# DDFV Ventcell Schwarz Algorithms

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## 1 Introduction

We are interested in this paper in anisotropic diffusion problems of the form

$$\mathcal{L}(u) := -\mathrm{div}(A\nabla u) + \eta u = f \text{ in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

$$\text{with } (x,y) \in \Omega \mapsto A(x,y) = \begin{pmatrix} A_{xx} & A_{xy} \\ A_{xy} & A_{yy} \end{pmatrix}. \tag{2}$$

Over the last five years, classical and optimized Schwarz methods have been developed for (1) discretized with Discrete Duality Finite Volume (DDFV) schemes. Like for Discontinuous Galerkin methods, it is not a priori clear how to appropriately discretize transmission conditions. Two versions have been proposed for Robin transmission conditions in [2] and [4]. Only the second one leads to the expected rapid convergence rate of the optimized Schwarz algorithm, see [1] for parabolic problems.

The DDFV method needs a dual set of unknowns located on both vertices and “centers” of the initial mesh, which leads to two meshes, the primal and the dual one. This permits the reconstruction of two-dimensional discrete gradients located on a third partition of $\Omega$, called the diamond mesh. A discrete divergence operator is also defined by duality. This method is particularly accurate in terms of gradient approximations, see the benchmark [6] for problem (1) with $\eta = 0$, and also an extensive bibliography.

A non-overlapping Schwarz method using Ventcell transmission conditions was first proposed in [7]. For the model problem (1), the algorithm with two

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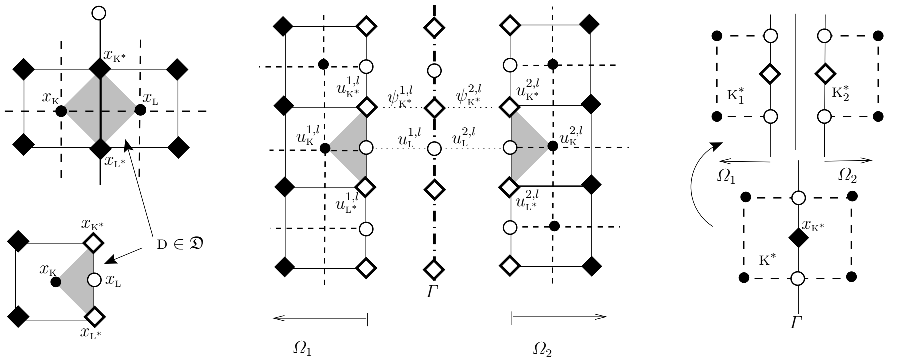


**Fig. 1** Diamond symbols are vertices of primal cells, circles are vertices of dual cells. Left: zoom on diamond cells in gray. Center: zoom on the interface $\Gamma$, and new unknowns needed to describe the DDFV scheme as the limit of the Schwarz algorithm. Right: zoom on a dual cell $K^*$ cut by $\Gamma$: $K^* = K_1^* \cup K_2^*$ with $K_i^* = \Omega_i \cap K^*$.

non-overlapping subdomains, $\Omega = \Omega_1 \cup \Omega_2$, and iteration index $l = 0, 1, \ldots$ is

$$\mathcal{L}(u_j^{l+1}) = f \quad \text{in } \Omega_j, \quad u = 0 \quad \text{on } \partial\Omega_j \cap \partial\Omega, \tag{3}$$

$$(A\nabla u_j^{l+1}, \mathbf{n}_{ji}) + \Lambda u_j^{l+1} = -(A\nabla u_i^l, \mathbf{n}_{ij}) + \Lambda u_i^l \quad \text{on } \Gamma = \partial\Omega_i \cap \partial\Omega_j, \tag{4}$$

with $\Lambda u = pu - q\partial_y(A_{yy}\partial_y u)$ (assuming that $\Gamma = \{x = 0\}$) and $\mathbf{n}_{ji}$ is the unit normal directed from $\Omega_j$ to $\Omega_i$. A FV4 finite volume discretization of this algorithm for an advection diffusion equation with isotropic diffusion is analyzed in [5]. We present here a DDFV discretization of (3)-(4), and prove convergence of the discretized algorithm.

## 2 DDFV schemes

**The meshes:** We now describe the DDFV Schwarz algorithm for general subdomains and decompositions using the notation from [2], see Figure 1. The primal mesh $\mathfrak{M}_j$ is a set of disjoint open polygonal control volumes $K \subset \Omega_j$ such that $\cup\overline{K} = \overline{\Omega_j}$. We denote by $\partial\mathfrak{M}_j$ the set of edges of the control volumes in $\mathfrak{M}_j$ included in $\partial\Omega_j$, and by $\partial\mathfrak{M}_{j,\Gamma}$ the set of edges of primal boundary cells related to the interface $\Gamma$. We use the same notations for the dual mesh, $\mathfrak{M}_j^*$, $\partial\mathfrak{M}_j^*$ and $\partial\mathfrak{M}_{j,\Gamma}^*$. We define the diamond cells $D_{\sigma,\sigma^*}$ as the quadrangles whose diagonals are a primal edge $\sigma = K|L = (x_{K^*}, x_{L^*})$ and a corresponding dual edge $\sigma^* = K^*|L^* = (x_K, x_L)$. The set of diamond cells is called the diamond mesh, denoted by $\mathfrak{D}_j$.

For any $V$ in $\mathfrak{M}_j \cup \partial\mathfrak{M}_j$ or $\mathfrak{M}_j^* \cup \partial\mathfrak{M}_j^*$, we denote by $m_V$ its Lebesgue measure, by $\mathcal{E}_V$ the set of its edges, and $\mathfrak{D}_V := \{D_{\sigma,\sigma^*} \in \mathfrak{D}_j,\ \sigma \in \mathcal{E}_V\}$. For

$\mathrm{D} = \mathrm{D}_{\sigma,\sigma^*}$ with vertices $(x_{\mathrm{K}}, x_{\mathrm{K}^*}, x_{\mathrm{L}}, x_{\mathrm{L}^*})$, we denote by $x_{\mathrm{D}}$ the center of D, that is the intersection of the primal edge $\sigma$ and the dual edge $\sigma^*$, by $m_{\mathrm{D}}$ its measure, by $m_\sigma$ the length of $\sigma$, by $m_{\sigma^*}$ the length of $\sigma^*$, by $m_{\sigma_{\mathrm{K}^*}}$ the length of $\partial\mathrm{K}^* \cap \Gamma$, by $m_{\sigma_{\mathrm{L}}}$ the length of $\mathrm{D} \cap \Gamma$, and by $m_{\sigma_{\mathrm{K}}}$ the length of $[x_{\mathrm{K}}, x_{\mathrm{D}}]$. $\mathbf{n}_{\sigma\mathrm{K}}$ is the unit vector normal to $\sigma$ oriented from $x_{\mathrm{K}}$ to $x_{\mathrm{L}}$, and $\mathbf{n}_{\sigma^*\mathrm{K}^*}$ is the unit vector normal to $\sigma^*$ oriented from $x_{\mathrm{K}^*}$ to $x_{\mathrm{L}^*}$.

**The unknowns:** the DDFV method associates to all primal control volumes $\mathrm{K} \in \mathfrak{M}_j \cup \partial\mathfrak{M}_j$ an unknown value $u_{j,\mathrm{K}}$, and to all dual control volumes $\mathrm{K}^* \in \mathfrak{M}^*_j \cup \partial\mathfrak{M}^*_j$ an unknown value $u_{j,\mathrm{K}^*}$. We denote the approximate solution on the mesh $\mathcal{T}_j$ by $u_{\mathcal{T}_j} = ((u_{j,\mathrm{K}})_{\mathrm{K}\in(\mathfrak{M}_j\cup\partial\mathfrak{M}_j)}, (u_{j,\mathrm{K}^*})_{\mathrm{K}^*\in(\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j)}) \in \mathbb{R}^{\mathcal{T}_j}$. DDFV schemes are described by two operators: a discrete gradient $\nabla^{\mathfrak{D}}$ and a discrete divergence $\mathrm{div}^{\mathcal{T}}$, which are dual to each other, see [2]. We define the discrete gradient $\nabla^{\mathfrak{D}} : u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} \mapsto \left(\nabla^{\mathrm{D}} u_{\mathcal{T}}\right)_{\mathrm{D}\in\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$ by

$$\nabla^{\mathrm{D}} u_{\mathcal{T}} := \frac{1}{2m_{\mathrm{D}}} \left( (u_{\mathrm{L}} - u_{\mathrm{K}}) m_\sigma \mathbf{n}_{\sigma\mathrm{K}} + (u_{\mathrm{L}^*} - u_{\mathrm{K}^*}) m_{\sigma^*} \mathbf{n}_{\sigma^*\mathrm{K}^*} \right), \quad \forall \mathrm{D} \in \mathfrak{D},$$

and the discrete divergence $\mathrm{div}^{\mathcal{T}} : \xi_{\mathfrak{D}} = (\xi_{\mathrm{D}})_{\mathrm{D}\in\mathfrak{D}} \mapsto \mathrm{div}^{\mathcal{T}} \xi_{\mathfrak{D}} \in \mathbb{R}^{\mathcal{T}}$ by

$$\mathrm{div}^{\mathrm{K}} \xi_{\mathfrak{D}} := \frac{1}{m_{\mathrm{K}}} \sum_{\mathrm{D}\in\mathfrak{D}_{\mathrm{K}}} m_\sigma (\xi_{\mathrm{D}}, \mathbf{n}_{\sigma\mathrm{K}}),\ \forall \mathrm{K} \in \mathfrak{M},\ \text{and } \mathrm{div}^{\mathrm{K}} \xi_{\mathfrak{D}} = 0, \forall \mathrm{K} \in \partial\mathfrak{M}, \quad (5)$$

$$\mathrm{div}^{\mathrm{K}^*} \xi_{\mathfrak{D}} := \frac{1}{m_{\mathrm{K}^*}} \sum_{\mathrm{D}\in\mathfrak{D}_{\mathrm{K}^*}} m_{\sigma^*} (\xi_{\mathrm{D}}, \mathbf{n}_{\sigma^*\mathrm{K}^*}),\ \forall \mathrm{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*. \quad (6)$$

We introduce additional flux unknowns $\psi_{j,\mathrm{K}^*}$ for $j = 1, 2$ on interface dual cells $\mathrm{K}^* \in \partial\mathfrak{M}^*_{j,\Gamma}$. Let $N$ be the number of edges on $\Gamma$. We sort these edges $\sigma_1, \ldots, \sigma_N$ such that $\sigma_s \cap \sigma_{s+1} \neq \emptyset$, and $x_{\mathrm{K}^*_s}, x_{\mathrm{K}^*_{s+1}}$ are the vertices of $\sigma_s$, where $x_{\mathrm{K}^*_s} = \sigma_s \cap \sigma_{s-1}$. For $u_{\mathcal{T}_j} \in \mathbb{R}^{\mathcal{T}_j}$, $\Psi_{\mathcal{T}_j} \in \mathbb{R}^{\partial\mathfrak{M}^*_{j,\Gamma}}$, $f_{\mathcal{T}_j} \in \mathbb{R}^{\mathcal{T}_j}$ and $h_{\mathcal{T}_j} \in \mathbb{R}^{\partial\mathfrak{M}_{j,\Gamma}\cup\partial\mathfrak{M}^*_{j,\Gamma}}$, we denote by $\mathcal{L}^{\mathcal{T}_j}_{\Omega_j,\Gamma}(u_{\mathcal{T}_j}, \Psi_{\mathcal{T}_j}, f_{\mathcal{T}_j}, h_{\mathcal{T}_j}) = 0$ the linear system

$$-\mathrm{div}^{\mathrm{K}} \left( A_{\mathfrak{D}} \nabla^{\mathfrak{D}} u_{\mathcal{T}_j} \right) + \eta_{\mathrm{K}} u_{j,\mathrm{K}} = f_{\mathrm{K}}, \quad \forall\ \mathrm{K} \in \mathfrak{M}_j, \quad (7)$$

$$-\mathrm{div}^{\mathrm{K}^*} \left( A_{\mathfrak{D}} \nabla^{\mathfrak{D}} u_{\mathcal{T}_j} \right) + \eta_{\mathrm{K}^*} u_{j,\mathrm{K}^*} = f_{\mathrm{K}^*}, \quad \forall\ \mathrm{K}^* \in \mathfrak{M}^*_j, \quad (8)$$

$$-\sum_{\mathrm{D}\in\mathfrak{D}_{\mathrm{K}^*}} \frac{m_{\sigma^*}}{m_{\mathrm{K}^*}} \left( A_{\mathrm{D}} \nabla^{\mathrm{D}} u_{\mathcal{T}_j}, \mathbf{n}_{\sigma^*\mathrm{K}^*} \right) - \frac{m_{\sigma_{\mathrm{K}^*}}}{m_{\mathrm{K}^*}} \psi_{j,\mathrm{K}^*} + \eta_{\mathrm{K}^*} u_{j,\mathrm{K}^*} = f_{\mathrm{K}^*}, \forall \mathrm{K}^* \in \partial\mathfrak{M}^*_{j,\Gamma}, \quad (9)$$

$$\left( A_{\mathrm{D}} \nabla^{\mathrm{D}} u_{\mathcal{T}_j}, \mathbf{n}_{\sigma\mathrm{L}_j} \right) + \Lambda^{\partial\mathfrak{M}_{j,\Gamma}}_{\mathrm{L}} (u_{\partial\mathfrak{M}_{j,\Gamma}}) = h_{j,\mathrm{L}}, \quad \forall\ \mathrm{L} \in \partial\mathfrak{M}_{j,\Gamma}, \quad (10)$$

$$\psi_{j,\mathrm{K}^*} + \Lambda^{\partial\mathfrak{M}^*_{j,\Gamma}}_{\mathrm{K}^*} (u_{\partial\mathfrak{M}^*_{j,\Gamma}}) = h_{j,\mathrm{K}^*}, \quad \forall\ \mathrm{K}^* \in \partial\mathfrak{M}^*_{j,\Gamma}, \quad (11)$$

$$u_{j,\mathrm{K}} = 0, \quad \forall\ \mathrm{K} \in \partial\mathfrak{M}_j \cap \partial\Omega, \qquad u_{j,\mathrm{K}^*} = 0, \quad \forall\ \mathrm{K}^* \in \partial\mathfrak{M}^*_j \cap \partial\Omega, \quad (12)$$

and for $s = 1, \cdots, N$

$$\Lambda^{\partial\mathfrak{M}_{j,\Gamma}}_{\mathrm{L}_s} (u_{\partial\mathfrak{M}_{j,\Gamma}}) = p u_{j,\mathrm{L}_s} - A_{yy} \frac{q}{m_{\sigma_s}} \left( \frac{u_{j,\mathrm{L}_{s+1}} - u_{j,\mathrm{L}_s}}{m_{\sigma_{\mathrm{K}^*_{s+1}}}} - \frac{u_{j,\mathrm{L}_s} - u_{j,\mathrm{L}_{s-1}}}{m_{\sigma_{\mathrm{K}^*_s}}} \right),$$

where $u_{j,\mathrm{L}_0} = u_{j,\mathrm{L}_{N+1}} = 0$, and for $s = 2, \cdots, N$

$$\Lambda_{\mathrm{K}^*_s}^{\partial\mathfrak{M}^*_{j,\Gamma}}(u_{\partial\mathfrak{M}^*_{j,\Gamma}}) = p u_{j,\mathrm{K}^*_s} - A_{yy}\frac{q}{m_{\sigma_{\mathrm{K}^*s}}}\left(\frac{u_{j,\mathrm{K}^*_{s+1}} - u_{j,\mathrm{K}^*_s}}{m_{\sigma_s}} - \frac{u_{j,\mathrm{K}^*_s} - u_{j,\mathrm{K}^*_{s-1}}}{m_{\sigma_{s-1}}}\right).$$

Note that $u_{j,\mathrm{K}^*_1} = u_{j,\mathrm{K}^*_{N+1}} = 0$ because of the homogeneous boundary condition on $\partial\Omega$. The unit normal $\mathbf{n}_{\sigma\mathrm{L}_j}$ is oriented from $\Omega_j$ to $\Omega_i$.

Equations (7)-(9) correspond to approximations of the equation after integration on $\mathfrak{M}_j$, $\mathfrak{M}^*_j$ and $\partial\mathfrak{M}^*_j$; equations (10) and (11) stem from the transmission condition on $\partial\mathfrak{M}_{j,\Gamma}$ and $\partial\mathfrak{M}^*_{j,\Gamma}$; equation (12) corresponds to the Dirichlet boundary condition on $\partial\Omega$.

The DDFV optimized Schwarz algorithm performs for an arbitrary initial guess $h^0_{\mathcal{T}_j} \in \mathbb{R}^{\partial\mathfrak{M}_{j,\Gamma}\cup\partial\mathfrak{M}^*_{j,\Gamma}}$, $j \in \{1,2\}$ and $l = 1, 2, \ldots$ the following steps:

- Compute for $j = 1, 2$ the solutions $(u^{l+1}_{\mathcal{T}_j}, \Psi^{l+1}_{\mathcal{T}_j}) \in \mathbb{R}^{\mathcal{T}_j} \times \mathbb{R}^{\partial\mathfrak{M}^*_{j,\Gamma}}$ of

$$\mathcal{L}^{\mathcal{T}_j}_{\Omega_j,\Gamma}(u^{l+1}_{\mathcal{T}_j}, \Psi^{l+1}_{\mathcal{T}_j}, f_{\mathcal{T}_j}, h^l_{\mathcal{T}_j}) = 0. \tag{13}$$

- Evaluate for $i, j \in \{1,2\}$, $j \neq i$ the new interface values $h^{l+1}_{\mathcal{T}_j}$ by

$$h^{l+1}_{j,\mathrm{L}} = -\left(A_{\mathrm{D}}\nabla^{\mathrm{D}} u^{l+1}_{\mathcal{T}_i}, \mathbf{n}_{\sigma\mathrm{L}_i}\right) + \Lambda^{\partial\mathfrak{M}_{j,\Gamma}}_{\mathrm{L}}(u^{l+1}_{\partial\mathfrak{M}_{i,\Gamma}}),\ \forall \mathrm{L} \in \partial\mathfrak{M}_{i,\Gamma}, \tag{14a}$$

$$h^{l+1}_{j,\mathrm{K}^*} = -\psi^{l+1}_{i,\mathrm{K}^*} + \Lambda^{\partial\mathfrak{M}^*_{j,\Gamma}}_{\mathrm{K}^*}(u^{l+1}_{\partial\mathfrak{M}^*_{i,\Gamma}}),\ \forall \mathrm{K}^* \in \partial\mathfrak{M}^*_{i,\Gamma}. \tag{14b}$$

**Theorem 1 (Well-posedness of subdomain problems).** *For any $f_{\mathcal{T}_j} \in \mathbb{R}^{\mathcal{T}_j}$ and $h_{\mathcal{T}_j} \in \mathbb{R}^{\partial\mathfrak{M}_{j,\Gamma}\cup\partial\mathfrak{M}^*_{j,\Gamma}}$, there exists a unique solution $(u_{\mathcal{T}_j}, \Psi_{\mathcal{T}_j}) \in \mathbb{R}^{\mathcal{T}_j} \times \mathbb{R}^{\partial\mathfrak{M}_{j,\Gamma}\cup\partial\mathfrak{M}^*_{j,\Gamma}}$ of the linear system $\mathcal{L}^{\mathcal{T}_j}_{\Omega_j,\Gamma}(u_{\mathcal{T}_j}, \Psi_{\mathcal{T}_j}, f_{\mathcal{T}_j}, h_{\mathcal{T}_j}) = 0$.*

*Proof.* By linearity, it is sufficient to prove that if $\mathcal{L}^{\mathcal{T}_j}_{\Omega_j,\Gamma}(u_{\mathcal{T}_j}, \Psi_{\mathcal{T}_j}, 0, 0) = 0$, then $u_{\mathcal{T}_j} = 0$ and $\Psi_{\mathcal{T}_j} = 0$. We multiply equation (7) by $m_{\mathrm{K}} u_{j,\mathrm{K}}$ and equations (8)-(9) by $m_{\mathrm{K}^*} u_{j,\mathrm{K}^*}$ and sum the results over all control volumes in $\mathfrak{M}_j$ and $\mathfrak{M}^* j \cup \partial\mathfrak{M}^*_{j,\Gamma}$. Reordering the different contributions over all diamond cells, we obtain

$$2\sum_{\mathrm{D}\in\mathfrak{D}} m_{\mathrm{D}}(A_{\mathrm{D}}\nabla^{\mathrm{D}} u_{\mathcal{T}_j}, \nabla^{\mathrm{D}} u_{\mathcal{T}_j}) + (\Lambda^{\partial\mathfrak{M}_\Gamma}(u_{\partial\mathfrak{M}_{j,\Gamma}}), u_{\partial\mathfrak{M}_{j,\Gamma}})$$
$$+ (\Lambda^{\partial\mathfrak{M}^*_\Gamma}(u_{\partial\mathfrak{M}^*_{j,\Gamma}}), u_{\partial\mathfrak{M}^*_{j,\Gamma}}) + \sum_{\mathrm{K}\in\mathfrak{M}_j} m_{\mathrm{K}}\eta_{\mathrm{K}} u^2_{j,\mathrm{K}} + \sum_{\mathrm{K}^*\in\mathfrak{M}^*_j} m_{\mathrm{K}^*}\eta_{\mathrm{K}^*} u^2_{j,\mathrm{K}^*} = 0.$$

The result thus follows by discrete Poincaré inequalities (see for example [2]) and the properties of $\Lambda^{\partial\mathfrak{M}_\Gamma}$ and $\Lambda^{\partial\mathfrak{M}^*_\Gamma}$.

**Theorem 2 (Convergence of the DDFV Schwarz algorithm).** *The solution of the Schwarz algorithm* (13)-(14) *converges as $l$ goes to $\infty$ to the solution of the DDFV scheme on the entire domain $\Omega$.*

*Proof.* We follow the ideas of [5]: we first rewrite the DDFV scheme for the problem on $\Omega$ as the limit of the Schwarz algorithm. To this end, we introduce new unknowns near the boundary $\Gamma$, see Figure 1:

- $\forall\ x_{\mathrm{K}} \in \Omega_j$ and $x_{\mathrm{K}^*} \in \Omega_j$, we set $u^\infty_{j,\mathrm{K}} = u_{\mathrm{K}}$ and $u^\infty_{j,\mathrm{K}^*} = u_{\mathrm{K}^*}$,
- $\forall\ x_{\mathrm{K}} \in \partial\Omega$ and $x_{\mathrm{K}^*} \in \partial\Omega$, we set $u^\infty_{j,\mathrm{K}} = 0$ and $u^\infty_{j,\mathrm{K}^*} = 0$,
- $\forall\ x_{\mathrm{L}} \in \Gamma$, choose $u^\infty_{j,\mathrm{L}}$ in such a way that $A_j \nabla^{\mathrm{D}} u^\infty_{\mathcal{T}_j} \cdot \mathbf{n}_{\sigma\mathrm{K}_j} = -A_i \nabla^{\mathrm{D}} u^\infty_{\mathcal{T}_i} \cdot \mathbf{n}_{\sigma\mathrm{K}_i}$:

$$u^\infty_{j,\mathrm{L}} = u^\infty_{i,\mathrm{L}} = \frac{m_{\sigma_{\mathrm{K}_j}} m_{\sigma_{\mathrm{K}_i}}}{\Big(A_j m_{\sigma_{\mathrm{K}_i}} + A_i m_{\sigma_{\mathrm{K}_j}}\Big)(\mathbf{n}_{\sigma\mathrm{K}_j}, \mathbf{n}_{\sigma\mathrm{K}_j})} \Bigg[ u_{\mathrm{K}_j} \frac{(A_j \mathbf{n}_{\sigma\mathrm{K}_j}, \mathbf{n}_{\sigma\mathrm{K}_j})}{m_{\sigma_{\mathrm{K}_j}}} + u_{\mathrm{K}_i} \frac{(A_i \mathbf{n}_{\sigma\mathrm{K}_j}, \mathbf{n}_{\sigma\mathrm{K}_j})}{m_{\sigma_{\mathrm{K}_i}}} + \frac{u_{\mathrm{L}^*} - u_{\mathrm{K}^*}}{m_\sigma} (A_i - A_j)(\mathbf{n}_{\sigma^*\mathrm{K}^*_j}, \mathbf{n}_{\sigma\mathrm{K}_j}) \Bigg],$$

- $\forall\ x_{\mathrm{K}^*} \in \Gamma$, $\mathrm{K}^* = \mathrm{K}^*_1 \cup \mathrm{K}^*_2$ with $\mathrm{K}^*_j \in \partial\mathfrak{M}^*_{j,\Gamma}$, choose $u^\infty_{j,\mathrm{K}^*} = u^\infty_{i,\mathrm{K}^*} = u_{\mathrm{K}^*}$ and

$$\begin{aligned}\psi^\infty_{j,\mathrm{K}^*} = -\psi^\infty_{i,\mathrm{K}^*} &= -\frac{1}{m_{\sigma_{\mathrm{K}^*}}} \sum_{\mathrm{D}\in\mathfrak{D}_{\mathrm{K}^*_j}} m_{\sigma^*} \Big(A_{\mathrm{D}} \nabla^{\mathrm{D}} u^\infty_{\mathcal{T}_j}, \mathbf{n}_{\sigma^*\mathrm{K}^*_j}\Big) + \frac{m_{\mathrm{K}^*_j}}{m_{\sigma_{\mathrm{K}^*}}} (\eta_{\mathrm{K}^*} u_{\mathrm{K}^*} - f_{\mathrm{K}^*}) \\ &= \frac{1}{m_{\sigma_{\mathrm{K}^*}}} \sum_{\mathrm{D}\in\mathfrak{D}_{\mathrm{K}^*_i}} m_{\sigma^*} \left(A_{\mathrm{D}} \nabla^{\mathrm{D}} u^\infty_{\mathcal{T}_i}, \mathbf{n}_{\sigma^*\mathrm{K}^*_i}\right) - \frac{m_{\mathrm{K}^*_i}}{m_{\sigma_{\mathrm{K}^*}}} (\eta_{\mathrm{K}^*} u_{\mathrm{K}^*} - f_{\mathrm{K}^*}).\end{aligned}$$

By linearity, it suffices to prove convergence of the DDFV Schwarz algorithm (7) to 0. We have constructed $(u^\infty_{\mathcal{T}_j}, \psi^\infty_{\mathcal{T}_j})$ from the solution $u_{\mathcal{T}}$ of the DDFV scheme on $\Omega$ such that

$$\mathcal{L}^{\mathcal{T}_j}_{\Omega_j,\Gamma}(u^\infty_{\mathcal{T}_j}, \psi^\infty_{\mathcal{T}_j}, f_{\mathcal{T}_j}, h^\infty_{\mathcal{T}_j}) = 0.$$

Observe that the errors $e^{l+1}_{\mathcal{T}_j} = u^{l+1}_{\mathcal{T}_j} - u^\infty_{\mathcal{T}_j}$, $\Psi^{l+1}_{\mathcal{T}_j} = \psi^{l+1}_{\mathcal{T}_j} - \psi^\infty_{\mathcal{T}_j}$ satisfy

$$\mathcal{L}^{\mathcal{T}_j}_{\Omega_j,\Gamma}(e^{l+1}_{\mathcal{T}_j}, \Psi^{l+1}_{\mathcal{T}_j}, 0, H^l_{\mathcal{T}_j}) = 0,$$

with

$$\begin{aligned}\forall\ \mathrm{K}^* \in \partial\mathfrak{M}^*_{i,\Gamma}, \quad & H^l_{j,\mathrm{K}^*} = -\Psi^l_{i,\mathrm{K}^*} + \Lambda^{\partial\mathfrak{M}^*_{i,\Gamma}}_{\mathrm{K}^*}(e^l_{\mathcal{T}_i}), \\ \forall\ l \in \partial\mathfrak{M}_{i,\Gamma}, \quad & H^l_{j,\mathrm{L}} = -(A_{\mathrm{D}} \nabla^{\mathrm{D}} e^l_{\mathcal{T}_i}, \mathbf{n}_{\sigma\mathrm{L}_i}) + \Lambda^{\partial\mathfrak{M}_{i,\Gamma}}_{\mathrm{L}}(e^l_{\mathcal{T}_i}).\end{aligned}$$

An a priori estimate using discrete duality leads to

$$\begin{aligned} 2 \sum_{\mathrm{D}\in\mathfrak{D}_j} & m_{\mathrm{D}}(A_{\mathrm{D}} \nabla^{\mathrm{D}} e^{l+1}_{\mathcal{T}_j}, \nabla^{\mathrm{D}} e^{l+1}_{\mathcal{T}_j}) \\ & - \sum_{\mathrm{L}\in\partial\mathfrak{M}_{j,\Gamma}} m_{\sigma_{\mathrm{L}}} (A_{\mathrm{D}} \nabla^{\mathrm{D}} e^{l+1}_{\mathcal{T}_j}, \mathbf{n}_{\sigma\mathrm{L}_j}) e^{l+1}_{j,\mathrm{L}} - \sum_{\mathrm{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\sigma_{\mathrm{K}^*}} \Psi^{l+1}_{j,\mathrm{K}^*} e^{l+1}_{j,\mathrm{K}^*} \\ & \qquad + \sum_{\mathrm{K}\in\mathfrak{M}_j} m_{\mathrm{K}} \eta_{\mathrm{K}} (e^{l+1}_{j,\mathrm{K}})^2 + \sum_{\mathrm{K}^*\in\mathfrak{M}^*_j \cup \partial\mathfrak{M}^*_{j,\Gamma}} m_{\mathrm{K}^*} \eta_{\mathrm{K}^*} (e^{l+1}_{j,\mathrm{K}^*})^2 = 0.\end{aligned}$$

Using the scalar product defined by $(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}$, we get

$$-\sum_{\mathrm{L}\in\partial\mathfrak{M}_{j,\Gamma}} m_{\sigma_\mathrm{L}}(A_\mathrm{D}\nabla^\mathrm{D} e^{l+1}_{\mathcal{T}_j},\mathbf{n}_{\sigma\mathrm{L}_j})e^{l+1}_{j,\mathrm{L}} = \Big((A_\mathfrak{D}\nabla^\mathfrak{D} e^{l+1}_{\mathcal{T}_j},\mathbf{n}_j),\Lambda^{\partial\mathfrak{M}_\Gamma}(e^{l+1}_{\partial\mathfrak{M}_{j,\Gamma}})\Big)_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}},$$

with $\mathbf{n}_j$ the unit outward normal of $\Omega_j$. The formula $-4ab=(a-b)^2-(a+b)^2$ now implies

$$\begin{aligned}-\sum_{\mathrm{L}\in\partial\mathfrak{M}_{j,\Gamma}} & m_{\sigma_\mathrm{L}}(A_\mathrm{D}\nabla^\mathrm{D} e^{l+1}_{\mathcal{T}_j},\mathbf{n}_{\sigma\mathrm{L}_j})e^{l+1}_{j,\mathrm{L}}\\ &=\frac{1}{4}\left\|-(A_\mathfrak{D}\nabla^\mathfrak{D} e^{l+1}_{\mathcal{T}_j},\mathbf{n}_j)+\Lambda^{\partial\mathfrak{M}_\Gamma}(e^{l+1}_{\partial\mathfrak{M}_{j,\Gamma}})\right\|^2_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}}\\ &\quad-\frac{1}{4}\left\|(A_\mathfrak{D}\nabla^\mathfrak{D} e^{l+1}_{\mathcal{T}_j},\mathbf{n}_j)+\Lambda^{\partial\mathfrak{M}_\Gamma}(e^{l+1}_{\partial\mathfrak{M}_{j,\Gamma}})\right\|^2_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}}.\end{aligned}$$

Using the Ventcell transmission condition, we now obtain

$$\begin{aligned}-\sum_{\mathrm{L}\in\partial\mathfrak{M}_{j,\Gamma}} & m_{\sigma_\mathrm{L}}(A_\mathrm{D}\nabla^\mathrm{D} e^{l+1}_{\mathcal{T}_j},\mathbf{n}_{\sigma\mathrm{L}_j})e^{l+1}_{j,\mathrm{L}}\\ &=\frac{1}{4}\left\|-(A_\mathfrak{D}\nabla^\mathfrak{D} e^{l+1}_{\mathcal{T}_j},\mathbf{n}_j)+\Lambda^{\partial\mathfrak{M}_\Gamma}(e^{l+1}_{\partial\mathfrak{M}_{j,\Gamma}})\right\|^2_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}}\\ &\quad-\frac{1}{4}\left\|-(A_\mathfrak{D}\nabla^\mathfrak{D} e^{l}_{\mathcal{T}_i},\mathbf{n}_i)+\Lambda^{\partial\mathfrak{M}_\Gamma}(e^{l}_{\partial\mathfrak{M}_{i,\Gamma}})\right\|^2_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}}.\end{aligned}$$

In a same way, we also obtain

$$\begin{aligned}-\sum_{\mathrm{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\sigma_{\mathrm{K}^*}}\Psi^{l+1}_{j,\mathrm{K}^*}e^{l+1}_{j,\mathrm{K}^*} &= \frac{1}{4}\left\|-\Psi^{l+1}_{\mathcal{T}_j}+\Lambda^{\partial\mathfrak{M}^*_\Gamma}(e^{l+1}_{\partial\mathfrak{M}^*_{j,\Gamma}})\right\|^2_{(\Lambda^{\partial\mathfrak{M}^*_\Gamma})^{-1}}\\ &\quad-\frac{1}{4}\left\|-\Psi^{l}_{\mathcal{T}_i}+\Lambda^{\partial\mathfrak{M}^*_\Gamma}(e^{l}_{\partial\mathfrak{M}^*_{i,\Gamma}})\right\|^2_{(\Lambda^{\partial\mathfrak{M}^*_\Gamma})^{-1}}.\end{aligned}$$

Summing over $l$ and $j$, the boundary terms cancel and we obtain the estimate

$$\begin{aligned}&2\sum_{l=0}^{l_{max}-1}\sum_{j=1,2}\sum_{\mathrm{D}\in\mathfrak{D}_j} m_\mathrm{D}(A_\mathrm{D}\nabla^\mathrm{D} e^{l+1}_{\mathcal{T}_j},\nabla^\mathrm{D} e^{l+1}_{\mathcal{T}_j})\\ &+\sum_{n=0}^{l_{max}-1}\sum_{j=1,2}\sum_{\mathrm{K}\in\mathfrak{M}_j} m_\mathrm{K}\eta_\mathrm{K}(e^{l+1}_{j,\mathrm{K}})^2+\sum_{n=0}^{l_{max}-1}\sum_{j=1,2}\sum_{\mathrm{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_{j,\Gamma}} m_{\mathrm{K}^*}\eta_{\mathrm{K}^*}(e^{l+1}_{j,\mathrm{K}^*})^2\\ &\le\sum_{j=1,2}\frac{1}{4}\left\|-(A_\mathfrak{D}\nabla^\mathfrak{D} e^{0}_{\mathcal{T}_j},\mathbf{n}_j)+\Lambda^{\partial\mathfrak{M}_\Gamma}(e^{0}_{\partial\mathfrak{M}_{j,\Gamma}})\right\|^2_{(\Lambda^{\partial\mathfrak{M}_\Gamma})^{-1}}\\ &+\sum_{j=1,2}\frac{1}{4}\left\|-\Psi^{0}_{\mathcal{T}_j}+\Lambda^{\partial\mathfrak{M}^*_\Gamma}(e^{0}_{\partial\mathfrak{M}^*_{j,\Gamma}})\right\|^2_{(\Lambda^{\partial\mathfrak{M}^*_\Gamma})^{-1}}.\end{aligned}$$

This shows that the total energy stays bounded as the iteration $l$ goes to infinity, and hence the algorithm converges.

## 3 Numerical experiments

We use the domain $\Omega = (-1,1) \times (0,1)$ with the two subdomains $x > 0$ and $x < 0$. For the first experiment, we choose the data such that the exact solution is $u(x,y) = \cos(2.5\pi x)\cos(2.5\pi y)$, where we set $\eta := 1$ and

$$A(x,y) := \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix} \text{ for } x < 0, \quad \text{and } A(x,y) := \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{pmatrix} \text{ for } x > 0.$$

Starting with a random initial guess, Figure 2 shows the convergence history

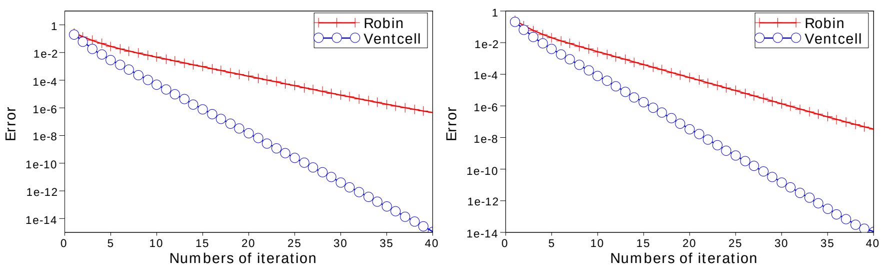


**Fig. 2** Convergence history $\frac{||u_n^{\mathcal{T}_i} - u^{\mathcal{T}_i}||_2}{||u^{\mathcal{T}_i}||_2}$ for non-conforming square meshes (left) and a conforming triangle-square mesh configuration (right).

of the algorithms using the Robin or Ventcell transmission conditions. For a fair comparison, the parameters $p$ and $q$ were numerically chosen to obtain the best convergence rate in each case. On the left, we used a non-conforming $32 \times 32$ square mesh on $\Omega_1$ and a $48 \times 48$ square mesh on $\Omega_2$ with $p = 11.2$ and $q = 0.007$ for the Ventcell transmission condition, and $p = 28$ and $q = 0$ for the Robin one. On the right, we used a conforming triangle-square mesh on $\Omega_1$-$\Omega_2$ with $p = 11.6$ and $q = 0.014$ for the Ventcell transmission condition, and $p = 23.5$ and $q = 0$ for the Robin one. We clearly see that the algorithm converges much faster with the Ventcell condition.

We next simulate the error equations, i.e. using homogeneous data, for a conforming square mesh ($2^i \times 2^i$ squares on $\Omega_j$, $j = 1, 2$). We start again with a random initial guess. On the left in Figure 3, we show the $p$ that worked best as $h$ is refined, and on the right the corresponding $q$. We also plot the asymptotic parameters from [3], which shows that the optimized parameters of the DDFV discretization behave asymptotically as expected.

In conclusion, we have shown how to discretize an optimized Schwarz algorithm with Ventcell transmission conditions using discrete duality finite

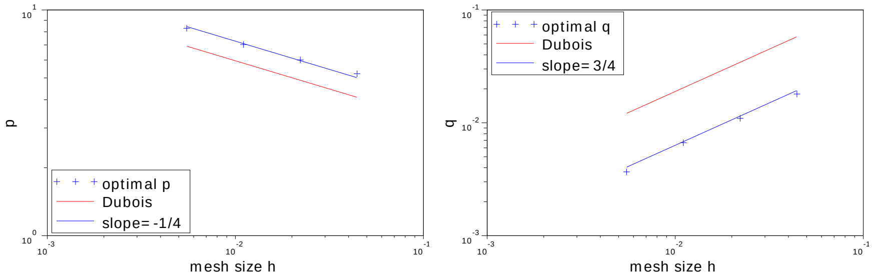


**Fig. 3** Behavior of the numerically optimized parameter $p$ on the left, $q$ on the right.

volumes. Using energy estimates, we proved that the algorithm converges, and we showed in numerical experiments that the convergence is substantially faster than for Robin transmission conditions. We also showed that the optimized parameters behave asymptotically as expected from a continuous analysis. We are currently working on an asymptotic analysis for the optimized parameters and associated contraction factor of the algorithm.

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