# Additive Schwarz Methods for DG Discretization of Elliptic Problems with Discontinuous Coefficient

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## 1 Introduction

In this paper we consider a second order elliptic problem defined on a polygonal region $\Omega$, where the diffusion coefficient is a discontinuous function. The problem is discretized by a symmetric interior penalty discontinuous Galerkin (DG) finite element method with triangular elements and piecewise linear functions. Our goal is to design and analyze an additive Schwarz method (ASM), see the book by Toselli and Widlund [2005], for solving the resulting discrete problem with rate of convergence independent of the jumps of the coefficient. The method is two-level and without overlap of the substructures into which the original region $\Omega$ is partitioned.

Usually, two level ASMs for discretizations on fine mesh of size $h$ are being built by introducing a partitioning of the domain into subdomains of size $H > h$, where local solvers are applied in parallel. A global coarse problem is then typically based on the same partitioning. This approach has been generalized for nonoverlapping domain decomposition methods for DG discretizations by Feng and Karakashian [2001] and further extended by Antonietti and Ayuso [2007] by allowing the coarse grid with mesh size $H$ to be a refinement of the original partitioning into subdomains where the local solvers are applied.

The ASM discussed here is a generalization to non-constant diffusion coefficient and very small subdomains of methods mentioned above and of those presented in Dryja and Sarkis [2010] and Dryja et al. [2014]. Other recent works towards domain decomposition preconditioning of DG discretizations of problems with strongly varying coefficients include Ayuso de Dios et al. [2014], Brix et al. [2013] and Canuto et al. [2014]. In this paper, local solvers act on subdomains which are equal to single elements of the fine mesh. By allowing single element subdomains we substantially increase the level of paral-

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lelism of the method. Very small and cheap to solve local systems come in huge quantities, which possibly can be an advantage on new multithreaded processors. Moreover, small subdomains give more flexibility in assigning them to processors in coarse grain parallel processing. The price to be paid for this in some sense extreme parallelism is worse condition number of the preconditioned system, which is of order $O(H^2/h^2)$, where $H$ and $h$ are the coarse and the fine mesh parameters, respectively. This bound is independent of the jumps of diffusion coefficient if its variation inside substructures is bounded. Numerical experiments confirm theoretical results.

The paper is organized as follows. In Section 2, differential and discrete DG problems are formulated. In Section 3, ASM for solving the discrete problem is designed and analyzed. Numerical experiments are presented in Section 4.

In the paper, for nonnegative scalars $x, y$, we shall write $x \lesssim y$ if there exits a positive constant $C$, independent of $x$, $y$ and the mesh parameters $h, H$, and of the jumps of the diffusion coefficient $\rho$ as well, such that $x \leq Cy$. If both $x \lesssim y$ and $y \lesssim x$, we shall write $x \simeq y$.

## 2 Differential and discrete DG problems

Let us consider the following variational problem in a polygonal region $\Omega$:

Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = (f, v)_\Omega, \qquad v \in H_0^1(\Omega), \tag{1}$$

where

$$a(u, v) = \int_\Omega \rho \nabla u \cdot \nabla v dx, \qquad (f, v)_\Omega = \int_\Omega f\, v.$$

We assume that $\rho \in L^\infty(\Omega)$ and that there exist constants $\alpha_0$ and $\alpha_1$ such that $0 < \alpha_0 \leq \rho \leq \alpha_1$ in $\Omega$. In addition we assume that $f \in L^2(\Omega)$.

### *2.1 Discrete problem*

Let $\mathcal{T}_H$ be a subdivision of $\Omega$ into $N_H$ disjoint open polygonal regions $\Omega_i$, $i = 1, \ldots, N_H$, such that $\bar{\Omega} = \bigcup_{i=1,\ldots,N_H} \bar{\Omega}_i$ and that the number of neighboring regions is uniformly bounded. We set $H_i = \mathrm{diam}(\Omega_i)$ and $H = \max_{i=1,\ldots,N_H} H_i$. Further, let $\mathcal{T}_h$ denote an affine, shape regular conforming triangulation (with triangles) of $\Omega$, $\bar{\Omega} = \bigcup_{\kappa \in \mathcal{T}_h} \bar{\kappa}$, which is derived from $\mathcal{T}_H$ by some refinement procedure. Thus, each $\Omega_i$ is a union of certain elements from $\mathcal{T}_h$. The diameter of a triangle $\kappa \in \mathcal{T}_h$ will be denoted by $h_\kappa$ and the mesh parameter is $h = \max_{\kappa \in \mathcal{T}_h} h_\kappa$.

In what follows we shall assume that $\rho$ is piecewise constant (possibly with large discontinuities) on $\mathcal{T}_h$, so that $\rho_{|_\kappa}$ is constant on each $\kappa \in \mathcal{T}_h$.

By $\mathcal{E}_h^0$ we denote the set of all common (internal) faces of elements in $\mathcal{T}_h$, so that $e_{ij} \in \mathcal{E}_h$ iff $e_{ij} = \kappa_i \cap \kappa_j$ is of positive measure. We will use symbol $\mathcal{E}_h$ to denote the set of all faces, that is those either in $\mathcal{E}_h^0$ or on the boundary $\partial\Omega$; for $e \in \mathcal{E}_h$, we also set $|e| = \mathrm{diam}(e)$. We shall assume local quasi-uniformity of the grid, i.e. if $e_{ij} \in \mathcal{E}_h^0$ is such that $e_{ij} = \kappa_i \cap \kappa_j$, then $h_i \simeq h_j$.

For $p \in \{0,1\}$, we denote by $\mathcal{P}_p(\kappa)$ the set of polynomials of degree not greater than $p$ on $\bar{\kappa}$. Then we define the finite element space $V_h$, in which we will approximate (1),

$$V_h = \{v \in L^2(\Omega) : v_{|_\kappa} \in \mathcal{P}_1(\kappa), \forall \kappa \in \mathcal{T}_h\}. \tag{2}$$

Note that the traces of the functions from $V_h$ are multi-valued on the interface $\mathcal{E}_h^0$.

We define the discrete problem as the symmetric interior penalty discontinuous Galerkin method, see for example Ern et al. [2009] or Dryja [2003]:

Find $u \in V_h$ such that

$$A_h(u,\, v) = (f, v)_\Omega, \qquad v \in V_h, \tag{3}$$

where

$$A_h(u,\, v) \equiv \sum_{\kappa\in\mathcal{T}_h} (\rho\,\nabla u, \nabla v)_\kappa + \sum_{e\in\mathcal{E}_h} \langle \gamma[u], [v]\rangle_e - \sum_{e\in\mathcal{E}_h} \Big( \langle [u], \{\rho\nabla v\}_\omega\rangle_e + \langle \{\rho\nabla u\}_\omega, [v]\rangle_e \Big),$$

and $\delta > 0$ is sufficiently large to ensure positive definiteness of $A_h(\cdot,\,\cdot)$, and on $e_{ij} = \kappa_i \cap \kappa_j$

$$\gamma = \frac{\delta}{|e_{ij}|}\frac{\rho_i\rho_j}{\rho_i+\rho_j}, \qquad \{\rho\nabla u\}_\omega = \omega_j\rho_i\nabla u_i + \omega_i\rho_j\nabla u_j, \qquad [u] = u_i\, n_i + u_j\, n_j,$$

with $\omega_j = \rho_j/(\rho_i+\rho_j)$. Here, for any function $\varphi$ we use the convention that $\varphi_i$ (resp.$\varphi_j$) refers to the value of $\varphi_{|_{\kappa_i}}$ (resp. $\varphi_{|_{\kappa_j}}$) on $e_{ij}$. The unit normal vector pointing outward $\kappa_i$ is denoted by $n_i$. On the boundary of $\Omega$, we set $\{\rho\nabla u\}_\omega = \rho\nabla u$ and $[u] = u\, n$.

Let us introduce a simplified form

$$D_h(u,\, v) = \sum_{\kappa\in\mathcal{T}_h} (\rho\,\nabla u, \nabla v)_\kappa + \sum_{e\in\mathcal{E}_h} \langle \gamma[u], [v]\rangle_e.$$

Then it is well known that $D_h(\cdot,\,\cdot)$ is spectrally equivalent to $A_h(\cdot,\,\cdot)$, i.e.

$$A_h(u,\, u) \simeq D_h(u,\, u) \quad \forall u \in V_h.$$

# 3 Additive Schwarz methods

## *3.1 Additive Schwarz method, version I*

Let $N_h$ be the number of elements in $\mathcal{T}_h$. We decompose $V_h$ as follows:

$$V_h = V_0 + \sum_{i=1}^{N_h} V_i$$

where

$$V_0 = \{v \in V_h : v_{|\kappa} \in \mathcal{P}_0(\kappa) \text{ on } \kappa \in \mathcal{T}_h\}$$

and

$$V_i = \{v \in V_h : v_{|\kappa} = 0 \text{ for all } \kappa \in \mathcal{T}_h \text{ such that } \kappa \neq \kappa_i\}. \tag{4}$$

Using the above decomposition we define local operators $T_i : V_h \to V_i$, $i = 1, \ldots, N_h$, with inexact solver

$$D_h(T_i u,\, v) = A_h(u,\, v) \qquad \forall v \in V_i,$$

so that we solve for $u_i = T_i u$ defined on $\kappa_i \in \mathcal{T}_h$ such that

$$(\rho_i \nabla u_i, \nabla v_i)_{\kappa_i} + \sum_{e \subset \partial \kappa_i} \int_e \gamma u_i v_i = A_h(u,\, v_i) \qquad \forall v_i \in V_i,$$

and set $(T_i u)_{|\kappa_j} = 0$ for $j \neq i$. The coarse solve operator is $T_0 : V_h \to V_0$ defined analogously as

$$D_h(T_0 u,\, v_0) = A_h(u,\, v_0) \qquad \forall v_0 \in V_0.$$

Note that on $V_0$, the approximate form $D_h(\cdot,\, \cdot)$ coincides with $A_h(\cdot,\, \cdot)$ and simplifies to

$$D_h(u_0,\, v_0) = \sum_{e \in \mathcal{E}_h} \langle \gamma [u_0], [v_0] \rangle_e \qquad \forall u_0, v_0 \in V_0.$$

**Theorem 1.** *Let* $T = T_0 + \sum_{i=1}^{N_h} T_i$. *Then*

$$A_h(Tu,\, u) \simeq A_h(u,\, u) \qquad \forall u \in V_h.$$

This means that the condition number of the resulting system is uniformly bounded independently of $h$, $H$ and $\rho$. However, the method is not robust, because $\dim V_0 = N_h$ is very large. The proof of Theorem 1 will appear elsewhere.

### 3.2 Additive Schwarz method, version II

Since version I described above suffers from the very large size of the coarse space $V_0$ (based on edges of the fine triangulation $\mathcal{T}_h$, with averaged coefficients on them), here we consider a coarse space which is set up on the edges of $\mathcal{T}_H$, the coarse partition. In this way the method regains high level of parallelism, as the coarse problem now can in principle be solved on a single processor. Note that this approach is similar to that of Feng and Karakashian [2001].

We decompose $V_h$ as follows:

$$V_h = \bar{V}_0 + \sum_{i=1}^{N_h} V_i$$

where

$$\bar{V}_0 = \{v \in V_h : v_{|\Omega_i} \in \mathcal{P}_0(\Omega_i),\, i = 1, \ldots, N_H\}$$

and the local spaces $V_i$, $i = 1, \ldots, N_h$, remain as defined in (4). Now, the coarse operator $\bar{T}_0 : V_h \to \bar{V}_0$ is defined such that $\bar{T}_0 u = \bar{u}_0$ where

$$D_h(\bar{u}_0, v) = A_h(u, v) \qquad \forall v \in \bar{V}_0.$$

In order to formulate the condition number result, we shall assume uniformly bounded level of variation of the coefficient within subdomain: there exist positive constants $c$ and $C$ such that

$$c\,\bar{\rho}_i \le \rho_{|\Omega_i} \le C\,\bar{\rho}_i, \qquad i = 1, \ldots, N_H, \tag{5}$$

where

$$\bar{\rho}_i := \frac{1}{|\Omega_i|}\int_{\Omega_i} \rho.$$

**Theorem 2.** *Let $H_i = \mathrm{diam}(\Omega_i)$ and let $T = \bar{T}_0 + \sum_{i=1}^{N_h} T_i$. Under the above assumptions,*

$$\beta^{-1} A_h(u, u) \lesssim A_h(Tu, u) \lesssim A_h(u, u)$$

*where $\beta = \max_{i=1,\ldots,N_H}\{\frac{H_i^2}{\min_{\kappa\in\mathcal{T}_h,\kappa\subset\Omega_i} h_\kappa^2}\}$.*

*Remark 1.* Detailed proofs of Theorems 1 and 2 will be provided elsewhere due to the page limits. Here we only briefly sketch the idea of the proof of Theorem 2. We follow the abstract theory from the book by Toselli and Widlund [2005]. Since the local stability and strengthened Schwarz inequality assumptions are straightforward, it remains to prove the existence of stable decomposition for any $v \in V_h$. To this end, we make use of the coarse space which makes it possible to extract subdomain average from $v$ and deal only with functions with zero average on each subdomain. Applying Friedrichs inequality for discontinuous functions, Brenner [2003], and making use of (5) we prove the stability constant of the decomposition is of order $\beta$.

## 4 Numerical experiments

Let us choose the unit square as the domain $\Omega$ and for some prescribed integer $M$ divide it into $N_H = 2^M \times 2^M$ smaller squares $\Omega_i$ $(i = 1, \ldots, N_H)$ of equal size. This decomposition of $\Omega$ is then further refined into a uniform triangulation $\mathcal{T}_h$ based on a square $2^m \times 2^m$ grid $(m \geq M)$ with each square split into two triangles of identical shape. Hence, the fine mesh parameter is $h = 2^{-m}$, while the coarse grid parameter is $H = 2^{-M}$. We discretize the problem (1) on the fine triangulation using the method (3) with $\delta = 7$.

In the following tables we report the number of Preconditioned Conjugate Gradient iterations for operator $T$ (defined in Section 3.2) which are required to reduce the initial Euclidean norm of the residual by a factor of $10^6$ and (in parentheses) the condition number estimate for $T$. We consider two sets of test problems: with either continuous or discontinuous coefficient $\rho$. We always choose a random vector for the right hand side and a zero as the initial guess.

### 4.1 ASM version II vs. "standard" ASM

First let us consider the performance of ASM version II against a more "standard" ASM, see [Dryja et al., 2014, Section 3.3], where the local solve is restricted not to a single element of size $h$, but to a single subdomain $\Omega_i$ of size $H$. For the diffusion coefficient we take a continuous function, $\rho(x) = x_1^2 + x_2^2 + 1$. As it turns out from Tables 1 and 2, the condition number of the method considered in Section 3.2 indeed shows an $O((H/h)^2)$ behavior, as predicted by Theorem 2, while methods which use local solves on subdomains of diameter at least $H$ (e.g. Dryja et al. [2014] or, similarly, Feng and Karakashian [2001], Antonietti and Ayuso [2007]) exhibit more favorable $O(H/h)$ dependence.

### 4.2 Discontinuous coefficient

Next, let us consider $\rho$ with discontinuities aligned with an auxiliary partitioning of $\Omega$ into $4 \times 4$ squares. Precisely, we introduce a red–black checkerboard coloring of this partitioning and set $\rho = 1$ in red regions, and the value of $\rho_1$ reported in Table 3 in black ones. In this way, our fine and coarse triangula-

| Fine ($m$) $\rightarrow$ <br> $\downarrow$ Coarse ($M$) | 4 | 5 | 6 | 7 |
|---|---|---|---|---|
| 4 | 29 (22) | 39 (40) | 59 ($1.1 \cdot 10^2$) | 96 ($3.8 \cdot 10^2$) |
| 5 | | 30 (23) | 39 (40) | 59 ($1.1 \cdot 10^2$) |
| 6 | | | 30 (23) | 38 (40) |
| 7 | | | | 30 (23) |

**Table 1** Dependence of the number of iterations and the condition number (in parentheses) on $H = 2^{-M}$ and $h = 2^{-m}$ for the method of Section 3.2.

| Fine ($m$) $\rightarrow$ <br> $\downarrow$ Coarse ($M$) | 4 | 5 | 6 | 7 |
|---|---|---|---|---|
| 4 | 27 (20) | 35 (34) | 46 (67) | 62 ($1.3 \cdot 10^2$ ) |
| 5 | | 28 (20) | 35 (34) | 46 (67) |
| 6 | | | 28 (20) | 35 (34) |
| 7 | | | | 28 (20) |

**Table 2** Dependence of the number of iterations and the condition number (in parentheses) on $H = 2^{-M}$ and $h = 2^{-m}$ for the method of [Dryja et al., 2014, Section 3.3].

tions, with $m = 7$ and $M = 4$, will always be aligned with the discontinuities. Table 3 shows the independence of the condition number on $\rho_1$ in this case.

Finally, we consider elementwise discontinuous coefficient, with $\rho = 1$ on odd and $\rho = \rho_1$ on even-numbered triangles. Table 4 shows that in this case the coarse space fails (a dash means the method did not converge in 600 iterations). This confirms the importance of the assumption of mild variation of the coefficient (5).

| $\rho_1$ | $10^0$ | $10^{-2}$ | $10^{-4}$ | $10^{-6}$ |
|---|---|---|---|---|
| iter (cond) | 134 ($3.8 \cdot 10^2$) | 141 ($3.7 \cdot 10^2$) | 161 ($3.7 \cdot 10^2$) | 179 ($3.8 \cdot 10^2$) |

**Table 3** Dependence of the number of iterations and the condition number (in parentheses) on the discontinuity when the coefficient is constant inside subdomains. Red–black $4 \times 4$ distribution of $\rho$, aligned with domain decomposition. Fixed $H/h = 8$.

| $\rho_1$ | $10^0$ | $10^{-2}$ | $10^{-4}$ | $10^{-6}$ |
|---|---|---|---|---|
| iter (cond) | 134 ($3.8 \cdot 10^2$) | 435 ($3.8 \cdot 10^3$) | – ($3.1 \cdot 10^5$) | – ($2.5 \cdot 10^7$) |

**Table 4** Dependence of the number of iterations and the condition number (in parentheses) on the discontinuity when the coefficient elementwise discontinuous. Fixed $H/h = 8$.

## 5 Conclusions

A nonoverlapping ASM for symmetric interior penalty DG discretization of 2nd order elliptic PDE with discontinuous coefficient has been presented, in which a very large number of very small local problems is solved in parallel, together with one coarse problem of moderate size. Under mild assumptions, the condition number of the resulting system is $O((H/h)^2)$, independently of the jumps of the coefficient.

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