Globally convergent multigrid method for variational inequalities with a nonlinear term

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1 Introduction

In Badea [2008], one- and two-level Schwarz methods have been proposed for variational inequalities with contraction operators. This type of inequalities generalizes the problems modeled by quasi-linear or semilinear inequalities. It is proved there that the convergence rates of the two-level methods are almost independent of the mesh and overlapping parameters. However, the original convex set, which is defined on the fine grid, is used to find the corrections on the coarse grid, too. This leads to a suboptimal computing complexity. A remedy can be found in adopting minimization techniques from the construction of multigrid methods for the constrained minimization of functionals. In this case, to avoid visiting the fine grid, some level convex sets for the corrections on the coarse levels have been proposed in Mandel [1984a], Mandel [1984b], Gelman and Mandel [1990], Kornhuber [1994], Kornhuber [1996] and the review article Graser and Kornhuber [2009] for complementarity problems, and in Badea [2014] for two two-obstacle problems. In this paper, we introduce and investigate the convergence of a new multigrid algorithm for the inequalities with contraction operators, and we have adopted the construction of the level convex sets which has been introduced in Badea [2014]. In this way, the introduced multigrid method has an optimal computing complexity of the iterations. Also, the convergence theorems for the methods introduced in Badea [2008] contain a convergence condition depending on the total number of the subdomains in the decompositions of the domain. The convergence condition of a direct extension of these methods to more than two-levels will introduce an upper bound for the number of mesh levels which can be used in the method. In comparison with these methods, the convergence condition of the algorithm introduced in this paper is less restrictive and depends neither

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on the number of the subdomains in the decompositions of the domain nor on the number of levels. Moreover, this convergence condition is very similar with the condition of existence and uniqueness of the solution of the problem.

The paper is organized as follows. In Section 2, the method is introduced as a subspace correction algorithm in a general reflexive Banach space. Under the same assumptions in Badea [2014] concerning the level convex sets where we are looking for the corrections, we prove that the algorithm is globally convergent and estimate the global convergence rate, provided that the convergence condition is satisfied. In Section 3, we show that the algorithm can be viewed as multilevel or multigrid methods if we associate finite element spaces to the level meshes and to the domain decompositions at each level. In Badea [2014], it has been proved that the assumptions made in the previous section hold for problems having the convex set of two-obstacle type. For this type of problems, we write the convergence rate of the proposed multigrid method in function of the number of level meshes.

2 Abstract convergence results

We consider a reflexive Banach space V and let $K \subset V$ be a nonempty closed convex set. Let $F : V \to \mathbf{R}$ be a Gâteaux differentiable functional and we assume that there exist two constants $\alpha,\ \beta > 0$ for which

$$\alpha||v-u||^2 \leq \langle F'(v) - F'(u), v-u\rangle \text{ and } ||F'(v) - F'(u)||_{V'} \leq \beta||v-u||, \quad (1)$$

for any $u, v \in V$. Above, we have denoted by F' the Gâteaux derivative of F, and V' is the dual space of V. Following the way in Glowinski et al. [1976], we can prove that

$$\langle F'(u), v-u\rangle + \frac{\alpha}{2}||v-u||^2 \leq F(v) - F(u) \leq \langle F'(u), v-u\rangle + \frac{\beta}{2}||v-u||^2, \quad (2)$$

for any $u, v \in V$. We point out that since F is Gâteaux differentiable and satisfies (1), then F is a convex functional (see Proposition 5.5 in Ekeland and Temam [1974], page 25). Also, let $T : V \to V'$ be an operator with the property that there exists a constant $\gamma > 0$ such that

$$||T(v) - T(u)||_{V'} \leq \gamma||v-u|| \text{ for any } v, u \in V. \quad (3)$$

Now, we consider the quasi-variational inequality

$$u \in K : \langle F'(u), v-u\rangle + \langle T(u), v-u\rangle \geq 0 \text{ for any } v \in K. \quad (4)$$

Using (2), we get

$$\tfrac{\alpha}{2}||v-u||^2 \leq F(v) - F(u) + \langle T(u), v-u\rangle \text{ for any } v \in K. \quad (5)$$

Problem (4) has a solution and it is unique (see Badea [2008], for instance) if

$$\gamma/\alpha < 1. \tag{6}$$

Now, let us assume that we have J closed subspaces of V, $V_1, \ldots, V_J$, and let V_{ji}, $i = 1, \ldots I_j$ be some closed subspaces of V_j, $j = J, \ldots, 1$. The subspaces V_j, $j = J, \ldots, 1$, will be associated with the grid levels, and, for each level $j = J, \ldots, 1$, V_{ji}, $i = 1, \ldots I_j$, will be associated with a domain decomposition. Let us write $I = \max_{j=J,\ldots,1} I_j$.

To introduce the algorithm, we make an assumption on choice of the convex sets $\mathcal{K}_j$, $j = 1, \ldots, J$, where we look for the level corrections. The chosen level convex sets depend on the current approximation in the algorithms.

Assumption 1 *For a given $w \in K$, we recursively introduce the convex sets $\mathcal{K}_j$, $j = J,\ J-1,\ \ldots, 1$, as*

- at level J: we assume that $0 \in \mathcal{K}_J$, $\mathcal{K}_J \subset \{v_J \in V_J : w + v_J \in K\}$ and consider a $w_J \in \mathcal{K}_J$,

- at a level $J-1 \geq j \geq 1$: we assume that $0 \in \mathcal{K}_j$ and $\mathcal{K}_j \subset \{v_j \in V_j : w + w_J + \ldots + w_{j+1} + v_j \in K\}$, and consider a $w_j \in \mathcal{K}_j$.

We now introduce the algorithm, which is of multiplicative type, and where the argument of T is kept unchanged for several iterations.

Algorithm 1 *We start the algorithm with an arbitrary $u^0 \in K$. Assuming that at iteration $n \geq 0$ we have $u^n \in K$, we write $\tilde{u}^n = u^n$ and carry out the following two steps:*

1. We perform $\kappa \geq 1$ multiplicative iterations, keeping the argument of T equal with u^n. We start with $\tilde{u}^n$ and having $\tilde{u}^{n+k-1}$ at iteration $1 \leq k \leq \kappa$, we successively calculate level corrections and compute $\tilde{u}^{n+k}$:

- at the level J, we construct the convex set $\mathcal{K}_J$ as in Assumption 1, with $w = \tilde{u}^{n+k-1}$. Then, we first write $w_J^k = 0$, and, for $i = 1, \ldots, I_J$, we successively calculate $w_{Ji}^{k+1} \in V_{Ji}$, $w_J^{k+\frac{i-1}{I_J}} + w_{Ji}^{k+1} \in \mathcal{K}_J$, the solution of the inequality

$$\langle F'(\tilde{u}^{n+k-1} + w_J^{k+\frac{i-1}{I_J}} + w_{Ji}^{k+1}), v_{Ji} - w_{Ji}^{k+1}\rangle + \langle T(u^n), v_{Ji} - w_{Ji}^{k+1}\rangle \geq 0,$$

for any $v_{Ji} \in V_{Ji}$, $w_J^{k+\frac{i-1}{I_J}} + v_{Ji} \in \mathcal{K}_J$, and write $w_J^{k+\frac{i}{I_J}} = w_J^{k+\frac{i-1}{I_J}} + w_{Ji}^{k+1}$,

- at a level $J-1 \geq j \geq 1$, we construct the convex set $\mathcal{K}_j$ as in Assumption 1 with $w = \tilde{u}^{n+k-1}$ and $w_J = w_J^{k+1}, \ldots, w_{j+1} = w_{j+1}^{k+1}$. Then, we write $w_j^{k+1} = 0$, and for $i = 1, \ldots, I_j$, we successively calculate $w_{ji}^{k+1} \in V_{ji}$, $w_j^{k+\frac{i-1}{I_j}} + w_{ji}^{k+1} \in \mathcal{K}_j$, the solution of the inequality

$$\langle F'(\tilde{u}^{n+k-1} + \textstyle\sum_{l=j+1}^{J} w_l^{k+1} + w_j^{k+\frac{i-1}{I_j}} + w_{ji}^{k+1}), v_{ji} - w_{ji}^{k+1}\rangle$$
$$+\langle T(u^n), v_{ji} - w_{ji}^{k+1}\rangle \geq 0,$$

for any $v_{ji} \in V_{ji}$, $w_j^{k+\frac{i-1}{I_j}} + v_{ji} \in \mathcal{K}_j$, and write $w_j^{k+\frac{i}{I_j}} = w_J^{k+\frac{i-1}{I_j}} + w_{ji}^{k+1}$,

- we write $\tilde{u}^{n+k} = \tilde{u}^{n+k-1} + \sum_{j=1}^{J} w_j^{k+1}$.

2. We write $u^{n+1} = \tilde{u}^{n+k}$.

In order to prove the convergence of the above algorithm, we shall make two new assumptions. In the case of the multigrid decompositions, the constants of some inequalities can be taken independent of the number J of levels, the classical Cauchy-Schwarz inequality can be strengthened, for instance. In this sense we make the following assumption.

Assumption 2 *1. There exist some constants* $0 < \beta_{jk} \leq 1$, $\beta_{jk} = \beta_{kj}$, $j,\ k = J, \dots, 1$, *such that* $\langle F'(v+v_{ji}) - F'(v), v_{kl}\rangle \leq \beta\beta_{jk}||v_{ji}||||v_{kl}||$, *for any* $v \in V$, $v_{ji} \in V_{ji}$, $v_{kl} \in V_{kl}$, $i = 1, \dots, I_j$ *and* $l = 1, \dots, I_k$.

2. There exists a constant C_1 *such that* $||\sum_{j=1}^{J}\sum_{i=1}^{I_j} w_{ji}|| \leq C_1(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}||^2)^{\frac{1}{2}}$, *for any* $w_{ji} \in V_{ji}$, $j = J, \dots, 1$, $i = 1, \dots, I_j$.

Evidently, for the moment, we can consider $C_1 = (IJ)^{\frac{1}{2}}$ and $\beta_{jk} = 1,\ j,\ k = J, \dots, 1$. The second new assumption refers to additional properties asked to the convex sets $\mathcal{K}_j$, $j = 1, \dots, J$, introduced in Assumption 1.

Assumption 3 *There exists a constant* $C_2 > 0$ *such that for any* $w \in K$, $w_{ji} \in V_{ji}$, $w_{j1} + \ldots + w_{ji} \in \mathcal{K}_j$, $j = J, \dots, 1$, $i = 1, \dots, I_j$, *and* $u \in K$, *there exist* $u_{ji} \in V_{ji}$, $j = J, \dots, 1$, $i = 1, \dots, I_j$, *which satisfy*

$u_{j1} \in \mathcal{K}_j$ *and* $w_{j1} + \ldots + w_{ji-1} + u_{ji} \in \mathcal{K}_j, i = 2, \dots, I_j,\ j = J, \dots, 1,$

$u - w = \sum_{j=1}^{J}\sum_{i=1}^{I_j} u_{ji}$, *and*

$\sum_{j=1}^{J}\sum_{i=1}^{I_j}||u_{ji}||^2 \leq C_2^2\left(||u-w||^2 + \sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}||^2\right)$.

The convex sets $\mathcal{K}_j$, $j = J, \dots, 1$, *are constructed as in Assumption 1 with the above* w *and* $w_j = \sum_{i=1}^{I_j} w_{ji}$, $j = J, \dots, 1$.

The global convergence of Algorithm 1 is proved by

Theorem 1. *Let* V *be a reflexive Banach space,* V_j, $j = 1, \dots, J$, *closed subspaces of* V, *and* V_{ji}, $i = 1, \dots, I_j$, *some closed subspaces of* V_j, $j = 1, \dots, J$. *Let* K *be a non empty closed convex subset of* V, *and we suppose that Assumptions 1-3 hold. Also, we assume that* F *is a Gâteaux differentiable functional which satisfies (1) and the operator* T *satisfies (3). On these conditions, if*

$$\gamma/\alpha < 1/2 \tag{7}$$

and κ *satisfies*

$$\left(\frac{\tilde{C}}{\tilde{C}+1}\right)^{\kappa} < \frac{1-2\frac{\gamma}{\alpha}}{1+3\frac{\gamma}{\alpha}+4\frac{\gamma^2}{\alpha^2}+\frac{\gamma^3}{\alpha^3}}, \tag{8}$$

where constant $\tilde{C}$ *is given by*

$$\tilde{C} = \frac{1}{C_2\varepsilon}\left[1 + C_2 + C_1C_2 + \frac{C_2}{\varepsilon}\right],\ \varepsilon = \frac{\alpha}{2\beta I(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})C_2}, \tag{9}$$

then Algorithm 1 is convergent and we have the following error estimations:

$$
\begin{aligned}
&F(u^{n+1}) + \langle T(u), u^{n+1}\rangle - F(u) - \langle T(u), u\rangle \\
&\le [2\tfrac{\gamma}{\alpha} + (\tfrac{\tilde{C}}{\tilde{C}+1})^{\kappa}(1 + 3\tfrac{\gamma}{\alpha} + 4\tfrac{\gamma^2}{\alpha^2} + \tfrac{\gamma^3}{\alpha^3})]^n \\
&\cdot[F(u^0) + \langle T(u), u^0\rangle - F(u) - \langle T(u), u\rangle],
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&\|u^n - u\|^2 \le \tfrac{2}{\alpha}[2\tfrac{\gamma}{\alpha} + (\tfrac{\tilde{C}}{\tilde{C}+1})^{\kappa}(1 + 3\tfrac{\gamma}{\alpha} + 4\tfrac{\gamma^2}{\alpha^2} + \tfrac{\gamma^3}{\alpha^3})]^n \\
&\cdot[F(u^0) + \langle T(u), u^0\rangle - F(u) - \langle T(u), u\rangle].
\end{aligned} \tag{11}
$$

Proof. First, we see that in view of (5), (11) can be obtained from (10). Now, for a fixed $n \ge 0$, let us consider the problem

$$
\tilde{u} \in K : \langle F'(\tilde{u}), v - \tilde{u}\rangle + \langle T(\tilde{u}^n), v - \tilde{u}\rangle \ge 0, \text{ for any } v \in K, \tag{12}
$$

where $\tilde{u}^n = u^n \in K$ is the approximation obtained from Algorithm 1 after n iterations. By applying Theorem 2.2 in Badea [2014] to variational inequality (12) we get that after κ iterations the following error estimation holds

$$
\begin{aligned}
&F(\tilde{u}^{n+\kappa}) + \langle T(\tilde{u}^n), \tilde{u}^{n+\kappa}\rangle - F(\tilde{u}) - \langle T(\tilde{u}^n), \tilde{u}\rangle \\
&\le (\tfrac{\tilde{C}}{\tilde{C}+1})^{\kappa}[F(\tilde{u}^n) + \langle T(\tilde{u}^n), \tilde{u}^n\rangle - F(\tilde{u}) - \langle T(\tilde{u}^n), \tilde{u}\rangle]
\end{aligned}
$$

or

$$
\begin{aligned}
&F(u^{n+1}) + \langle T(u^n), u^{n+1}\rangle - F(\tilde{u}) - \langle T(u^n), \tilde{u}\rangle \\
&\le (\tfrac{\tilde{C}}{\tilde{C}+1})^{\kappa}[F(u^n) + \langle T(u^n), u^n\rangle - F(\tilde{u}) - \langle T(u^n), \tilde{u}\rangle],
\end{aligned} \tag{13}
$$

where $\tilde{C}$ is given in (9). From (2), (12) and (3), we have

$$
\begin{aligned}
&F(\tilde{u}) + \langle T(u), \tilde{u}\rangle - F(u) - \langle T(u), u\rangle + \tfrac{\alpha}{2}||\tilde{u} - u||^2 \\
&\le \langle F'(\tilde{u}), \tilde{u} - u\rangle + \langle T(u^n), \tilde{u} - u\rangle + \langle T(u) - T(u^n), \tilde{u} - u\rangle \\
&\le \langle T(u) - T(u^n), \tilde{u} - u\rangle \le \gamma||u - u^n||||u - \tilde{u}|| \le \tfrac{\gamma}{2}||u - u^n||^2 + \tfrac{\gamma}{2}||u - \tilde{u}||^2.
\end{aligned}
$$

From (4) and using again (2), we get

$$
\begin{aligned}
&\tfrac{\alpha}{2}||u - u^n||^2 \le \langle F'(u), u - u^n\rangle + F(u^n) - F(u) \\
&\le F(u^n) + \langle T(u), u^n\rangle - F(u) - \langle T(u), u\rangle.
\end{aligned} \tag{14}
$$

From the last two equations, in view of (7), we get

$$
\begin{aligned}
&F(\tilde{u}) + \langle T(u), \tilde{u}\rangle - F(u) - \langle T(u), u\rangle \\
&\le \tfrac{\gamma}{\alpha}[F(u^n) + \langle T(u), u^n\rangle - F(u) - \langle T(u), u\rangle].
\end{aligned} \tag{15}
$$

Now, we have

$$
\begin{aligned}
&F(u^{n+1}) + \langle T(u), u^{n+1}\rangle - F(u) - \langle T(u), u\rangle \\
&= F(u^{n+1}) + \langle T(u^n), u^{n+1}\rangle - F(\tilde{u}) - \langle T(u^n), \tilde{u}\rangle \\
&+ F(\tilde{u}) + \langle T(u), \tilde{u}\rangle - F(u) - \langle T(u), u\rangle \\
&+ \langle T(u) - T(u^n), u^{n+1} - \tilde{u}\rangle.
\end{aligned} \tag{16}
$$

But, in view of (13), we get

$$\begin{aligned}
&F(u^{n+1}) + \langle T(u^n), u^{n+1}\rangle - F(\tilde{u}) - \langle T(u^n), \tilde{u}\rangle \\
&\le (\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa [F(u^n) + \langle T(u^n), u^n\rangle - F(\tilde{u}) - \langle T(u^n), \tilde{u}\rangle] \\
&= (\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa [F(u^n) + \langle T(u), u^n\rangle - F(u) - \langle T(u), u\rangle \\
&+F(u) + \langle T(u), u\rangle - F(\tilde{u}) - \langle T(u), \tilde{u}\rangle] \\
&+(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa \langle T(u^n) - T(u), u^n - \tilde{u}\rangle.
\end{aligned} \tag{17}$$

It follows from (16), (17), (15) and (3) that

$$\begin{aligned}
&F(u^{n+1}) + \langle T(u), u^{n+1}\rangle - F(u) - \langle T(u), u\rangle \\
&\le (\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa [F(u^n) + \langle T(u), u^n\rangle - F(u) - \langle T(u), u\rangle] \\
&+[1 - (\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa][F(\tilde{u}) + \langle T(u), \tilde{u}\rangle - F(u) - \langle T(u), u\rangle] \\
&+(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa \langle T(u^n) - T(u), u^n - \tilde{u}\rangle + \langle T(u) - T(u^n), u^{n+1} - \tilde{u}\rangle \\
&\le [(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa - \tfrac{\gamma}{\alpha}(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa + \tfrac{\gamma}{\alpha}][F(u^n) + \langle T(u), u^n\rangle - F(u) - \langle T(u), u\rangle] \\
&+\gamma(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa ||u^n - u|| ||u^n - \tilde{u}|| + \gamma ||u^n - u|| ||u^{n+1} - \tilde{u}||.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa ||u^n - u|| ||u^n - \tilde{u}|| + ||u^n - u|| ||u^{n+1} - \tilde{u}|| \\
&\le (\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa (||u^n - u||^2 + ||u^n - u|| ||u - \tilde{u}||) + ||u^n - u|| ||u^{n+1} - \tilde{u}|| \\
&\le \tfrac{1}{2}[3(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa + 1]||u^n - u||^2 + \tfrac{1}{2}(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa ||u - \tilde{u}||^2 + \tfrac{1}{2}||u^{n+1} - \tilde{u}||^2.
\end{aligned}$$

Therefore, from the last two equation, we have

$$\begin{aligned}
&F(u^{n+1}) + \langle T(u), u^{n+1}\rangle - F(u) - \langle T(u), u\rangle \\
&\le [(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa - \tfrac{\gamma}{\alpha}(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa + \tfrac{\gamma}{\alpha}][F(u^n) + \langle T(u), u^n\rangle - F(u) - \langle T(u), u\rangle] \\
&+\tfrac{\gamma}{2}[3(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa + 1]||u^n - u||^2 + \tfrac{\gamma}{2}(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa ||u - \tilde{u}||^2 + \tfrac{\gamma}{2}||u^{n+1} - \tilde{u}||^2.
\end{aligned} \tag{18}$$

From (2), (4) and (15) we have

$$\begin{aligned}
&\tfrac{\alpha}{2}||\tilde{u} - u||^2 \le \langle F'(u), u - \tilde{u}\rangle + F(\tilde{u}) - F(u) \le F(\tilde{u}) + \langle T(u), \tilde{u}\rangle \\
&-F(u) - \langle T(u), u\rangle \le \tfrac{\gamma}{\alpha}[F(u^n) + \langle T(u), u^n\rangle - F(u) - \langle T(u), u\rangle].
\end{aligned} \tag{19}$$

In view of (2), (12), (17) and (3), we get

$$\begin{aligned}
&\tfrac{\alpha}{2}||u^{n+1} - \tilde{u}||^2 \le \langle F'(\tilde{u}), \tilde{u} - u^{n+1}\rangle + F(u^{n+1}) - F(\tilde{u}) \\
&\le (\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa [F(u^n) + \langle T(u), u^n\rangle - F(u) - \langle T(u), u\rangle \\
&+F(u) + \langle T(u), u\rangle - F(\tilde{u}) - \langle T(u), \tilde{u}\rangle] + \gamma(\tfrac{\tilde{C}}{\tilde{C}+1})^\kappa ||u^n - u|| ||u^n - \tilde{u}||.
\end{aligned}$$

As previously, using (14) and (19), we get

$$\begin{aligned}
&||u^n - u|| ||u^n - \tilde{u}|| \le \tfrac{3}{2}||u^n - u||^2 + \tfrac{1}{2}||u - \tilde{u}||^2 \\
&\le [\tfrac{3}{\alpha} + \tfrac{\gamma}{\alpha^2}][F(u^n) + \langle T(u), u^n\rangle - F(u) - \langle T(u), u\rangle].
\end{aligned}$$

From the last two equations, since $F(u) - F(\tilde{u}) + \langle T(u), u - \tilde{u}\rangle \leq 0$, we have

$$\begin{array}{l}\frac{\alpha}{2}||u^{n+1} - \tilde{u}||^2 \leq (\frac{\tilde{C}}{\tilde{C}+1})^\kappa [1 + 3\frac{\gamma}{\alpha} + \frac{\gamma^2}{\alpha^2}] \\ \cdot[F(u^n) + \langle T(u), u^n\rangle - F(u) - \langle T(u), u\rangle].\end{array} \tag{20}$$

Finally, from (18), (14), (19) and (20), we get

$$\begin{array}{l}F(u^{n+1}) + \langle T(u), u^{n+1}\rangle - F(u) - \langle T(u), u\rangle \\ \leq [2\frac{\gamma}{\alpha} + (\frac{\tilde{C}}{\tilde{C}+1})^\kappa (1 + 3\frac{\gamma}{\alpha} + 4\frac{\gamma^2}{\alpha^2} + \frac{\gamma^3}{\alpha^3})][F(u^n) + \langle T(u, u^n) - F(u) - \langle T(u, u)].\end{array}$$

Remark 1. Theorem 1 shows that if the convergence condition (7) is satisfied and the number κ of the intermediate iterations is sufficiently large then Algorithm 1 converges and error estimation (11) holds.

3 Multilevel and multigrid methods

We consider a family of regular meshes $\mathcal{T}_{h_j}$ of mesh sizes h_j, $j = 1, \ldots, J$ over the domain $\Omega \subset \mathbf{R}^d$ and assume that $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$, $j = 1, \ldots, J-1$. Also, at each level $j = 1, \ldots, J$, we consider an overlapping decomposition $\{\Omega_j^i\}_{1\leq i\leq I_j}$ of Ω, and assume that the mesh partition $\mathcal{T}_{h_j}$ supplies a mesh partition for each Ω_j^i, $1 \leq i \leq I_j$.

At each level $j = 1, \ldots, J$, we introduce the linear finite element spaces V_{h_j} whose elements vanish on $\partial\Omega$. Also, for $i = 1, \ldots, I_j$, we consider the subspaces $V_{h_j}^i$ of V_{h_j} whose elements vanish on $\Omega\backslash\Omega_j^i$. With these spaces, Algorithm 1 becomes a multilevel method. In Badea [2014], for a problem of two-obstacle type, $K = [\varphi, \psi]$, level convex sets $\mathcal{K}_j = [\varphi_j, \psi_j]$, $j = 1, \ldots, J$, satisfying Assumption 1 have been constructed. Also, it has been proved there that Assumption 3 holds with the constant

$C_2 = CI^2(J-1)^{\frac{1}{2}}[\sum_{j=2}^J C_d(h_{j-1}, h_J)^2]^{\frac{1}{2}}$,

where

$C_d(H, h) :=$ 1 if $d = 1$, $(\ln\frac{H}{h} + 1)^{\frac{1}{2}}$ if $d = 2$ and $(\frac{H}{h})^{\frac{d-2}{2}}$ if $2 < d$,

d being the Euclidean dimension of the space where the domain Ω lies and C is a constant independent of J and I_j, $i = 1, \cdots, J$. Consequently, Theorem 1 shows that the multilevel method corresponding to Algorithm 1 is convergent and we can explicitly write its convergence rate.

If the level decompositions of the domain are given by the supports of the nodal basis functions of the spaces V_{h_j}, $j = J, \ldots, 1$, Algorithm 1 becomes a multigrid method. In this case, it is proved in Badea [2014] that we can take $C_1 = C$ and $\max_{k=1,\ldots,J}\sum_{j=1}^J \beta_{kj} = C$, where $C \geq 1$ is a constant independent of the number of meshes. By expressing the constant C_2 only in function of J, the following result is a direct consequence of Theorem 1,

Corollary 1. *As a function of the number J of levels, the error estimate of the multigrid method obtained from Algorithm 1 can be written as*

$$\|u^n - u\|_1^2 \le C\left[2\frac{\gamma}{\alpha} + \left(\frac{\tilde{C}(J)}{\tilde{C}(J)+1}\right)^{\kappa}\left(1 + 3\frac{\gamma}{\alpha} + 4\frac{\gamma^2}{\alpha^2} + \frac{\gamma^3}{\alpha^3}\right)\right]^n,$$

where $||\cdot||_1$ is the norm of $H^1(\Omega)$ and $\tilde{C}(J) = CJS_d(J)^2$, in which $S_d(J)$ is $\left[\sum_{j=2}^{J} C_d(h_{j-1}, h_J)^2\right]^{\frac{1}{2}}$ expressed in function of J,

$$S_d(J) := (J-1)^{\frac{1}{2}} \text{ if } d = 1,\ CJ \text{ if } d = 2 \text{ and } C^J \text{ if } d = 3,$$

constant C being independent of the number of levels J.

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