Schwarz methods for a Crouzeix-Raviart finite volume discretization of elliptic problems

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1 Introduction

In this paper, we present two variants of the Additive Schwarz Method (ASM) for a Crouzeix-Raviart finite volume (CRFV) discretization of the second order elliptic problem with discontinuous coefficients, where the discontinuities are only across subdomain boundaries. The resulting system, which is nonsymmetric, is solved using the preconditioned GMRES iteration, where in one variant of the ASM the preconditioner is symmetric while in the other variant it is nonsymmetric. The proposed methods are almost optimal, in the sense that the convergence of the GMRES iteration, in the both cases, depend only poly-logarithmically on the mesh parameters.

In the CRFV method, the equations are discretized on a mesh which is dual to a primal mesh where the nonconforming Crouzeix-Raviart finite element space is defined, it is the space in which we seek for an approximation of the solution, cf. Chatzipantelidis [1999].

There are many results concerning Additive Schwarz Methods (ASM) for solving symmetric systems, those arising from the finite element discretization of second order elliptic problems, cf. e.g. Toselli and Widlund [2005], but only a few papers that consider the FV discretization using the standard finite element space, cf. Chou and Huang [2003], Zhang [2006]. There is also a number of results focused on iterative methods for the CR finite element for second order problems; cf. Brenner [1996], Marcinkowski and Rahman [2008], Sarkis [1997].

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The purpose of this paper is to construct two parallel algorithms based on the edge based discrete space decomposition in the abstract Schwarz scheme. The algorithms are very similar in application.

We present almost optimal estimates for the convergence of the GMRES iteration applied to the preconditioned system, showing that the minimum eigenvalue of the preconditioned operator in the estimate, grows like $(1 + \log(H/h))^{-2}$, where H is the maximal diameter of the subdomains and h is the fine mesh size parameter. Some preliminary results of numerical tests are also presented.

2 Discrete problem

In this section we present the Crouzeix-Raviart finite element (CRFE) and finite volume (CRFV) discretizations of a model second order elliptic problem with discontinuous coefficients across prescribed substructures boundaries.

Let Ω be a polygonal domain in the plane. We assume that there exists a partition of Ω into disjoint polygonal subdomains Ω_k such that $\overline{\Omega} = \bigcup_{k=1}^{N} \overline{\Omega}_k$ with $\overline{\Omega}_k \cap \overline{\Omega}_l$ being an empty set, an edge or a vertex (crosspoint). We also assume that these subdomains form a coarse triangulation of the domain which is shape regular as in Brenner and Sung [1999]. We introduce a global interface $\Gamma = \bigcup_i \overline{\partial\Omega_i \setminus \partial\Omega}$ which plays an important role in our study.

Our model differential problem is to find u^* such that

$$\begin{aligned} -\nabla A(x)\nabla u^*(x) &= f(x) \qquad x \in \Omega \\ u^*(s) &= 0 \qquad s \in \partial\Omega, \end{aligned} \tag{1}$$

where $A(x)$ is the symmetric coefficients matrix.

The standard variational (weak) formulation is to find $u^* \in H_0^1(\Omega)$ such that $a(u^*, v) = \int_\Omega fv\,dx$ for all $v \in H_0^1(\Omega)$, where $f \in L^2(\Omega)$, and $a(u, v) = \sum_{k=1}^{N} \int_{\Omega_k} \nabla u^T A(x) \nabla v\,dx$. We assume that the restriction of the symmetric coefficients matrices to Ω_k: $A_k = A_{|\Omega_k}$ is in $W^{1,\infty}(\Omega_k)$ and bounded and positive definite, i.e.

$$\exists \alpha_k > 0\ \forall x \in \Omega_k\ \forall \xi \in \mathbb{R}^2 \quad \xi^T A(x)\xi \geq \alpha_k |\xi|^2, \tag{2}$$

$$\exists M_k > 0\ \forall x \in \Omega_k\ \forall \xi, \mu \in \mathbb{R}^2 \quad \mu^T A(x)\xi \leq M_k |\mu||\xi|. \tag{3}$$

Here $|\xi| = \sqrt{\xi^T \xi}$. We can always scale the matrix functions A in such a way that all $\alpha_k \geq 1$. Thus we assume that the restriction of the coefficient matrices to Ω_k: $A_k = A_{|\Omega_k}$ is in $W^{1,\infty}(\Omega_k)$ with the following bounds: $\|A_k\|_{W^{1,\infty}(\Omega_k)} \leq C$, and $M_k \leq C_e \alpha_k$, i.e. we assume that the coefficient matrix locally is smooth, isotropic and not too much varying.

We assume that there exists a sequence of quasiuniform triangulations: $T_h = T_h(\Omega) = \{\tau\}$, of Ω such that any element τ of T_h is contained in only one subdomain, as a consequence any subdomain Ω_k inherits a sequence of local triangulations: $T_h(\Omega_k) = \{\tau\}_{\tau\subset\Omega_k,\tau\in T_h}$.

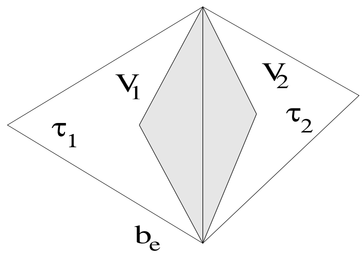


Fig. 1: The control volume b_e for an edge e which is the common edge to the triangles τ_1 and τ_2.

Let $h = \max_{\tau\in T_h(\Omega)} \mathrm{diam}(\tau)$ be the mesh size parameter of the triangulation. We introduce the following sets of Crouzeix-Raviart (CR) nodal points or nodes: let $\Omega_h^{CR}, \partial\Omega_h^{CR}, \Omega_{k,h}^{CR}, \partial\Omega_{k,h}^{CR}$, Γ_h^{CR}, and $\Gamma_{kl,h}^{CR}$ be the midpoints of edges of elements in T_h which are on $\Omega, \partial\Omega, \Omega_k, \partial\Omega_k$, Γ, and Γ_{kl}, respectively. Here Γ_{kl} is an interface, an open edge, which is shared by the two subdomains, Ω_k and Ω_l. Note that $\Gamma_h^{CR} = \bigcup_{\Gamma_{kl}\subset\Gamma} \Gamma_{kl,h}^{CR}$. Now we define a

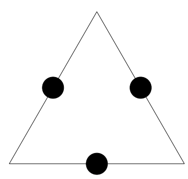


Fig. 2: Edge midpoint corresponding to the degrees of freedom of the nonconforming Crouzeix-Raviart element.

dual triangulation T_h^* to the initial one. For an edge e of an element not on $\partial\Omega$ i.e. being the common edge of two elements τ_1 and τ_2 i.e. $e = \partial\tau_1 \cap \partial\tau_2$ we introduce two triangles: $V_k \subset \tau_k$ obtained by connecting the ends of e to the centroid (barycenter) of τ_k for $k = 1, 2$. Then, let the control volume $b_e = V_1 \cup e \cup V_2$, cf. Figure 1. For an edge of an element τ contained in $\partial\Omega$ let the control volume be the triangle V obtained analogously i.e. by connecting the ends of e with the centroid of τ. Then let $T_h^* = \{b_e\}_{e\in E_h}$, where E_h is the set of all edges of elements in T_h.

Next we introduce two discrete spaces contained in $L^2(\Omega)$:

$$V_h := \{v \in L^2(\Omega) : v_{|\tau} \in P_1, \quad \tau \in T_h \quad v(m) = 0 \quad m \in \partial\Omega_h^{CR}\},$$
$$V_h^* := \{v \in L^2(\Omega) : v_{|b_e} \in P_0, \quad b_e \in T_h^* \quad v(m) = 0 \quad m \in \partial\Omega_h^{CR}\}.$$

The first space is the classical nonconforming Crouzeix-Raviart finite element space, cf. Figure 2, and the second space is the space of piecewise constant functions which are zero on the boundary of the domain.

Let $\{\phi_m\}_{m\in\Omega_h^{CR}}$ be the standard CR nodal basis of V^h and $\{\psi_m\}_{m\in\Omega_h^{CR}}$ be the standard basis of V_h^* consisting of characteristic functions of the control volumes.

We also introduce two interpolation operators, I_h and I_h^*, defined for any function that has properly defined and unique values at each midpoint $m \in \Omega_h^{CR}$:

$$I_h(u) = \sum_{m\in\Omega_h^{CR}} u(m)\phi_m, \qquad I_h^*(u) = \sum_{m\in\Omega_h^{CR}} u(m)\psi_m.$$

Note that $I_h I_h^* u = u$ for any $u \in V_h$ and $I_h^* I_h u = u$ for any $u \in V_h^*$. We also define a nonsymmetric in general bilinear form $a_h : V_h \times V_h^* \to \mathbb{R}$:

$$a_h^{CRFV}(u,v) = -\sum_{e\in E_h^{in}} v(m_e)\int_{\partial b_e} \mathbf{n}^T A(s)\nabla u \, ds, \tag{4}$$

where $\mathbf{n}$ is a normal unit vector outer to ∂b_e, m_e is the median (midpoint) of the edge e and $E_h^{in} \subset E_h$ is the set of all interior edges, i.e. those which are not on $\partial\Omega$.

Then our discrete CRFV problem is to find $u^* \in V_h$ such that:

$$a_h^{FV}(u^*,v) = f(I_h^* v) \qquad \forall v \in V_h \tag{5}$$

for $a_h^{FV}(u,v) := a_h^{CRFV}(u, I_h^* v)$. In general the problem is nonsymmetric unless the coefficients matrix is a piecewise constant matrix over T_h. One can prove that there exists $h_0 > 0$ such that for all $h \le h_0$ the form $a_h^{FV}(u,v)$ is positive definite over V_h. Thus this problem has a unique solution. Some error estimates are also proven, cf. Loneland et al. [2014a] or Chatzipantelidis [1999] in the case of the smooth coefficients.

3 Additive Schwarz method

In this section, we construct our ASM based on the abstract framework for additive Schwarz methods, see Toselli and Widlund [2005].

First we introduce the local spaces being the restriction of V_h to $\overline{\Omega}_k$ and its subspace with discrete CR zero boundary conditions:

$$W_k := \{v_{|\overline{\Omega}_k} : v \in V_h\}, \qquad W_{k,0} := \{v \in W_k : w(m) = 0 \quad m \in \partial\Omega_{k,h}^{CR}\} \subset W_k.$$

Let $P_k : W_k \to W_{k,0}$ be the orthogonal projection onto $W_{k,0}$ in terms of the local bilinear form: $a_{k,h}^{FE}(u,v) = \sum_{\tau\in T_h(\Omega_k)} \int_\tau \nabla u^T A \nabla v \, dx$, i.e.

$$a_{k,h}^{FE}(P_k u, v) = a_{k,h}^{FE}(u, v) \quad \forall v \in W_{k,0}.$$

Then $H_k u = u - P_k u$ will be the discrete harmonic part of $u \in W_k$. If $u = H_k u$ then we say that $u \in W_k$ is discrete harmonic. A function $u \in V_h$ is discrete harmonic if its all restrictions to subdomains are discrete harmonic i.e. $u_{|\Omega_k} = H_k u_{|\Omega_k}$ for $k = 1, \ldots, N$. We also define an edge function $\theta_{\Gamma_{kl}} \in V_h$ as a discrete harmonic function such that it is equal to one at CR nodes interior to Γ_{kl} and zero at all other CR nodes on the interface.

We now define the decomposition of V_h. Let $V_0 = \mathrm{Span}(\theta_{\Gamma_{kl}})_{\Gamma_{kl} \subset \Gamma}$ be the coarse space, V_{kl} be the edge space associated with the interface Γ_{kl} formed by discrete harmonic functions that are zero at each $x \in \Gamma_h^{CR} \setminus \Gamma_{kl,h}^{CR}$. Finally let V_k be the space $W_{k,0}$ extended by zero to all remaining subdomains. Thus we have the following decomposition: $V_h = V_0 + \sum_{\Gamma_{kl} \subset \Gamma} V_{kl} + \sum_{k=1}^{N} V_k$. Note that this is a direct sum and that the subspace $V_0 + \sum_{\Gamma_{kl} \subset \Gamma} V_{kl}$ is $a_h^{FE}(u, v) = \sum_k a_{k,h}^{FE}(u, v)$ orthogonal to $\sum_{k=1}^{N} V_k$. Now we define the first type of projection like operators: the coarse and the local operators, $T_k^{sym} : V_h \to V_k$, as

$$a_h^{FE}(T_k^{sym} u, v) = a_h^{FV}(u, v) \qquad \forall v \in V_k, \quad k = 0, 1, \ldots, N,$$

the edge related operators, $T_{kl}^{sym} : V_h \to V_{kl}$, as

$$a_h^{FE}(T_{kl}^{sym} u, v) = a_h^{FV}(u, v) \qquad \forall v \in V_{kl}, \quad \Gamma_{kl} \subset \Gamma.$$

Note that $T_k^{sym} u$ can be computed by solving a symmetric local discrete CRFE Dirichlet problem and then extended by zero to the other subdomains.

The second type of operators is based solely on the nonsymmetric bilinear form $a_h^{FV}(u, v)$. We define the coarse and the local operators, $T_k^{nsym} : V_h \to V_k$, as

$$a_h^{FV}(T_k^{nsym} u, v) = a_h^{FV}(u, v) \qquad \forall v \in V_k, \quad k = 0, 1, \ldots, N,$$

and the edge related operators, $T_{kl}^{nsym} : V_h \to V_{kl}$, as

$$a_h^{FV}(T_{kl}^{nsym} u, v) = a_h^{FV}(u, v) \qquad \forall v \in V_{kl}, \quad \Gamma_{kl} \subset \Gamma.$$

We define the two ASM operators as follows:

$$T^{type} := \sum_{\Gamma_{kl} \subset \Gamma} T_{kl}^{type} + \sum_{k=0}^{N} T_k^{type},$$

where the super-index *type* is either *sym* or *nsym*. We can replace our discrete CRFV equation (5) by the following system:

$$T^{type} u_h^* = g^{type}, \tag{6}$$

where $g^{type} = g_0^{type} + \sum_{\Gamma_{kl} \subset \Gamma} g_{kl}^{type} + \sum_{k=1}^{N} g_k^{type}$, $g_0^{type} = T_0^{type} u_h^*$, $g_{kl}^{type} = T_{kl}^{type} u_h^*$ $g_k^{type} = T_k^{type} u_h^*$, and $type \in \{sym, nsym\}$.

We apply the GMRES method in the inner product $a_h^{FE}(u,v)$, to the new system (6), and get the the following estimate (see Eisenstat et al. [1983] for the case of standard l_2 inner product, and Cai and Widlund [1992] for the general case):

$$\|g - T^{type} u_j\|_a \leq \left(1 - \frac{\alpha_{min}^2}{\alpha_{max}^2}\right)^{j/2} \|g - T^{type} u_0\|_a. \tag{7}$$

where $\alpha_{min} = \min_{u \in V_h \setminus \{0\}} \frac{a_h^{FE}(T^{type}u,u)}{\|u\|_a^2}$ and $\alpha_{max} = \max_{u \in V_h \setminus \{0\}} \frac{\|T^{type}u\|_a}{\|u\|_a}$, $\|v\|_a := \sqrt{a_h^{FE}(v,v)}$, and T^{type} is either T^{sym} or T^{nsym}.

Next, we present the main theoretical result of this paper, namely an estimate of the convergence rate of the GMRES method, which is the same for both preconditioned systems (6). The proof of this theorem is an extension of the proof in Marcinkowski et al. [2014] to the case of CRFV and will be published in Loneland et al. [2014b].

Theorem 1. *There exists $h_0 > 0$ such that for all $h < h_0$ and $u \in V_h$*

$$\|T^{type}u\|_a \leq C\|u\|_a, \qquad a^{FE}(T^{type}u, u) \geq c\left(1 + \log\left(\frac{H}{h}\right)\right)^{-2} \|u\|_a^2$$

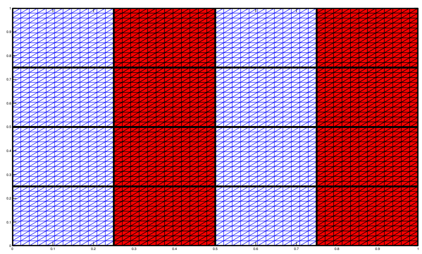
where T^{type} is either T^{sym} or T^{nsym}, C and c are positive constants independent of h, $H = \max_{k=1,\ldots,N} diam(\Omega_k)$, and the magnitudes of α_k and M_k, but they depend on $\frac{M_k}{\alpha_k} \leq C_e$, cf. (2)-(3).

This theorem together with (7) gives as an estimate of the rate of convergence of the GMRES iteration for the two cases showing that the rates slow down very slowly - poly-logarithmically.

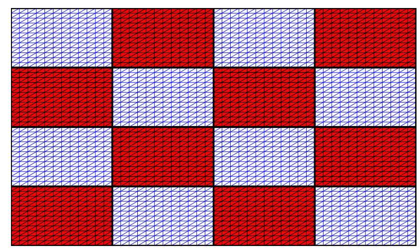
4 Numerical results

In this section, we present some preliminary numerical results for the proposed method. All experiments are done for the symmetric preconditioner, that is for T^{sym}, but we expect a similar performance for T^{nsym}. In all cases Ω is a unit square domain. The coefficient A is equal to $2+\sin(100\pi x)\sin(100\pi y)$, except for areas (subdomains) marked with red where A equals $\alpha_1(2 + \sin(100\pi x)\sin(100\pi y))$ with α_1 being a parameter (cf. Figure 3 and Table 1). The right hand side is chosen as $f = 1$. The numerical solution is found by using the generalized minimal residual method (GMRES).

For the paper, we consider two test problems as shown in Figure 3. We run the method until the l_2 norm of the residual is reduced by a factor of 10^6,

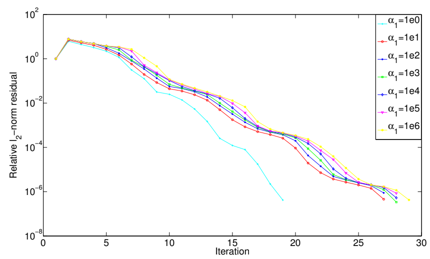


(a) Problem 1.

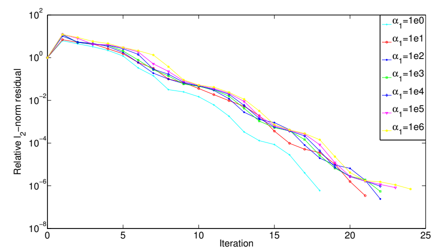


(b) Problem 2.

Fig. 3: Test problems 1 and 2. Regions (subdomains) marked with red are where A depends on α_1. Fine mesh consists of 48×48 rectangular blocks, while coarse mesh consists of 4×4 rectangular subdomains



(a) Problem 1.



(b) Problem 2.

Fig. 4: Relative residual norms for GMRES minimizing the A-norm for different values of α_1.

that is when $\|r_i\|_2/\|r_0\|_2 \leq 10^{-6}$. Number of iterations, for the problems under consideration, for different values of α_1, are shown in Table 1. The

α_1	$1e0$	$1e1$	$1e2$	$1e3$	$1e4$	$1e5$	$1e6$
Problem 1.	18	26	26	27	27	27	28
Problem 2.	18	21	22	22	22	23	24

Table 1: Number of GMRES iterations until convergence for the solution of (5), with different values of α_1 describing the coefficient A in the red regions, cf. figures 3a and 3b.

results show that the methods are robust for the present distribution of the coefficients, and supports our theory.

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