A deluxe FETI-DP method for full DG discretization of elliptic problems

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1 Introduction, differential and discrete problems

In this paper we consider a boundary value problem for elliptic second order partial differential equations with highly discontinuous coefficients in a 2D polygonal region Ω. The problem is discretized by a (full) DG method on triangular elements using the space of piecewise linear functions. The goal of this paper is to study a special version of FETI-DP preconditioner, called *deluxe*, for the resulting discrete system of this discretization. The deluxe version for continuous FE discretization is considered in Dohrmann and Widlund [2013], for standard FETI-DP methods for composite DG method, see Dryja et al. [2014], for full DG, see Dryja et al. [2014], and for conforming FEM, see the book Toselli and Widlund [2005].

Now we discuss the continuous and discrete problems we take into consideration for preconditioning.

Differential problem: *Find* $u_{ex}^* \in H_0^1(\Omega)$ *such that*

$$a(u_{ex}^*, v) = f(v) \quad \text{for all } v \in H_0^1(\Omega), \tag{1}$$

$$a(u,v) := \sum_{i=1}^N \int_{\Omega_i} \rho_i \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) := \int_\Omega f v \, dx,$$

where the ρ_i are positive constants and $f \in L^2(\Omega)$.

We assume that $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$ and the substructures Ω_i are disjoint shaped regular polygonal subregions of diameter $O(H_i)$. We assume that the parti-

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tion $\{\Omega_i\}_{i=1}^N$ is geometrically conforming, i.e., for all i and j with $i \neq j$, the intersection $\partial\Omega_i \cap \partial\Omega_j$ is either empty, a common corner or a common edge of Ω_i and Ω_j. For clarity we stress that here and below the identifier *edge* means a curve of continuous intervals and its two endpoints are called corners. The collection of these corners on $\partial\Omega_i$ are referred as the set of corners of Ω_i. Let us denote $\bar{E}_{ij} := \partial\Omega_i \cap \partial\Omega_j$ as an edge of $\partial\Omega_i$ and $\bar{E}_{ji} := \partial\Omega_j \cap \partial\Omega_i$ as an edge of $\partial\Omega_j$. Let us denote by $\mathcal{J}_H^{i,0}$ the set of indices j such that Ω_j has a common edge E_{ji} with Ω_i. To take into account edges of Ω_i which belong to the global boundary $\partial\Omega$, let us introduce a set of indices $\mathcal{J}_H^{i,\partial}$ to refer these edges. The set of indices of all edges of Ω_i is denoted by $\mathcal{J}_H^i = \mathcal{J}_H^{i,0} \cup \mathcal{J}_H^{i,\partial}$.

Discrete problem: Let us introduce a shape regular and quasiuniform triangulation (with triangular elements) $\mathcal{T}_h^i$ on Ω_i and let h_i represent its mesh size. The resulting triangulation on Ω is matching across $\partial\Omega_i$. Let $X_i(\Omega_i) := \prod_{\tau\in\mathcal{T}_h^i} X_\tau$ be the product space of finite element (FE) spaces X_τ which consists of linear functions on the element τ belonging to $\mathcal{T}_h^i$. We note that a function $u_i \in X_i(\Omega_i)$ allows discontinuities across elements of $\mathcal{T}_h^i$. We also note that we do not assume that functions in $X_i(\Omega_i)$ vanish on $\partial\Omega$. The global DG finite element space we consider is defined by $X(\Omega) = \prod_{i=1}^N X_i(\Omega_i) \equiv X_1(\Omega_1) \times X_2(\Omega_2) \times \cdots \times X_N(\Omega_N)$.

We define $\mathcal{E}_h^{i,0}$ as the set of edges of the triangulation $\mathcal{T}_h^i$ which are inside Ω_i, and by $\mathcal{E}_h^{i,j}$, for $j \in \mathcal{J}_H^i$, the set of edges of the triangulation $\mathcal{T}_h^i$ which are on E_{ij}. An edge $e \in \mathcal{E}_h^{i,0}$ is shared by two elements denoted by τ_+ and τ_- of $\mathcal{T}_h^i$ with outward unit normal vectors $\mathbf{n}^+$ and $\mathbf{n}^-$, respectively, and denote $\{\nabla u\} = \frac{1}{2}(\nabla u_{\tau_+} + \nabla u_{\tau_-})$ and $[u] = u_{\tau_+}\mathbf{n}^+ + u_{\tau_-}\mathbf{n}^-$.

The discrete problem we consider by the DG method is of the form: *Find* $u^* = \{u_i^*\}_{i=1}^N \in X(\Omega)$ *where* $u_i^* \in X_i(\Omega_i)$, *such that*

$$a_h(u^*, v) = f(v) \qquad \text{for all } v = \{v_i\}_{i=1}^N \in X(\Omega), \tag{2}$$

where the global bilinear from a_h and the right hand side f are assembled as

$$a_h(u,v) := \sum_{i=1}^N a_i'(u,v) \ \text{ and } \ f(v) := \sum_{i=1}^N \int_{\Omega_i} f v_i \, dx.$$

Here, the local bilinear forms a_i', $i = 1, \dots, N$, are defined as

$$a_i'(u,v) := a_i(u_i,v_i) + s_{0,i}(u_i,v_i) + p_{0,i}(u,v) + s_{\partial,i}(u,v) + p_{\partial,i}(u,v) \tag{3}$$

where a_i, $s_{0,i}$ and $p_{0,i}$ are defined by,

$$a_i(u_i,v_i) := \sum_{\tau\in\mathcal{T}_h^i} \int_\tau \rho_i \nabla u_i \cdot \nabla v_i \, dx,$$

$$s_{0,i}(u_i,v_i) := -\sum\nolimits_{e\in\mathcal{E}_h^{i,0}} \int_e \left(\rho_i\{\nabla u_i\}\cdot[v_i] + \rho_i\{\nabla v_i\}\cdot[u_i]\right) ds, \qquad \text{and}$$

$p_{0,i}(u,v) := \sum_{e\in\mathcal{E}_h^{i,0}} \int_e \delta \frac{\rho_i}{h_e}[u_i].[v_i]\, ds$. The corresponding forms over the local interface edges are given by

$$s_{\partial,i}(u,v) := \sum_{j\in\mathcal{J}_H^i} \sum_{e\in\mathcal{E}_h^{i,j}} \int_e \frac{1}{l_{ij}} \left(\rho_{ij} \frac{\partial u_i}{\partial n}(v_j - v_i) + \rho_{ij} \frac{\partial v_i}{\partial n}(u_j - u_i) \right) ds,$$

$$p_{\partial,i}(u,v) := \sum_{j\in\mathcal{J}_H^i} \sum_{e\in\mathcal{E}_h^{i,j}} \int_e \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_e}(u_i - u_j)(v_i - v_j)\, ds,$$

respectively. Here $\rho_{ij} = 2\rho_i\rho_j/(\rho_i + \rho_j)$, h_e denotes the length of the edge e. When $j \in \mathcal{J}_H^{i,0}$ we set $l_{ij} = 2$, when $j \in \mathcal{J}_H^{i,\partial}$ we denote the boundary edges $E_{ij} \subset \partial\Omega_i$ by $E_{i\partial}$ and set $l_{i\partial} = 1$, and on the artificial edge $E_{ji} \equiv E_{\partial i}$ we set $u_\partial = 0$ and $v_\partial = 0$. The partial derivative $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega_i$ and δ is the penalty positive parameter.

The discrete formulation used here is convenient for our FETI-DP method. We also mention that problem (2) has a unique solution for sufficiently large δ and its error bound is known, see for example, Dryja et al. [2013, 2014].

2 Schur complement matrices and harmonic extensions

In this section, we describe the elimination of unknowns interior to the subdomains required on the FETI-DP formulation for DG discretizations.

Let the set of degrees of freedom associated to subdomain Ω_i be defined by

$$\Omega_i' := \overline{\Omega}_i \bigcup \{\cup_{j\in\mathcal{J}_H^{i,0}} \bar{E}_{ji}\}$$

i.e., it is the union of $\overline{\Omega}_i$ and the $\bar{E}_{ji} \subset \partial\Omega_j$ such that $j \in \mathcal{J}_H^{i,0}$. Define $\Gamma_i := \overline{\partial\Omega_i \backslash \partial\Omega}$ and $\Gamma_i' := \Gamma_i \bigcup \{\cup_{j\in\mathcal{J}_H^{i,0}} \bar{E}_{ji}\}$. We also introduce the sets

$$\Gamma := \bigcup_{i=1}^N \Gamma_i, \quad \Gamma' := \prod_{i=1}^N \Gamma_i', \quad I_i := \Omega_i' \backslash \Gamma_i' \quad \text{and} \quad I := \prod_{i=1}^N I_i. \tag{4}$$

Let $W_i(\Omega_i')$ be the FE space of functions defined by nodal values on Ω_i'

$$W_i(\Omega_i') = W_i(\overline{\Omega}_i) \times \prod_{j\in\mathcal{J}_H^{0,i}} W_i(\bar{E}_{ji}), \tag{5}$$

where $W_i(\overline{\Omega}_i) := X_i(\Omega_i)$ and $W_i(\bar{E}_{ji})$ is the trace of the DG space $X_j(\Omega_j)$ on $\bar{E}_{ji} \subset \partial\Omega_j$ for all $j \in \mathcal{J}_H^{i,0}$. A function $u_i' \in W_i(\Omega_i')$ is defined by the nodal values on Ω_i', i.e., by the nodal values on $\overline{\Omega}_i$ and the nodal values on all adjacent faces $\bar{E}_{ji}$ for all $j \in \mathcal{J}_H^{i,0}$. Below, we denote u_i' by u_i if it is not confused with functions of $X_i(\Omega_i)$. A function $u_i \in W_i(\Omega_i')$ is represented as $u_i = \{(u_i)_i, \{(u_i)_j\}_{j\in\mathcal{J}_H^{i,0}}\}$, where $(u_i)_i := u_{i\,|\overline{\Omega}_i}$ (u_i restricted to $\overline{\Omega}_i$) and $(u_i)_j := u_{i\,|\bar{E}_{ji}}$ (u_i restricted to $\bar{E}_{ji}$). Here and below we use the same notation to identify both DG functions and their vector representations. Note that

$a_i'(\cdot,\cdot)$, see (3), is defined on $W_i(\Omega_i') \times W_i(\Omega_i')$ with corresponding stiffness matrix A_i' defined by

$$a_i'(u_i, v_i) = \langle A_i' u_i, v_i \rangle \quad u_i, v_i \in W_i(\Omega_i'), \tag{6}$$

where $\langle u_i, v_i \rangle$ denotes the ℓ_2 inner product of nodal values associated to the vector space in consideration. We also represent $u_i \in W_i(\Omega_i')$ as $u_i = (u_{i,I}, u_{i,\Gamma'})$ where $u_{i,\Gamma'}$ represents values of u_i at nodal points on Γ_i' and $u_{i,I}$ at the interior nodal points in I_i, see (4). Hence, let us represent $W_i(\Omega_i')$ as the vector spaces $W_i(I_i) \times W_i(\Gamma_i')$. Using the representation $u_i = (u_{i,I}, u_{i,\Gamma'})$, the matrix A_i' can be represented as

$$A_i' = \begin{pmatrix} A'_{i,II} & A'_{i,I\Gamma'} \\ A'_{i,\Gamma' I} & A'_{i,\Gamma'\Gamma'} \end{pmatrix}. \tag{7}$$

The Schur complement of A_i' with respect to $u_{i,\Gamma'}$ is of the form

$$S_i' := A'_{i,\Gamma'\Gamma'} - A'_{i,\Gamma' I}(A'_{i,II})^{-1} A'_{i,I\Gamma'} \tag{8}$$

and introduce the block diagonal matrix $S' = \mathrm{diag}\{S_i'\}_{i=1}^N$.

Let us introduce the product space

$$W(\Omega') := \prod_{i=1}^N W_i(\Omega_i'),$$

i.e., $u \in W(\Omega')$ means that $u = \{u_i\}_{i=1}^N$ where $u_i \in W_i(\Omega_i')$; see (5) for the definition of $W_i(\Omega_i')$. Recall that we write $(u_i)_i = u_{i\,|\overline{\Omega}_i}$ (u_i restricted to $\overline{\Omega}_i$) and $(u_i)_j = u_{i\,|\bar{E}_{ji}}$ (u_i restricted to $\bar{E}_{ji}$). Using the representation $u_i = (u_{i,I}, u_{i,\Gamma'})$ where $u_{i,I} \in W_i(I_i)$ and $u_{i,\Gamma'} \in W_i(\Gamma_i')$ were used in (7), let us introduce the product space

$$W(\Gamma') := \prod_{i=1}^N W_i(\Gamma_i'),$$

i.e., $u_{\Gamma'} \in W(\Gamma')$ means that $u_{\Gamma'} = \{u_{i,\Gamma'}\}_{i=1}^N$ where $u_{i,\Gamma'} \in W_i(\Gamma_i')$. The space $W(\Gamma')$ which was defined on Γ' only, is also interpreted below as the subspace of $W(\Omega')$ of functions which are discrete $\mathcal{H}_i'$-harmonic in each Ω_i.

3 FETI-DP with corner constraints

We now design a FETI-DP method for solving (2). We follow the abstract approach described in pages 160-167 in Toselli and Widlund [2005].

We introduce the nodal points associated to the corner unknowns by

$$\mathcal{V}_i' := \mathcal{V}_i \bigcup \{\cup_{j\in\mathcal{J}_H^{i,0}} \partial E_{ji}\} \qquad \text{where} \qquad \mathcal{V}_i := \{\cup_{j\in\mathcal{J}_H^{i,0}} \partial E_{ij}\}.$$

We now consider the subspace $\tilde{W}(\Omega') \subset W(\Omega')$ (and $\tilde{W}(\Gamma') \subset W(\Gamma')$) as the space of functions which are continuous on all the $\mathcal{V}_i'$ as follows.

Definition 1 (Subspaces $\tilde{W}(\Omega')$ and $\tilde{W}(\Gamma')$). We say that $u = \{u_i\}_{i=1}^N \in \widetilde{W}(\Omega')$ if it is continuous at the corners $\mathcal{V}_i'$, that is, if for $1 \leq i \leq N$ we have

$$(u_i)_i(x) = (u_j)_i(x) \quad \text{at} \;\; x \in \partial E_{ij} \;\; \text{for all} \;\; j \in \mathcal{J}_H^{i,0}, \quad \text{and} \tag{9}$$

$$(u_i)_j(x) = (u_j)_j(x) \quad \text{at} \;\; x \in \partial E_{ji} \;\; \text{for all} \;\; j \in \mathcal{J}_H^{i,0}. \tag{10}$$

Analogously we define $\tilde{W}(\Gamma')$.

Note that $\tilde{W}(\Gamma') \subset W(\Gamma'.)$ Let $\tilde{A}$ be the stiffness matrix which is obtained by assembling the matrices A_i' for $1 \leq i \leq N$, from $W(\Omega')$ to $\tilde{W}(\Omega')$. Note that the matrix $\tilde{A}$ is no longer block diagonal since there are couplings between variables at the corners $\mathcal{V}_i'$ for $1 \leq i \leq N$. We represent $u \in \tilde{W}(\Omega')$ as $u = (u_I, u_\Pi, u_\triangle)$ where the subscript I refers to the interior degrees of freedom at nodal points $I = \prod_{i=1}^N I_i$, the Π refers to the corners $\mathcal{V}_i'$ for all $1 \leq i \leq N$, and the $\triangle$ refers to the remaining nodal points, i.e., the nodal points of $\Gamma_i' \backslash \mathcal{V}_i'$, for all $1 \leq i \leq N$. The vector $u = (u_I, u_\Pi, u_\triangle) \in \tilde{W}(\Omega')$ is obtained from the vector $u = \{u_i\}_{i=1}^N \in W(\Omega')$ using the equations (9) and (10), i.e., the continuity of u on $\mathcal{V}_i'$ for all $1 \leq i \leq N$. Using the decomposition $u = (u_I, u_\Pi, u_\triangle) \in \tilde{W}(\Omega')$ we can partition $\tilde{A}$ as

$$\tilde{A} = \begin{pmatrix} A_{II}' & A_{I\Pi}' & A_{I\triangle}' \\ A_{\Pi I}' & \tilde{A}_{\Pi\Pi} & A_{\Pi\triangle}' \\ A_{\triangle I}' & A_{\triangle\Pi}' & A_{\triangle\triangle}' \end{pmatrix}.$$

We note that the only couplings across subdomains are through the variables Π where the matrix $\tilde{A}$ is subassembled.

A Schur complement of $\tilde{A}$ with respect to the $\triangle$-unknowns (eliminating the I- and the Π-unknowns) is of the form

$$\tilde{S} := A_{\triangle\triangle}' - (A_{\triangle I}' \; A_{\triangle\Pi}') \begin{pmatrix} A_{II}' & A_{I\Pi}' \\ A_{\Pi I}' & \tilde{A}_{\Pi\Pi} \end{pmatrix}^{-1} \begin{pmatrix} A_{I\triangle}' \\ A_{\Pi\triangle}' \end{pmatrix}. \tag{11}$$

A vector $u \in \tilde{W}(\Gamma')$ can uniquely be represented by $u = (u_\Pi, u_\triangle)$, therefore, we can represent $\tilde{W}(\Gamma') = \hat{W}_\Pi(\Gamma') \times W_\triangle(\Gamma')$, where $\hat{W}_\Pi(\Gamma')$ refers to the Π-degrees of freedom of $\tilde{W}(\Gamma')$ while $W_\triangle(\Gamma')$ to the $\triangle$-degrees of freedom of $\tilde{W}(\Gamma')$. The vector space $W_\triangle(\Gamma')$ can be decomposed as

$$W_\triangle(\Gamma') = \prod_{i=1}^N W_{i,\triangle}(\Gamma_i') \tag{12}$$

where the local space $W_{i,\triangle}(\Gamma'_i)$ refers to the degrees of freedom associated to the nodes of $\Gamma'_i \backslash \mathcal{V}'_i$ for $1 \leq i \leq N$. Hence, a vector $u \in \tilde{W}(\Gamma')$ can be represented as $u = (u_\Pi, u_\triangle)$ with $u_\Pi \in \hat{W}_\Pi(\Gamma')$ and $u_\triangle = \{u_{i,\triangle}\}_{i=1}^N \in W_\triangle(\Gamma')$ where $u_{i,\triangle} \in W_{i,\triangle}(\Gamma'_i)$. Note that $\tilde{S}$, see (11), is defined on the vector space $W_\triangle(\Gamma')$.

In order to measure the jump of $u_\triangle \in W_\triangle(\Gamma')$ across the $\triangle$-nodes let us introduce the space $\hat{W}_\triangle(\Gamma)$ defined by

$$\hat{W}_\triangle(\Gamma) = \prod_{i=1}^N X_i(\Gamma_i \backslash \mathcal{V}_i),$$

where $X_i(\Gamma_i \backslash \mathcal{V}_i)$ is the restriction of $X_i(\Omega_i)$ to $\Gamma_i \backslash \mathcal{V}_i$. To define the jumping matrix $B_\triangle : W_\triangle(\Gamma') \rightarrow \hat{W}_\triangle(\Gamma)$, let $u_\triangle = \{u_{i,\triangle}\}_{i=1}^N \in W_\triangle(\Gamma')$ and let $v := B_\triangle u$ where $v = \{v_i\}_{i=1}^N \in \hat{W}_\triangle(\Gamma)$ is defined by

$$v_i = (u_{i,\triangle})_i - (u_{j,\triangle})_i \;\text{ on }\; E_{ijh} \;\text{ for all }\; j \in \mathcal{J}_H^{i,0}, \tag{13}$$

where E_{ijh} is the set of interior nodal points on E_{ij}. The jumping matrix $B_\triangle$ can be written as

$$B_\triangle = (B_\triangle^{(1)}, B_\triangle^{(2)}, \cdots, B_\triangle^{(N)}), \tag{14}$$

where the rectangular matrix $B_\triangle^{(i)}$ consists of columns of $B_\triangle$ attributed to the (i) components of functions of $W_{i,\triangle}(\Gamma'_i)$ of the product space $W_\triangle(\Gamma')$, see (12). The entries of the rectangular matrix consist of values of $\{0, 1, -1\}$. It is easy to see that the Range $B_\triangle = \hat{W}_\triangle(\Gamma)$, so $B_\triangle$ is full rank.

We can reformulate the problem (2) as the variational problem with constraints in $W_\triangle(\Gamma')$ space: *Find* $u^*_\triangle \in W_\triangle(\Gamma')$ *such that*

$$J(u^*_\triangle) = \min J(v_\triangle) \tag{15}$$

subject to $v_\triangle \in W_\triangle(\Gamma')$ with constraints $B_\triangle v_\triangle = 0$. Here $J(v_\triangle) := \frac{1}{2}\langle \tilde{S} v_\triangle, v_\triangle\rangle - \langle \tilde{g}_\triangle, v_\triangle\rangle$ with $\tilde{S}$ given in (11) and $\tilde{g}_\triangle$ is easily obtained using the fact that it can be represented as $f = (f_I, f_\Pi, f_{\Gamma\backslash\Pi})$. Note that $\tilde{S}$ is symmetric and positive definite since $\tilde{A}$ has these properties. Introducing Lagrange multipliers $\lambda \in \hat{W}_\triangle(\Gamma)$, the problem (15) reduces to the saddle point problem of the form: *Find* $u^*_\triangle \in W_\triangle(\Gamma')$ *and* $\lambda^* \in \hat{W}_\triangle(\Gamma)$ *such that*

$$\begin{cases} \tilde{S} u^*_\triangle + B_\triangle^T \lambda^* = \tilde{g}_\triangle \\ B_\triangle u^*_\triangle \qquad\quad\;\; = 0. \end{cases} \tag{16}$$

Hence, (16) reduces to

$$F\lambda^* = g \tag{17}$$

where $F := B_\triangle \tilde{S}^{-1} B_\triangle^T$ and $g := B_\triangle \tilde{S}^{-1} \tilde{g}_\triangle$.

3.1 Dirichlet Preconditioner

We now define the FETI-DP preconditioner for F, see (17). Let $S'_{i,\triangle}$ be the Schur complement of S'_i, see (8), restricted to $W_{i,\triangle}(\Gamma'_i) \subset W_i(\Gamma'_i)$, i.e., taken S'_i on functions in $W_i(\Gamma'_i)$ which vanish on $\mathcal{V}'_i$. Let

$$S'_\triangle := \text{diag}\{S'_{i,\triangle}\}_{i=1}^N.$$

In other words, $S'_{i,\triangle}$ is obtained from S'_i by deleting rows and columns corresponding to nodal values at nodal points of $\mathcal{V}'_i \subset \Gamma'_i$.

Let us introduce diagonal scaling operators $D^{(i)}_\triangle : W_{i,\triangle}(\Gamma'_i) \to W_{i,\triangle}(\Gamma'_i)$, for $1 \le i \le N$. They are based on partial Schur complements of $S'_{i,\triangle}$ used in Dohrmann and Widlund [2013] for continuous FE discretization and this is know in the literature as the deluxe version of FETI-DP preconditioner. We first introduce $W_{i,\triangle,E_{ij}}(\Gamma'_i)$ as the space of $u_i \in W_{i,\triangle}(\Gamma'_i)$ which vanish on $\partial\Omega_i \backslash E_{ij}$ and $E_{ki} \subset \partial\Omega_k$ for $k \neq j$. Let $S'_{i,\triangle,E_{ij}}$ denote the Schur complement of $S'_{i,\triangle}$ restricted to $W_{i,\triangle,E_{ij}}$. In a similar way it is defined the restricted Schur complement $S'_{j,\triangle,E_{ji}}$. The operator $D^{(i)}_\triangle$ on $E_{ij} \subset \partial\Omega_i$ is defined as

$$D^{(i)}_{\triangle,E_{ij}} = (S'_{i,\triangle,E_{ij}} + S'_{j,\triangle,E_{ji}})^{-1} S'_{j,\triangle,E_{ji}}. \tag{18}$$

Let $B_{D,\triangle} = (B^{(1)}_\triangle D^{(1)}_\triangle, \cdots, B^{(N)}_\triangle D^{(N)}_\triangle)$ and $P_\triangle := B^T_{D,\triangle} B_\triangle$, which maps $W_\triangle(\Gamma')$ into itself. It can be checked straightforwardly that $P_\triangle$ preserves jumps in the sense that $B_\triangle P_\triangle = B_\triangle$ and $P^2_\triangle = P_\triangle$.

In the FETI-DP method, the preconditioner M^{-1} is defined as follows:

$$M^{-1} = B_{D,\triangle} S'_\triangle B^T_{D,\triangle} = \sum_{i=1}^N B^{(i)}_\triangle D^{(i)}_\triangle S'_{i,\triangle} (D^{(i)}_\triangle)^T (B^{(i)}_\triangle)^T.$$

Note that M^{-1} is a block-diagonal matrix, and each block is invertible since $S'_{i,\triangle}$ and $D^{(i)}_\triangle$ are invertible and $B^{(i)}_\triangle$ is a full rank matrix. The following theorem holds.

Theorem 1. *For any* $\lambda \in \hat{W}_\triangle(\Gamma)$ *it holds that*

$$\langle M\lambda, \lambda\rangle \;\le\; \langle F\lambda, \lambda\rangle \;\le\; C\left(1 + \log\frac{H}{h}\right)^2 \langle M\lambda, \lambda\rangle$$

where $\log(\frac{H}{h}) := \max_{i=1}^N \log(\frac{H_i}{h_i})$, *$C$ is a positive constant independent of* h_i, h_i/h_j, H_i, λ *and the jumps of* ρ_i.

The complete proof of Theorem 1 will be presented elsewhere.

Remark 3: The FETI-DP method is introduced for a composite DG discretization in the 3-D case in (Dryja and Sarkis [2014]). In order to extend

the deluxe scaling FETI-DP method for 3-D DG discretizations, we need to introduce the averaging of the deluxe operators for faces and edges. The face operators are introduced similarly as described as in (18) by replacing edges E_{ij} by faces F_{ij}. For the edge operators, consider for instance that E_{ijk} is an edge of Ω_i common to Ω_j and Ω_k. Let E_{jik} and E_{kij} be edges equal to E_{ijk} but belonging to Ω_j and Ω_k, respectively. Let $W_{i,\Delta,E_{ijk}}(\Gamma_i')$ be a subspace of $W_{i,\Delta}(\Gamma_i')$ with nonzero data on E_{ijk}, E_{jik} and E_{kij} only. Let $S'_{i,\Delta,E_{ijk}}$ be the restriction of $S'_{i,\Delta}$ to the space $W_{i,\Delta,E_{ijk}}$. In the same way we introduce $S'_{j,\Delta,E_{jik}}$ and $S'_{k,\Delta,E_{kij}}$. For the deluxe FETI-DP method with non-redundant Lagrange multipliers on edges, see Toselli and Widlund [2005], it is enough to define the edge averaging operators as follows:
$D^{(i)}_{\Delta,E_{ijk},1} = (S'_{i,\Delta,E_{ijk}} + S'_{j,\Delta,E_{jik}} + S'_{k,\Delta,E_{kij}})^{-1} S'_{j,\Delta,E_{jik}}$, and
$D^{(i)}_{\Delta,E_{ijk},2} = (S'_{i,\Delta,E_{ijk}} + S'_{j,\Delta,E_{jik}} + S'_{k,\Delta,E_{kij}})^{-1} S'_{k,\Delta,E_{kij}}$.
In the 3-D case $B_{D,\triangle}$ is modified by setting $B_{D,\triangle} = (B_\triangle D_\triangle B_\triangle^T)^{-1} B_\triangle D_\triangle$ and $M^{-1} = B_{D,\triangle} S'_\triangle B^T_{D,\triangle}$ where $D_\triangle = \mathrm{diag}\{D^{(i)}_\triangle\}$ and $D^{(i)}_\triangle$ is a block diagonal containing the averaging operators corresponding to faces and edges defined above. The operator $P_\triangle = B^T_{D,\triangle} B_\triangle$ preserves the jumps and is a projection.

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