Adaptive coarse spaces for BDDC with a transformation of basis

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1 Introduction

We describe a BDDC algorithm, see e.g., Dohrmann [2003], and an adaptive coarse space enforced by a transformation of basis for the iterative solution of scalar diffusion problems with a discontinuous diffusion coefficient. The coefficient varies over several orders of magnitude both inside of the subdomains and along the interface. A related algorithm for FETI-DP with a balancing preconditioner has been already described in Klawonn et al. [2013b,a]. Other adaptive coarse space constructions for FETI, FETI-DP, and BDDC methods have been proposed in Spillane and Rixen [2013], Mandel and Sousedík [2007]. We also present some preliminary numerical results for different scalings, including the recent deluxe scaling; cf., Dohrmann and Widlund [2013].

We consider the following model problem. Let $\Omega \subset \mathbb{R}^2$ be a bounded polyhedral domain. We subdivide $\partial\Omega$ into a subset of positive measure $\partial\Omega_D$ where Dirichlet boundary conditions are imposed and $\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$ where general Neumann boundary conditions are prescribed. Define the Sobolev space $H_0^1(\Omega, \partial\Omega_D) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}$ and consider the piecewise linear finite element approximation of the scalar diffusion problem: Find $u \in H_0^1(\Omega, \partial\Omega_D)$, such that $a(u,v) = f(v)$ holds for all $v \in H_0^1(\Omega, \partial\Omega_D)$. The bilinear form $a(u,v)$ and the functional $f(v)$ are defined by

$$a(u,v) = \int_\Omega \rho(x) \nabla u \nabla v \, dx \quad \text{and} \quad f(v) = \int_\Omega f v \, dx + \int_{\partial\Omega_N} g_N v \, ds,$$

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where g_N is the Neumann boundary data on $\partial\Omega_N$. The model problem is discretized with linear finite elements. We assume $\rho(x)$ to be positive and piecewise constant on Ω and constant on single elements of the triangulation.

The remainder of the paper is organized as follows. We describe the transformation of basis which is performed in our BDDC algorithm to introduce additional coarse constraints in Section 2. The characterization how these constraints are chosen via the solution of local eigenvalue problems and an overview over our theoretical results is given in Section 3. For a more detailed analysis, see Klawonn et al. [2013b]. In Section 4 we consider some examples and present numerical results.

2 Transformation of Basis and Scaling in the BDDC algorithm

As a domain decomposition method we use BDDC. Due to space limitation, for a description of the algorithm and the notation, we refer the reader to Klawonn et al. [2008]. Given a set of primal vertex variables, in the next section, we describe a way to obtain adaptively additional primal variables in the form of weighted edge averages. To implement these edge averages, we transform our local stiffness matrices $K^{(i)}$ and right hand sides $f^{(i)}$ with a transformation matrix $T^{(i)}$. The resulting transformed stiffness matrices $\overline{K}^{(i)} = T^{(i)T} K^{(i)} T^{(i)}$ and right hand sides $\overline{f}^{(i)} = T^{(i)T} f^{(i)}$ then replace $K^{(i)}$ and $f^{(i)}$ in the BDDC algorithm; see, e.g., Klawonn et al. [2008] for more details. We construct the transformation matrices $T^{(i)}$ edge by edge. Consider an edge E of Ω_i and the restriction of $T^{(i)}$ to this edge, denoted by T_E. Suppose we have selected a set of weighted edge averages with weights described by orthonormal column vectors $\{v_{E,1}^{(i)}, \dots, v_{E,m}^{(i)}\}$. We augment this set to an orthonormal basis $\{v_{E,1}^{(i)}, \dots, v_{E,m}^{(i)}, v_{E,m+1}^{(i)}, \dots, v_{E,n_E}^{(i)}\}$ of $\mathbb{R}^{n_E}$, where n_E denotes the number of nodes of the edge E. The transformation matrix T_E is defined by $T_E = [v_{E,1}^{(i)}, \dots, v_{E,m}^{(i)}, v_{E,m+1}^{(i)}, \dots, v_{E,n_E}^{(i)}]$ and describes the change of basis from the new to the original nodal basis. The first m columns of T_E correspond to the new additional primal variables and the remaining columns correspond to the new dual unknowns. Denoting the edge unknowns in the new basis by $\widehat{u}_E$ and the unknowns in the original basis by u_E, we have $u_E = T_E \widehat{u}_E$. We denote by $T_E^{(i)}$ the transformation matrix which operates on all edges of $\partial\Omega_i$. The transformation matrix $T^{(i)}$ is then defined by $T^{(i)} = \mathrm{diag}(I_I, I_V, T_E^{(i)})$, where I_I and I_V denote the identity on inner variables and on vertex variables, respectively. The transformed stiffness matrices are of the form

$$T^{(i)T}K^{(i)}T^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{IV}^{(i)} & K_{IE}^{(i)}T_E^{(i)} \\ K_{VI}^{(i)} & K_{VV}^{(i)} & K_{VE}^{(i)}T_E^{(i)} \\ T_E^{(i)T}K_{EI}^{(i)} & T_E^{(i)T}K_{EV}^{(i)} & T_E^{(i)T}K_{EE}^{(i)}T_E^{(i)} \end{bmatrix},$$

with right hand sides $T^{(i)T}f^{(i)} = [f_I^{(i)T} \quad f_V^{(i)T} \quad f_E^{(i)T}T_E^{(i)}]^T$. We can now perform our BDDC algorithm with the transformed problem; see, e.g., Klawonn et al. [2008] for a detailed desription. In our algorithm we will use two different scalings. Let φ_i be the nodal finite element function associated with the node x_i and define $\widehat{\rho}_j(x_i) = \max_{T \in \text{supp}(\varphi_i) \cap \Omega_j} \rho_{j|T}(x_i)$. Our scaling weights are now defined as $\delta_j^\dagger(x) = \widehat{\rho}_j(x) / \sum_{k \in N_x} \widehat{\rho}_k(x)$, where N_x is the set of indices of the subdomains that have the node x on their boundary. The scaling matrices $D^{(j)}$ are diagonal matrices in this case with the weights $\delta_j^\dagger(x)$ on the diagonal. This approach is usually referred to as ρ-scaling. We consider another scaling variant, also known as deluxe scaling, see e.g., Dohrmann and Widlund [2013]. In this case the restriction $D_{E_{ij}}^{(k)}$ of $D^{(k)}$ to an edge $\mathcal{E}_{ij}$ is defined by $D_{E_{ij}}^{(k)} = (S_{E_{ij}E_{ij}}^{(i)} + S_{E_{ij}E_{ij}}^{(j)})^{-1} S_{E_{ij}E_{ij}}^{(k)}$, $k = i, j$, where $S_{E_{ij}E_{ij}}^{(k)}$ is the restriction of $S^{(k)}$ to the edge $\mathcal{E}_{ij}$ after the transformation of basis.

3 Choice of Weighted Edge Averages

In the following we will consider two different eigenvalue problems to compute weighted edge averages for our algorithm; see also Klawonn et al. [2013b]. The first eigenvalue problem is a replacement for the weighted Poincaré inequalities in the case of non-quasimonotone coefficient functions; see Klawonn et al. [2013b,a]. The second is related to an extension theorem; see Klawonn et al. [2013b]. For a common edge $\mathcal{E}_{ij}$ of the subdomains Ω_i and Ω_j we define $S_{\mathcal{E}_{ij},\rho}^{(l)}$, $l = i, j$, as the Schur complement which is obtained after eliminating all variables of $K^{(l)}$ except of the variables on the closure of $\mathcal{E}_{ij}$, denoted by $\overline{\mathcal{E}}_{ij}$. We define the mass matrix $(M_{\mathcal{E}_{ij},\rho}^{(l)})_{pq} := \int_{\mathcal{E}_{ij}} \rho_l \varphi_p \varphi_q ds$, $p, q = 1, \ldots, n_{\mathcal{E}_{ij}}$, where $n_{\mathcal{E}_{ij}}$ denotes the number of degrees of freedom on $\overline{\mathcal{E}}_{ij}$ and φ_p is the nodal finite element basis function associated with a node $x_p \in \overline{\mathcal{E}}_{ij}$. We also introduce the bilinear forms $s_{\mathcal{E}_{ij},\rho}^{(l)}(u,v) := u^T S_{\mathcal{E}_{ij},\rho}^{(l)} v$ and $m_{\mathcal{E}_{ij},\rho}^{(l)}(u,v) := u^T M_{\mathcal{E}_{ij},\rho}^{(l)} v$. If the coefficient $\rho(x)$ of the diffusion problem varies over several orders of magnitude inside of subdomains and over the interface of the decomposition and is non-quasimonotone the constant in the Poincaré inequality is polluted by the contrast of the coefficient. For a definition of quasimonotone coefficients and a detailed analysis of weighted Poincaré inequalities, see Pechstein and Scheichl [2013]. The Poincaré constant also appears in the bound of the condition number estimate of substructuring methods equipped with a classical coarse space, e.g., a coarse space consisting of vertices and standard edge averages only. To circumvent this problem we introduce a generalized eigenvalue problem to compute new weighted averages which will be used to enhance our coarse space. Note, that related eigenvalue problems are also used in Galvis and Efendiev [2010] and in Dolean et al. [2012] in the context of overlapping Schwarz methods. However, our approach is more local. We denote the finite element trace space on $\mathcal{E}_{ij}$ by $W^h(\mathcal{E}_{ij})$.

Eigenvalue Problem 1 (EVP 1) *Find* $(u_k^{(i)}, \mu_k^{(i)}) \in W^h(\mathcal{E}_{ij}) \times \mathbb{R}$ *such that*

$$s_{\mathcal{E}_{ij},\rho}^{(i)}(u_k^{(i)}, v) = \mu_k^{(i)} m_{\mathcal{E}_{ij},\rho}^{(i)}(u_k^{(i)}, v) \qquad \forall v \in W^h(\mathcal{E}_{ij}). \tag{1}$$

For $L \in \{1, \dots, n_{\mathcal{E}_{ij}}\}$, where $n_{\mathcal{E}_{ij}}$ is the number of degrees of freedom on $\overline{\mathcal{E}}_{ij}$, and for $l = i, j$ we introduce the projection

$$I_L^{\mathcal{E}_{ij},(l)} = \sum_{k=1}^{L} m_{\mathcal{E}_{ij},\rho}^{(l)}(u_k^{(l)}, v) u_k^{(l)}, \quad l = i, j,$$

with the eigenvectors $u_k^{(l)}$ of Eigenvalue Problem 1. The next lemma provides a generalized Poincaré inequality and is needed to estimate weighted L^2-norms of projected finite element functions on edges; for a proof, see Klawonn et al. [2013b].

Lemma 1. *For* $v \in W^h(\mathcal{E}_{ij})$ *and* $w := \left(v - I_L^{\mathcal{E}_{ij},(l)} v\right) \in W^h(\mathcal{E}_{ij})$, *we have*

$$||v - I_L^{\mathcal{E}_{ij},(l)} v||^2_{L^2_{\rho_l}(\mathcal{E}_{ij})} = m_{\mathcal{E}_{ij},\rho}^{(l)}(w, w) \le \frac{1}{\mu_{L+1}^{(l)}} s_{\mathcal{E}_{ij},\rho}^{(l)}(v, v) \tag{2}$$

$$\text{and} \qquad s_{\mathcal{E}_{ij},\rho}^{(l)}(w, w) \le s_{\mathcal{E}_{ij},\rho}^{(l)}(v, v). \tag{3}$$

In our BDDC coarse space we will enforce the equality of the projected functions $I_L^{\mathcal{E}_{ij},(i)} v^{(i)} = I_L^{\mathcal{E}_{ij},(i)} v^{(j)}$ and $I_L^{\mathcal{E}_{ij},(j)} v^{(i)} = I_L^{\mathcal{E}_{ij},(j)} v^{(j)}$ on the interface. We cannot directly enforce this equality, but instead we guarantee that $m_{\mathcal{E}_{ij},\rho}^{(l)}(u_k^{(l)}, v_{\mathcal{E}_{ij}}^{(i)}) = m_{\mathcal{E}_{ij},\rho}^{(l)}(u_k^{(l)}, v_{\mathcal{E}_{ij}}^{(j)})$, for $k = 1, .., L$, by a transformation of basis. To do so, we first build $M_{\mathcal{E}_{ij},\rho}^{(l)} u_k^{(l)}$ and discard the entries related to primal vertices. Then, this vector defines those columns of the local transformation matrices $T_E^{(i)}$ and $T_E^{(j)}$ which are related to the corresponding primal variable in the new basis. We choose all eigenvectors of Eigenvalue Problem 1 whose corresponding eigenvalues satisfy $\mu \le \tau_\mu$ with a chosen tolerance τ_μ.

To guarantee that certain extensions can be bounded with constants independent of coefficient jumps, we introduce a second eigenvalue problem.

Eigenvalue Problem 2 (EVP 2)

$$s_{\mathcal{E}_{ij},\rho_j}^{(j)}(v, w_\kappa) = \nu_\kappa^{(i)} s_{\mathcal{E}_{ij},\rho_i}^{(i)}(v, w_\kappa), \quad \kappa = 1, \dots, n_{\mathcal{E}_{ij}}. \tag{4}$$

Remark 1. If $\ker(s^{(j)}_{\mathcal{E}_{ij},\rho_j}) = \ker(s^{(i)}_{\mathcal{E}_{ij},\rho_i})$, instead of solving Eigenvalue Problem 2 on range$(s^{(j)}_{\mathcal{E}_{ij},\rho_j})$, we solve

$$\overline{\Pi} S^{(j)}_{\mathcal{E}_{ij},\rho_j} \overline{\Pi}\,\overline{w} = \nu \left(\overline{\Pi} S^{(i)}_{\mathcal{E}_{ij},\rho_i} \overline{\Pi} + \sigma \left(I - \overline{\Pi} \right) \right) \overline{w},$$

where σ is any positive constant and $\overline{\Pi}$ is an orthogonal projection onto range$(s^{(i)}_{\mathcal{E}_{ij},\rho_i})$. In our computations we have chosen σ as the maximum diagonal entry of $\overline{\Pi} S^{(i)}_{\mathcal{E}_{ij},\rho_i} \overline{\Pi}$. The right-hand side of this problem is positive definite; see also Mandel and Sousedík [2007].

We introduce a second projection operator

$$\Pi^{(l)}_K v := \sum_{k=1}^{K} s^{(l)}_{\mathcal{E}_{ij},\rho}(w^{(l)}_k, v) w^{(l)}_k, \quad l = i, j,$$

with $K \in \{1, \ldots, n_{\mathcal{E}_{ij}}\}$ and obtain the following lemma; see Klawonn et al. [2013b] for a proof.

Lemma 2. *We have* $\forall\, w^{(j)} \in W^h(\mathcal{E}_{ij})$

$$s^{(i)}_{\mathcal{E}_{ij},\rho_i} \left(w^{(j)} - \Pi^{(i)}_K w^{(j)}, w^{(j)} - \Pi^{(i)}_K w^{(j)} \right) \leq \frac{1}{\nu^{(i)}_{K+1}} s^{(j)}_{\mathcal{E}_{ij},\rho_j} \left(w^{(j)}, w^{(j)} \right).$$

To take advantage of Lemma 2 we need to introduce a second set of primal constraints of the form $\Pi^{(i)}_K w^{(i)} = \Pi^{(i)}_K w^{(j)}$ and $\Pi^{(j)}_K w^{(i)} = \Pi^{(j)}_K w^{(j)}$. For both generalized eigenvalue problems 1 and 2 we introduce tolerances to decide which eigenvectors are chosen to enhance our coarse space. Additionally to the eigenvectors of Eigenvalue Problem 1 we choose all eigenvectors of Eigenvalue Problem 2 whose corresponding eigenvalues satisfy $\nu \leq \tau_\nu$. with a chosen tolerance τ_ν.

Definition 1. By an η-patch $\omega \subset \Omega$ we denote an open set which can be represented as a union of shape regular finite elements and which has diam$(\omega) \in \mathcal{O}(\eta)$ and a measure of $\mathcal{O}(\eta^2)$. Let $\mathcal{E}_{ij} \subset \partial\Omega_i$ be an edge. Then, a slab $\widetilde{\Omega}_{i\eta}$ is a subset of Ω_i of width η with $\mathcal{E}_{ij} \subset \partial\widetilde{\Omega}_{i\eta}$ which can be represented as the union of η-patches ω_{ik}, $k = 1, \ldots, n$, such that $\mathcal{E}^{(k)}_{ij} := (\partial\omega_{ik} \cap \mathcal{E}_{ij})^\circ \neq \emptyset$, $k = 1, \ldots, n$.

For each edge $\mathcal{E}_{ij}$ let $\widetilde{\Omega}_{i\eta} \subset \Omega_i$ be a slab of width η, such that $\mathcal{E}_{ij} \subset \partial\widetilde{\Omega}_{i\eta}$. Let $\omega_{ik} \subset \widetilde{\Omega}_{i\eta}$, $k = 1, \ldots, n$, be a set of η-patches such that $\widetilde{\Omega}_{i\eta} = \cup^n_{k=1}\omega_{ik}$, and the coefficient function $\rho_{i|\omega_{ik}} = \rho_{ik}$ is constant on each ω_{ik}. Let $\omega_{ik} \cap \omega_{il} = \emptyset$, $k \neq l$. We obtain the following condition number estimate which is proven in Klawonn et al. [2013b].

Theorem 1. *The condition number for our BDDC algorithm satisfies*

$$\kappa(M_{BDDC}^{-1}S) \leq C\left(1+\log\left(\frac{\eta}{h}\right)\right)^2 \frac{1}{\nu_{K+1}}\left(1+\frac{1}{\eta\mu_{L+1}}\right).$$

Here, $C > 0$ is a constant independent of H, h, and η and

$$\frac{1}{\mu_{L+1}} = \max_{k=1,\ldots,N}\left\{\frac{1}{\mu_{L_k+1}^{(k)}}\right\}, \quad \frac{1}{\nu_{K+1}} = \max\left\{1, \max_{k=1,\ldots,N}\frac{1}{\nu_{K+1}^{(k)}}\right\}.$$

4 Numerical Results

We now present a few numerical examples that support our theory. We choose $\Omega = [0,1]^2$ with Dirichlet boundary conditions on $\partial\Omega$ and a constant right hand side $f = 0.1$. The coefficient distributions are depicted in Fig. 1. Alg. A corresponds to a FETI-DP method using only vertex constraints. In Tab. 1 we vary the number of elements for each subdomain. In Tab. 2 we vary the coefficient in the channels. In both cases the coefficient distribution is symmetric with respect to the interface, and thus the extension from EVP 2 is not needed. Indeed, the results in Tab. 1 and 2 support that EVP 1 is sufficient, here. In Tab. 3 we vary the number of subdomains. In Tab. 4 we apply the adaptive method using EVP 1 for the coefficient distribution in Fig. 1 (middle) using standard ρ-scaling and deluxe scaling. The coefficient distribution is mildly unsymmetric and a good condition number is obtained using only EVP 1. This is different for Fig. 1 (right); see Tab. 5. Here, EVP 2 seems to be necessary. It interesting to note that, in Tab. 5, using deluxe scaling a relatively low condition number can be obtained using Alg. A. This is not the case in Tab. 4.

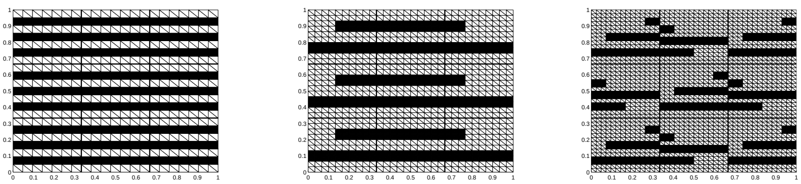

Fig. 1 Coefficient distribution for 3×3 domain decomposition: Three channels (left), two shorter and displaced channels (middle), three shorter and displaced channels (right). Black corresponds to a high coefficient $\rho = 1e+06$, white corresponds to $\rho = 1$.

	Algorithm A ($\tau_\mu = -\infty$, $\tau_\nu = -\infty$)			**Adaptive Method** EVP 1 ($\tau_\mu = 1$)			**Adaptive Method** EVP 1+2 ($\tau_\mu = 1$, $\tau_\nu = 1e-01$)		
H/h	cond	its	# primal	cond	its	# primal	cond	its	# primal
14	1.227e05	13	4	**1.0387**	**2**	24	**1.0387**	**2**	24
28	1.545e05	17	4	**1.1507**	**3**	24	**1.1507**	**3**	24
42	1.730e05	16	4	**1.2471**	**3**	24	**1.2462**	**4**	28
56	1.861e05	16	4	**1.3272**	**3**	24	**1.3272**	**3**	24
70	1.962e05	16	4	**1.3954**	**3**	24	**1.3954**	**5**	28

Table 1 Three channels for each subdomain; see Figure 1 (left). We have $\rho_1 = 1e06$ in the channel, and $\rho_2 = 1$ elsewhere. The number of additional constraints is clearly determined by the structure of the heterogeneity and independent of the mesh size. $1/H = 3$.

	Algorithm A ($\tau_\mu = -\infty$, $\tau_\nu = -\infty$)			**Adaptive Method** EVP 1 ($\tau_\mu = 1$)			**Adaptive Method** EVP 1+2 ($\tau_\mu = 1$, $\tau_\nu = 1e-01$)		
ρ_2/ρ_1	cond	# its	# primal	cond	# its	# primal	cond	# its	# primal
1e00	3.207	5	4	**1.6376**	**5**	8	**1.6376**	**5**	8
1e01	5.581	7	4	**1.5663**	**7**	8	**1.5663**	**7**	8
1e02	$1.998e+01$	9	4	**1.4599**	**7**	12	**1.4567**	**7**	16
1e03	$1.591e+02$	10	4	**1.1505**	**4**	24	**1.1505**	**4**	32
1e04	$1.550e+03$	13	4	**1.1507**	**3**	24	**1.1476**	**4**	31
1e05	$1.545e+04$	15	4	**1.1507**	**3**	24	**1.1507**	**3**	28
1e06	$1.545e+05$	17	4	**1.1507**	**3**	24	**1.1507**	**3**	24

Table 2 Three channels for each subdomain; see Figure 1 (left). Adaptive method using Eigenvalue Problem 1+2. We have ρ_2 in the channels, and $\rho_1 = 1$ elsewhere. $H/h = 28$. The number of additional constraints is bounded for increasing contrast ρ_2/ρ_1. $1/H = 3$.

	Algorithm A ($\tau_\mu = -\infty$, $\tau_\nu = -\infty$)			**Adaptive Method** EVP 1 ($\tau_\mu = 1$)			**Adaptive Method** EVP 1+2 ($\tau_\mu = 1$, $\tau_\nu = 1e-01$)		
$1/H$	cond	# its	# primal	cond	# its	# primal	cond	# its	# primal
2	1	1	1	**1.0000**	**1**	1	**1.0000**	**1**	1
3	$1.545e+05$	17	4	**1.1507**	**3**	24	**1.1507**	**3**	24
4	$2.734e+05$	26	9	**1.1507**	**3**	51	**1.1502**	**4**	59
5	$3.475e+05$	65	16	**1.1507**	**3**	88	**1.1507**	**3**	90
6	$4.078e+05$	65	25	**1.1507**	**3**	135	**1.1507**	**3**	152

Table 3 Three channels for each subdomain; see Figure 1 (left). Increasing number of subdomains and channels. We have $\rho_2 = 1e06$ in the channels, and $\rho_1 = 1$ elsewhere. $H/h = 28$.

	Algorithm A ($\tau_\mu = -\infty$, $\tau_\nu = -\infty$)					**Adaptive Method** EVP 1 ($\tau_\mu = 1$)				
	ρ-scaling		Deluxe			ρ-scaling		Deluxe		
H/h	cond	its	cond	its	# primal	cond	its	cond	its	# primal
10	6.201e4	25	6.200e4	20	4	**1.1480**	**6**	**1.1421**	**5**	24
20	7.684e4	25	7.683e4	20	4	**1.1978**	**7**	**1.1948**	**6**	24
30	8.544e4	25	8.544e4	23	4	**1.2630**	**7**	**1.2618**	**6**	24

Table 4 Adaptive method for the coefficient distribution in Figure 1 (middle). $1/H = 3$. Deluxe scaling and standard ρ-scaling is used.

	τ_μ	τ_ν	H/h	Multiplicity-scaling cond	Multiplicity-scaling # its	Deluxe-scaling cond	Deluxe-scaling # its	# primal
Alg. A	$-\infty$	$-\infty$	42	$2.492e5$	161	24.4261	17	4
EVP 1	1	$-\infty$	42	$2.496e5$	128	$9.760e4$	40	24
EVP 1+2	1	1/10	42	**1.5184**	**10**	**1.4306**	**9**	126

Table 5 Adaptive method for the heterogenous problem from the image in Figure 1 (right) with a coefficient of 10^6 (black) and 1 (white) respectively. $1/H = 3$. Either multiplicity or deluxe scaling are used.

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