partial\mathfrak{M}_\Gamma} (e_{\partial\mathfrak{M}_{i,\Gamma}}^{0}) \right\|_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}}^2 \\
& + \sum_{i=1,2} \frac{1}{4} \left\| -\Phi_{\mathcal{T}_i}^{0} + \Lambda^{\partial\mathfrak{M}_\Gamma^*} (e_{\partial\mathfrak{M}_{i,\Gamma}^*}^{0}) \right\|_{(\Lambda^{\partial\mathfrak{M}_\Gamma^*})^{-1}}^2 .
\end{aligned}
$$

- This shows that the total energy stays bounded as the iteration $n \to +\infty$, and hence the algorithm converges.



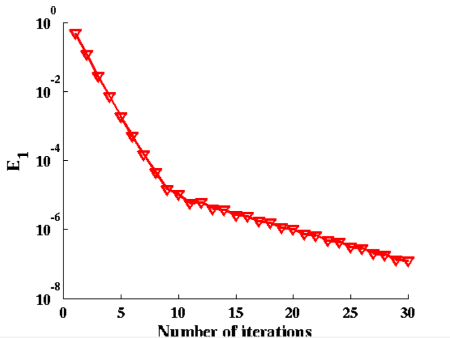
$$u_e(x,y) = \sin(\pi x)\sin(\pi y)\sin(\pi(x+y)),$$

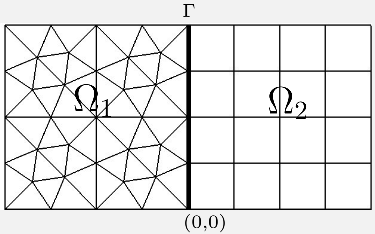
$$A(x,y) = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}.$$

$$\eta(x,y) = 1$$

- $p = 2.5$
- $q = 3.10^{-2}$

Convergence $E_1 = \dfrac{||u_n^{\mathcal{T}_i} - u^{\mathcal{T}_i}||_2}{||u^{\mathcal{T}_i}||_2}$





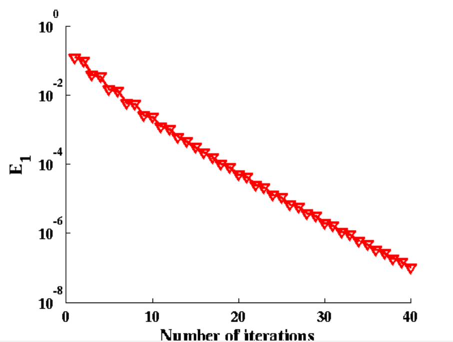
$$u_e(x,y) = \cos(2.5\pi x)\cos(2.5\pi y),$$

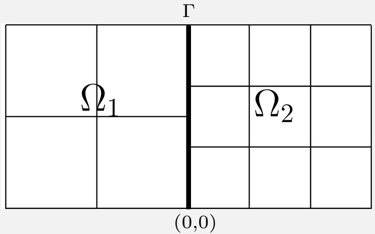
$$A(x,y) = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}.$$

$$\eta(x,y) = 1$$

- $p = 2.5$
- $q = 3.10^{-2}$

Convergence $E_1 = \dfrac{||u_n^{\mathcal{T}_i} - u^{\mathcal{T}_i}||_2}{||u^{\mathcal{T}_i}||_2}$



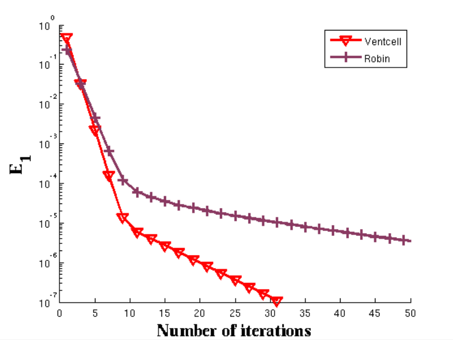


$$u_e(x,y) = \sin(\pi x)\sin(\pi y)\sin(\pi(x+y)),$$

$$A(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \eta(x,y) = 1$$

- $p = C_{opt}(A,\eta)h^{-\frac{1}{4}} \quad q = C_{opt}(A,\eta)h^{\frac{3}{4}}$
- $p = \widetilde{C}_{opt}(A,\eta)h^{-\frac{1}{2}} \qquad q = 0$

Convergence $E_1 = \dfrac{||u_n^{\mathcal{T}_i} - u^{\mathcal{T}_i}||_2}{||u^{\mathcal{T}_i}||_2}$



▶ Available when using **cartesian grids**

- $A = \mathrm{Id}$

$$p_{opt} \sim \frac{(2-\sqrt{2})(2\pi^2+2\eta)^{3/8}}{\sqrt{2}-1} h^{-1/4}$$

$$q_{opt} \sim \frac{2^{3/8}}{2(\pi^2+\eta)^{1/8}} h^{3/4}$$

- A diagonal

$$p_{opt} \sim 2^{3/8}(A_{yy}\pi^2+\eta)^{3/8}\frac{A_{xx}-a}{a} b h^{-1/4},$$

$$q_{opt} \sim \frac{2^{3/8}}{2(A_{yy}\pi^2+\eta)^{1/8}} c h^{3/4},$$

where a, b, c depend only on the diagonal coefficient of A.

- Comparison numerical/theoritical optimized parameters for Laplace equation.
- Comparison with Robin condition.
- Optimized parameters for anisotropic operator.
- Optimization of the Ventcell parameters.

Thank you for your attention!