Multiplicative overlapping Schwarz smoothers for H^{div}-conforming discontinuous Galerkin methods for the Stokes problem

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Abstract We present numerical results for a multigrid method employing overlapping Schwarz smoothers in various V-cycle configurations. The method is based on finite element discretizations of the Stokes problem employing H^{div}-conforming velocity spaces and matching pressure spaces. The method acts on the combined velocity and pressure spaces and thus does not need a Schur complement approximation.

Key words: multigrid, smoother, overlapping Schwarz, discontinuous Galerkin methods, divergence-conforming

1 Introduction

The efficient solution of the Stokes equations is an important step in the development of fast flow solvers. The saddle point structure due to the divergence constraint makes the solution process more complicated. Block preconditioners are often employed, but their performance is limited by the inf-sup constant of the problem and by the difficulty of finding a good preconditioner for the pressure Schur complement. This could be avoided, if the multigrid method operated on the divergence free subspace directly. Recently in Kanschat and Mao [2014], we introduced and analyzed a multigrid method with an additive overlapping Schwarz smoother. The main ingredients of our method are a smoother which implicitly operates on the divergence free subspace and a grid transfer operator from coarse to fine mesh which

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maps the coarse divergence free subspace into the fine one. In this contribution here, we now employ the multiplicative version of this Schwarz method and present numerical results for it.

We consider discretizations of the Stokes equations with no-slip boundary conditions

$$\begin{aligned} -\triangle u + \nabla p &= f && \text{in} \quad \Omega, \\ \nabla\cdot u &= 0 && \text{in} \quad \Omega, \\ u &= u^B && \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

on a bounded domain $\Omega \subset \mathbb{R}^d$ of dimension $d = 2,3$. The natural solution spaces for this problem are $V = H^1_0(\Omega;\mathbb{R}^d)$ for the velocity u and the space of mean value free square integrable functions $Q = L^2_0(\Omega)$ for the pressure p. We point out that other well-posed boundary conditions do not pose a problem.

In order to obtain a finite element discretization, we partition the domain Ω into a hierarchy of meshes $\{\mathbb{T}_\ell\}_{\ell=0,\dots,L}$ of parallelogram and parallelepiped cells in two and three dimensions, respectively. By $\mathbb{F}_\ell$ we denote the set of all faces of the mesh $\mathbb{T}_\ell$. The set $\mathbb{F}_\ell$ is composed of the set of interior faces $\mathbb{F}^i_\ell$ and the set of all boundary faces $\mathbb{F}^\partial_\ell$.

In order to discretize (1) on the mesh $\mathbb{T}_\ell$, we choose discrete subspaces $X_\ell = V_\ell \times Q_\ell$, where $Q_\ell \subset Q$. FollowingCockburn et al. [2007], we employ discrete subspaces V_ℓ of the space $H^{\text{div}}_0(\Omega)$, where

$$\begin{aligned} H^{\text{div}}(\Omega) &= \left\{ v \in L^2(\Omega;\mathbb{R}^d) \middle| \nabla\cdot v \in L^2(\Omega) \right\}, \\ H^{\text{div}}_0(\Omega) &= \left\{ v \in H^{\text{div}}(\Omega) \middle| v\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega \right\}. \end{aligned}$$

On each mesh cell T, we choose the Raviart–Thomas Raviart and Thomas [1977] space of degree k with $k \geq 1$, mapped by the Piola transformation if necessary and denoted by V_T. We point out that any pair of velocity spaces V_ℓ and pressure spaces Q_ℓ is admissible, if the key relation

$$\nabla\cdot V_\ell = Q_\ell \tag{2}$$

holds. We obtain the finite element spaces

$$\begin{aligned} V_\ell &= \left\{ v \in H^{\text{div}}_0(\Omega) \middle| \forall T \in \mathbb{T}_\ell : v_{|T} \in V_T \right\}, \\ Q_\ell &= \left\{ q \in L^2_0(\Omega) \middle| \forall T \in \mathbb{T}_\ell : q_{|T} \in Q_T \right\}. \end{aligned}$$

1.1 Discontinuous Galerkin discretization

While the fact that V_ℓ is a subspace of $H^{\text{div}}_0(\Omega)$ implies continuity of the normal component of its functions across interfaces between cells, this is not true for tangential components. Thus, $V_\ell \not\subset H^1(\Omega;\mathbb{R}^d)$, and it cannot be used immediately to discretize (1). We follow the example in for instance Cockburn et al. [2007] and apply a DG formulation to the discretization of the elliptic operator. Here, we focus on the interior penalty methodArnold [1982]. Let T_1 and T_2 be two mesh cells with a joint face F, and let u_1 and u_2 be the traces of a function u on F from T_1 and T_2, respectively. On this face F, we introduce the averaging operator

$$\{\{u\}\} = \frac{u_1 + u_2}{2}. \tag{3}$$

Using the notation, that every integral form over a set of mesh cells or faces is the sum of the integrals over all objects in the set, the interior penalty bilinear form reads

$$\begin{aligned} a_\ell(u,v) = &(\nabla u, \nabla v)_{\mathbb{T}_\ell} + 4\,\langle \sigma_L \{\{u\otimes \mathbf{n}\}\}, \{\{v\otimes \mathbf{n}\}\}\rangle_{\mathbb{F}_\ell^i} \\ &- 2\,\langle \{\{\nabla u\}\}, \{\{\mathbf{n}\otimes v\}\}\rangle_{\mathbb{F}_\ell^i} - 2\,\langle \{\{\nabla v\}\}, \{\{\mathbf{n}\otimes u\}\}\rangle_{\mathbb{F}_\ell^i} \\ &+ 2\,\langle \sigma_L u, v\rangle_{\mathbb{F}_\ell^\partial} - \langle \partial_n u, v\rangle_{\mathbb{F}_\ell^\partial} - \langle \partial_n v, u\rangle_{\mathbb{F}_\ell^\partial}. \end{aligned} \tag{4}$$

The operator "$\otimes$" denotes the Kronecker product of two vectors. We note that the term $4\{\{u\otimes \mathbf{n}\}\} : \{\{v\otimes \mathbf{n}\}\}$ actually denotes the product of the jumps of u and v.

The discrete weak formulation of (1) reads now: find $(u_\ell, p_\ell) \in V_\ell \times Q_\ell$, such that for all test functions $v_\ell \in V_\ell$ and $q_\ell \in Q_\ell$ there holds

$$\mathscr{A}_\ell\left(\begin{pmatrix} u_\ell \\ p_\ell \end{pmatrix}, \begin{pmatrix} v_\ell \\ q_\ell \end{pmatrix}\right) \equiv a_\ell(u_\ell, v_\ell) + (p_\ell, \nabla\cdot v_\ell) - (q_\ell, \nabla\cdot u_\ell) = \mathscr{F}(v_\ell, q_\ell) \equiv (f, v_\ell)\,. \tag{5}$$

Discussion on the existence and uniqueness of such solutions can be found for instance in Cockburn et al. [2002]. Here, we summarize, that $a_\ell(.,.)$ is symmetric and, if σ_L is sufficiently large, it is positive definite. Thus, we can define a norm on V_ℓ by

$$\|v_\ell\|_{V_\ell} = \sqrt{a_\ell(v_\ell, v_\ell)}. \tag{6}$$

In order to obtain optimal convergence results, σ_L is chosen as $\overline{\sigma}/h_L$, where h_L is mesh size on the finest level L and $\overline{\sigma}$ is a positive constant depending on the polynomial degree. A key result in the convergence analysis of this discretization as well as in the analysis of the additive Schwarz smoother is the inf-sup condition

$$\inf_{v\in V_\ell} \sup_{q\in Q_\ell} \frac{(q, \nabla\cdot v)}{\|v\|_{V_\ell} \|q\|_{Q_\ell}} \geq \gamma_\ell > 0 \tag{7}$$

where $\gamma_\ell = c\sqrt{\frac{h_L}{h_\ell}} = c\sqrt{2^{\ell-L}}$ and c is a constant independent of the grid level ℓ.

2 Multigrid method

In this section we define a V-cycle multigrid preconditioner $\mathscr{B}_\ell$ for the operator $\mathscr{A}_\ell$. We define the action of the multigrid preconditioner $\mathscr{B}_\ell : X_\ell \to X_\ell$ recursively as the multigrid V-cycle with $m(\ell) \geq 1$ pre- and post-smoothing steps. Let $\mathscr{R}_\ell$ be a suitable smoother. Let $\mathscr{B}_0 = \mathscr{A}_0^{-1}$. For $\ell \geq 1$, define the action of $\mathscr{B}_\ell$ on a vector $\mathscr{L}_\ell = (f_\ell, g_\ell)$ by

1. Pre-smoothing: begin with $(u_0, p_0) = (0,0)$ and let

$$\begin{pmatrix} u_i \\ p_i \end{pmatrix} = \begin{pmatrix} u_{i-1} \\ p_{i-1} \end{pmatrix} + \mathscr{R}_\ell \left(\mathscr{L}_\ell - \mathscr{A}_\ell \begin{pmatrix} u_{i-1} \\ p_{i-1} \end{pmatrix} \right) \quad i = 1, \ldots, m(\ell), \tag{8a}$$

2. Coarse grid correction:

$$\begin{pmatrix} u_{m(\ell)+1} \\ p_{m(\ell)+1} \end{pmatrix} = \begin{pmatrix} u_{m(\ell)} \\ p_{m(\ell)} \end{pmatrix} + \mathscr{B}_{\ell-1} \mathscr{I}_{\ell-1}^t \left(\mathscr{L}_\ell - \mathscr{A}_\ell \begin{pmatrix} u_{m(\ell)} \\ p_{m(\ell)} \end{pmatrix} \right), \tag{8b}$$

3. Post-smoothing:

$$\begin{pmatrix} u_i \\ p_i \end{pmatrix} = \begin{pmatrix} u_{i-1} \\ p_{i-1} \end{pmatrix} + \mathscr{R}_\ell \left(\mathscr{L}_\ell - \mathscr{A}_\ell \begin{pmatrix} u_{i-1} \\ p_{i-1} \end{pmatrix} \right), \quad i = m(\ell)+2, \ldots, 2m(\ell)+1 \tag{8c}$$

4. Assign:

$$\mathscr{B}_\ell \mathscr{L}_\ell = \begin{pmatrix} u_{2m(\ell)+1} \\ p_{2m(\ell)+1} \end{pmatrix} \tag{8d}$$

We distinguish between the standard V-cycle with $m(\ell) = m(L)$ and the variable V-cycle with $m(\ell) = m(L)2^{L-\ell}$, where the number $m(L)$ of smoothing steps on the finest level is a free parameter. We refer to $\mathscr{B}_L$ as the V-cycle preconditioner of $\mathscr{A}_L$. The iteration

$$\begin{pmatrix} u_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} u_k \\ p_k \end{pmatrix} + \mathscr{B}_L \left(\mathscr{L}_L - \mathscr{A}_L \begin{pmatrix} u_k \\ p_k \end{pmatrix} \right) \tag{9}$$

is the V-cycle iteration.

2.1 Overlapping Schwarz smoothers

In this subsection, we define a class of smoothing operators $\mathscr{R}_\ell$ based on a subspace decomposition of the space X_ℓ. Let $\mathscr{N}_\ell$ be the set of vertices in the triangulation $\mathbb{T}_\ell$, and let $\mathbb{T}_{\ell,\upsilon}$ be the set of cells in $\mathbb{T}_\ell$ sharing the vertex υ. They form a subdivision of Ω with N overlapping subdomains (also called patches) which we denote by $\{\Omega_{\ell,\upsilon}\}_{\upsilon=1}^{N}$.

The subspace $X_{\ell,\upsilon} = V_{\ell,\upsilon} \times Q_{\ell,\upsilon}$ consists of the functions in X_ℓ with support in $\Omega_{\ell,\upsilon}$. Note that this implies homogeneous slip boundary conditions on $\partial\Omega_{\ell,\upsilon}$ for the velocity subspace $V_{\ell,\upsilon}$ and zero mean value on $\Omega_{\ell,\upsilon}$ for the pressure subspace $Q_{\ell,\upsilon}$. The Ritz projection $\mathscr{P}_{\ell,\upsilon} : X_\ell \to X_{\ell,\upsilon}$ is defined by the equation

$$\mathscr{A}_\ell\left(\mathscr{P}_{\ell,\upsilon}\begin{pmatrix}u_\ell\\p_\ell\end{pmatrix},\begin{pmatrix}v_{\ell,\upsilon}\\q_{\ell,\upsilon}\end{pmatrix}\right) = \mathscr{A}_\ell\left(\begin{pmatrix}u_\ell\\p_\ell\end{pmatrix},\begin{pmatrix}v_{\ell,\upsilon}\\q_{\ell,\upsilon}\end{pmatrix}\right) \qquad \forall\begin{pmatrix}v_{\ell,\upsilon}\\q_{\ell,\upsilon}\end{pmatrix} \in X_{\ell,\upsilon}. \quad (10)$$

Note that each cell belongs to no more than four (eight in 3D) patches $\mathbb{T}_{\ell,\upsilon}$, one for each of its vertices.

We recall the additive Schwarz smoother

$$\mathscr{R}_{a,\ell} = \eta \sum_{\upsilon \in \mathscr{N}_\ell} \mathscr{P}_{\ell,\upsilon}\mathscr{A}_\ell^{-1}$$

where $\eta \in (0,1]$ is a scaling factor, $\mathscr{R}_\ell$ is L^2 symmetric and positive definite. In Kanschat and Mao [2014], it was shown based on arguments from Arnold et al. [1997], Schöberl [1999], that this smoother yields a uniformly convergent multigrid method if η is chosen appropriately.

Here, we use the symmetric multiplicative Schwarz smoother $\mathscr{R}_{m,\ell}$ associated with the spaces $X_{\ell,\upsilon}$, defined by

$$\begin{aligned}\mathscr{R}_{m,\ell} &= (\mathscr{I} - \mathscr{E}_\ell)\mathscr{A}_\ell^{-1},\\ \mathscr{E}_\ell &= \left(\mathscr{I} - \mathscr{P}_{\ell,1}\right)\dots\left(\mathscr{I} - \mathscr{P}_{\ell,N}\right)\dots\left(\mathscr{I} - \mathscr{P}_{\ell,1}\right).\end{aligned}$$

We proved uniform convergence for the variable V-cycle iteration with the smoother $\mathscr{R}_{a,\ell}$ in Kanschat and Mao [2014] and showed its efficiency by numerical experiments. Since standard arguments from domain decomposition theory like stable decomposition and strengthened Cauchy-Schwarz inequalities are used, we conjecture that the analysis applies to the multiplicative version in the usual fashion. We note that the use of the variable V-cycle is induced by the level dependence of the inf-sup condition (7). Since optimality of this estimate has not been established, we study standard cycles as well.

3 Numerical results

We present numerical results for the multiplicative Schwarz method in various V-cycle methods and different solvers in order to show that the contraction numbers are not only bounded away from one, but are actually small enough to make this method very efficient. The following results were produced using the deal.II library Bangerth et al. [2007, 2015] and its multigrid capabilities Janssen and Kanschat [2011].

	$m(\ell)=2^{L-\ell}$			$m(\ell)=1$			$m(\ell)=2$		
L	RT_1	RT_2	RT_3	RT_1	RT_2	RT_3	RT_1	RT_2	RT_3
3	5	5	5	5	5	5	3	3	3
4	6	6	7	6	6	7	5	5	5
5	6	6	6	6	6	7	5	5	6
6	5	5	6	6	6	7	5	5	6
7	5	5	6	7	7	7	5	5	6
8	5	5	6	7	7	7	6	6	6

Table 1 Number of iterations n_8 to reduce the residual by 10^{-8} with the variable V-cycle and the standard V-cycle iteration with one and two pre- and post-smoothing steps. Penalty parameter dependent of the finest level mesh size 2^{1-L}.

	variable			standard		
level	RT_1	RT_2	RT_3	RT_1	RT_2	RT_3
3	6	6	6	6	6	6
4	6	6	6	6	6	7
5	6	6	6	6	6	7
6	5	5	6	6	6	7
7	5	5	6	6	6	7
8	5	5	6	6	6	7

Table 2 Penalty parameter dependent on the mesh size of each level. Number of iterations n_8 to reduce the residual by 10^{-8} with variable and standard V-cycle iterations with $m(L)=1$.

The experimental setup for most of the tables is as follows: the domain is $\Omega = [-1,1]^2$, the coarsest mesh $\mathbb{T}_0$ consists of a single cell $T=\Omega$. The mesh $\mathbb{T}_\ell$ on level ℓ is obtained by dividing all cells in $\mathbb{T}_{\ell-1}$ into four quadrilaterals by connecting the edge midpoints. Thus, a mesh on level ℓ has 4^ℓ cells, and the length of their edges is $2^{1-\ell}$. The right hand side is $f=(1,1)$.

In Table 1, we first study convergence of the linear multigrid method (preconditioned Richardson iteration) with the multiplicative Schwarz smoother using a variable V-cycle algorithm on a square domain with no-slip boundary condition. The penalty constant in the DG form (4) is chosen as $\bar{\sigma}/h_L$, where $\bar{\sigma}=(k+1)(k+2)$, on the finest level L and all lower levels ℓ. Results for pairs of RT_k/Q_k with orders k between one and three are reported in the table which show the fast and uniform convergence. On the right of this table, we keep the same experimental setup and present iteration counts for the standard V-cycle algorithm with one and two pre- and post-smoothing steps, respectively. Although not proven for this case, we still observe uniform convergence results. We also see that the variable V-cycle with a single smoothing step on the finest level is as fast as the standard V-cycle with two smoothing steps, and thus the variable V-cycle is more efficient.

In Table 2, we test the variable and standard V-cycles with penalty parameters depending on the mesh level ℓ, namely $\bar{\sigma}/h_\ell$ (where $\bar{\sigma}$ is a positive constant depending on the polynomial degree) in the DG form (4). This is the typical situation when the operators are assembled independently on each grid level.

In Table 3, we provide results with GMRES solver and $\mathscr{B}_L$ as preconditioner for experimental setups as in Tables 1 and 2, respectively. The second to fourth columns

	variable			standard			noninherited		
level	RT_1	RT_2	RT_3	RT_1	RT_2	RT_3	RT_1	RT_2	RT_3
3	2	2	2	2	2	2	3	3	3
4	3	3	4	4	4	4	5	5	5
5	5	5	5	5	5	5	5	5	5
6	4	4	5	5	5	5	5	5	5
7	4	4	5	5	5	5	5	5	5
8	5	4	5	5	5	5	5	5	5

Table 3 Number of iterations n_8 to reduce the residual by 10^{-8} with GMRES solver and preconditioner $\mathscr{B}_L$; variable and standard V-cycle with inherited forms, variable V-cycle with noninherited forms. One pre- and post-smoothing step on the finest level.

	Richardson		GMRES	
level	RT_1	RT_2	RT_1	RT_2
2	1	1	1	1
3	5	5	4	4
4	6	5	4	4
5	6	5	4	4

Table 4 Three-dimensional domain. Number of iterations n_8 to reduce the residual by 10^{-8} with the variable V-cycle algorithm with penalty parameter dependent of the finest level mesh size.

are results for the variable V-cycle with penalty parameter dependent of the finest level mesh size. The fifth and seventh columns are the results for the standard V-cycle with penalty parameter dependent of the finest level mesh size. The last three columns are the results for the standard V-cycle with penalty parameter depend on the mesh size of each level. From this table, we see that the GMRES method, as expected, is faster in every case.

In Table 4, we provide results in three dimensions for variable V-cycle methods with the same penalty parameter as we choose in Table 1. We keep the similar experimental setups: domain $\Omega = [-1,1]^3$ and right hand side $f = (1,1,1)$. We observe the similar fast and uniform convergence performance as in two dimensions.

We finish our experiments by applying our method to a non-simply connected domain. We choose a square with a square hole, namely the domain $\Omega = [-1,1] \setminus [-\frac{1}{3}, \frac{1}{3}]$. The coarse grid on level $\ell = 0$ consists of the squares of the form $[-1 + \frac{2i}{3}, -1 + \frac{2i+2}{3}] \times [-1 + \frac{2j}{3}, -1 + \frac{2j+2}{3}]$ with $0 \le i,j \le 2$, and with the index pair $(i,j) = (1,1)$ missing. We note that the Hodge decomposition in this case is more complicated due to the presence of a harmonic form. Nevertheless, the results with the multiplicative Schwarz method in Table 5 exhibit the same performance we observed in the simply connected case.

	Richardson		GMRES	
level	RT_1	RT_2	RT_1	RT_2
2	6	6	4	4
3	6	6	4	4
4	6	6	4	4
5	5	5	4	4
6	5	5	4	4
7	5	5	4	4

Table 5 Number of iterations n_8 to reduce the residual by 10^{-8}, different finite element orders and solvers on the domain with hole $[-1,1]^2 \setminus [-1/3,1/3]^2$

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