A BDDC Preconditioner for problems posed in $\boldsymbol{H}$ (div) with Deluxe Scaling

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Summary. The purpose of this paper is to introduce a BDDC method for vector field problems discretized with the lowest order Raviart-Thomas finite elements. Our method is based on a new type of weighted average, a deluxe scaling, developed to deal with more than one variable coefficient. Numerical experiments show that the deluxe scaling is robust and more powerful than traditional methods.

Key words: BDDC method, Raviart–Thomas finite elements, deluxe scaling

1 Introduction

Let Ω be a bounded polyhedral domain in $\mathbb{R}^3$. We will work with the Hilbert space $H(\mathrm{div};\Omega)$, the subspace of vector valued functions $\boldsymbol{u} \in (L^2(\Omega))^3$ with $\mathrm{div}\,\boldsymbol{u} \in L^2(\Omega)$. The space $H_0(\mathrm{div};\Omega)$ is the subspace of $H(\mathrm{div};\Omega)$ with a vanishing normal component on the boundary $\partial\Omega$.

We will consider the following problem: Find $\boldsymbol{u} \in H_0(\mathrm{div};\Omega)$, such that

$$a(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega (\alpha\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v} + \beta\,\boldsymbol{u}\cdot\boldsymbol{v})dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx, \quad \boldsymbol{v} \in H_0(\mathrm{div};\Omega). \tag{1}$$

We will assume that the coefficient $\alpha \in L^\infty(\Omega)$ is nonnegative, that $\beta \in L^\infty(\Omega)$ is strictly positive, and that the right hand side $\boldsymbol{f} \in (L^2(\Omega))^3$.

The model problem (1) is equivalent to the variational forms of mixed or first order system least-squares formulations as in [3]. There are also other applications of $H(\mathrm{div})$, e.g., in the sequential regularization method for the Navier-Stokes equations; see [12].

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The main purpose of this paper is to construct a BDDC preconditioner for vector field problems discretized with Raviart-Thomas finite elements. Iterative substructuring methods for such problems were first considered in [25]. Other iterative substructuring methods for these types of problems have been developed in [19]. Overlapping Schwarz methods have also been introduced; see [1, 14, 15, 16]. Other methods such as multigrid methods have been applied successfully in [2, 8, 10]. We also remark that domain decomposition methods for $H(\mathbf{curl})$ problems were introduced in [5, 7, 9, 20, 21, 22]. BDDC methods for other problems related to $H\,(\mathrm{div})$ can be found in [18, 23, 24].

In the construction of a BDDC preconditioners, a set of primal constraints and a weighted averaging technique have to be chosen and these choices will very directly affect the performance. Effective primal constraints are very simple for the Raviart-Thomas elements; we choose the average value of the normal component over the subdomain faces as primal variables. However, the choice of averaging is much more intricate. We will use a new type of weighted averaging technique introduced in [6] for three dimensional $H(\mathbf{curl})$ problems.

2 Preliminary

We first introduce a triangulation $\mathcal{T}_h$ of Ω of hexahedral elements. We will consider the lowest order Raviart-Thomas elements on mesh $\mathcal{T}_h$. We then decompose the domain Ω into N nonoverlapping subdomains Ω_i. We also define the global interface Γ and the local interface Γ_i by

$$\Gamma := \left(\bigcup_{i=1}^{N} \partial\Omega_i\right) \backslash \partial\Omega, \qquad \Gamma_i := \Gamma \cap \partial\Omega_i,$$

respectively.

Let $W^{(i)}$ be the space of the finite elements on Ω_i with a zero normal component on $\partial\Omega \cap \partial\Omega_i$. We decompose $W^{(i)}$ into two subspaces, $W_\Gamma^{(i)}$ and $W_I^{(i)}$. Here, $W_\Gamma^{(i)}$ is the interface space which consists of degrees of freedom corresponding to Γ_i and $W_I^{(i)}$ is the space of discrete unknowns of the interior of Ω_i. The space $W_\Gamma^{(i)}$ can be decomposed into a primal space $W_\Pi^{(i)}$ and a dual space $W_\Delta^{(i)}$. In general, the functions in $W_\Gamma := \prod_{i=1}^{N} W_\Gamma^{(i)}$ have discontinuous normal components across the interface while those of the finite element solutions are continuous. We denote the continuous subspace by $\widehat{W}_\Gamma\ (\subset W_\Gamma)$. We next define operators $R_\Gamma^{(i)} : \widehat{W}_\Gamma \to W_\Gamma^{(i)}$ which extract the degrees of freedom associated with Γ_i. Similarly, we define a space $\widetilde{W}_\Gamma$, for which all the primal constraints are enforced. We next define local operators $\overline{R}_\Gamma^{(i)}$: $\widetilde{W}_\Gamma \to W_\Gamma^{(i)}$ which extract the degrees of freedom corresponding to Γ_i. We also define the global operator $\widetilde{R}_\Gamma : \widehat{W}_\Gamma \to \widetilde{W}_\Gamma$. Finally, we introduce the scaled operator $\widetilde{R}_{D,\Gamma} : \widehat{W}_\Gamma \to \widetilde{W}_\Gamma$ obtained by pre-multiplying the entries of $\widetilde{R}_\Gamma$ associated with $W_\Delta^{(i)}$ by a scaling matrix $D^{(i)}$. The discrete form of our problem is written in terms of local stiffness matrices as

$$\begin{bmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Gamma \end{bmatrix} = \sum_{i=1}^N \begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_I^{(i)} \\ \boldsymbol{u}_\Gamma^{(i)} \end{bmatrix} = \sum_{i=1}^N \begin{bmatrix} \boldsymbol{f}_I^{(i)} \\ \boldsymbol{f}_\Gamma^{(i)} \end{bmatrix}. \tag{2}$$

Before we introduce the BDDC algorithm, we eliminate all interior unknowns locally. After this step, we obtain these local Schur complements:

$$S_\Gamma^{(i)} := A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)-1} A_{I\Gamma}^{(i)}.$$

By using the local Schur complements, we can build a reduced interface problem. The global problem is given by

$$\widehat{S}_\Gamma \boldsymbol{u}_\Gamma = \boldsymbol{g}_\Gamma, \tag{3}$$

where

$$\widehat{S}_\Gamma := \sum_{i=1}^N R_\Gamma^{(i)T} S_\Gamma^{(i)} R_\Gamma^{(i)} \quad \text{and} \quad \boldsymbol{g}_\Gamma := \sum_{i=1}^N R_\Gamma^{(i)T} \left(\boldsymbol{f}_\Gamma - A_{\Gamma I}^{(i)} A_{II}^{(i)-1} \boldsymbol{f}_I^{(i)} \right).$$

Moreover, we have the partially assembled Schur complement $\widetilde{S}_\Gamma$:

$$\widetilde{S}_\Gamma = \sum_{i=1}^N {\overline{R}_\Gamma^{(i)}}^T S_\Gamma^{(i)} \overline{R}_\Gamma^{(i)}. \tag{4}$$

3 BDDC

We consider a BDDC preconditioner to solve the interface problem (3). We can find background information and a description of the algorithm in [4, 11]. The BDDC preconditioner has the following form:

$$M^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}. \tag{5}$$

It is convenient to make a change of variables by introducing a basis for the primal degrees of freedom and a complementary basis for the dual subspace $W_\Delta^{(i)}$. Here we can follow the recipes of [11, subsection 3.3] closely. For our problem, the only primal variables will be the averages of the normal component over the subdomain faces.

In order to specify the algorithm completely, we need to define the weighted averaging operator $D^{(i)}$. Conventional weighted averaging techniques, known as stiffness and ρ scalings, are described in [4, 13]. However, these methods are designed for constant coefficients or for one variable coefficient. For more than one variable coefficient, we need a different approach and we will use the new weighted averaging technique introduced in [6] for $H(\mathbf{curl})$ problems.

Let F_{ij} be the common face of two adjacent subdomains Ω_i and Ω_j. Moreover, let $R^{(i)}_{F_{ij}}$ be the restriction operator which extracts the degrees of freedom on F_{ij} from those on Γ_i. Then, the two Schur complements associated with F_{ij} are given by $S^{(i)}_{F_{ij}} = R^{(i)}_{F_{ij}} S^{(i)}_{\Gamma} R^{(i)}_{F_{ij}}{}^T$ and $S^{(j)}_{F_{ij}} = R^{(j)}_{F_{ij}} S^{(j)}_{\Gamma} R^{(j)}_{F_{ij}}{}^T$. We will use the scaling matrices $D^{(i)}_j := \left(S^{(i)}_{F_{ij}} + S^{(j)}_{F_{ij}}\right)^{-1} S^{(i)}_{F_{ij}}$. We note that we can apply the operator $\left(S^{(i)}_{F_{ij}} + S^{(j)}_{F_{ij}}\right)^{-1}$ by solving a Dirichlet problem on $\Omega_i \cup F_{ij} \cup \Omega_j$ with zero Dirichlet boundary conditions. The scaling operator $D^{(i)}$ is then given by a block diagonal matrix with the diagonal components $D^{(i)}_{j_1}, D^{(i)}_{j_2}, \cdots, D^{(i)}_{j_k}$, where $j_1, j_2, \ldots, j_k \in \mathcal{N}_i$ and $\mathcal{N}_i$ is the set of indices of the Ω_j's $(i \neq j)$ which share a subdomain face with Ω_i.

The condition number of $M^{-1}\widehat{S}_\Gamma$ is bounded by $C\left(1 + \log H/h\right)^2$, where the constant C does not depend on the size of subdomain and mesh size as well as the coefficients and their jumps between subdomains. Due to space restriction, a detailed analysis will not be reported here. Further details are provided in [17].

4 Numerical results

We have applied the BDDC algorithm to our model problem (1). For algorithmic details, we follow [11]. We set $\Omega = (0,1)^3$ and decompose the unit cube into N^3 identical cubic subdomains. Each subdomain has a side length $H = 1/N$. Moreover, we assume that the coefficients α and β have jumps across the interface between the subdomains with a checkerboard pattern in which (α, β) for a subdomain is either (α_b, β_b) or (α_w, β_w). We discretize the model problem (1) by using the lowest order hexahedral Raviart-Thomas finite elements and use the preconditioned conjugate gradient method to solve the discretized problem. The iteration is stopped when the l^2-norm of the residual has been reduced by a factor of 10^{-6}.
We first fix the value of β and vary α. Second, we fix the value of α and vary β. Tables 1 and 2 show the first two sets of results. We next use a different distribution, instead of the checkerboard distribution. We first generate $2N^3$ random numbers $\{r_{\alpha_i}\}_{i=1,\ldots,N^2}$ and $\{r_{\beta_i}\}_{i=1,\ldots,N^2}$ in $[-3, 3]$ with a uniform distribution. We then assign $10^{r_{\alpha_i}}$ and $10^{r_{\beta_i}}$ for α_i and β_i, respectively. The third set of results can be found in Table 3. We see that the condition number is insensitive to the jumps of coefficients.

We next report on numerical experiments for the case where coefficients have jumps inside the subdomains. For each subdomain Ω_i, we let $\Omega_i^o =$

Table 1 Condition numbers and iteration counts (in parentheses). Checkerboard pattern and $N = 4$.

(α_b, β_b)	(α_w, β_w)	$H/h = 2$	$H/h = 4$	$H/h = 8$	$H/h = 16$
$(10^{-2}, 1)$	$(1, 1)$	1.64 (7)	2.32 (9)	3.26 (11)	4.37 (13)
$(10^{-1}, 1)$	$(1, 1)$	1.80 (7)	2.64 (9)	3.70 (12)	4.94 (13)
$(1, 1)$	$(1, 1)$	1.83 (7)	2.69 (10)	3.75 (11)	5.01 (14)
$(10^{1}, 1)$	$(1, 1)$	1.83 (7)	2.69 (10)	3.76 (11)	5.02 (14)
$(10^{2}, 1)$	$(1, 1)$	1.83 (7)	2.69 (10)	3.76 (11)	5.02 (14)

Table 2 Condition numbers and iteration counts (in parentheses). Checkerboard pattern and $N = 4$.

(α_b, β_b)	(α_w, β_w)	$H/h = 2$	$H/h = 4$	$H/h = 8$	$H/h = 16$
$(1, 10^{-2})$	$(1, 1)$	1.03 (3)	1.06 (4)	1.09 (4)	1.12 (4)
$(1, 10^{-1})$	$(1, 1)$	1.28 (5)	1.53 (6)	1.89 (8)	2.31 (9)
$(1, 10^{1})$	$(1, 1)$	1.27 (5)	1.51 (6)	1.85 (7)	2.27 (9)
$(1, 10^{2})$	$(1, 1)$	1.02 (3)	1.05 (4)	1.08 (4)	1.12 (4)

Table 3 Condition numbers and iteration counts (in parentheses). Random coefficients and $N = 4$.

	$H/h = 2$	$H/h = 4$	$H/h = 8$	$H/h = 16$
Set 1	1.80 (8)	2.69 (11)	3.76 (13)	5.01 (16)
Set 2	1.65 (8)	2.37 (9)	3.39 (11)	4.61 (14)
Set 3	1.78 (8)	2.50 (10)	3.49 (12)	4.82 (14)
Set 4	1.67 (8)	2.50 (10)	3.50 (12)	4.68 (14)
Set 5	1.74 (8)	2.49 (10)	3.45 (13)	4.54 (15)

$\{(x, y, z) \,|\, 1/4 \le x^o, y^o, z^o \le 1/2, \text{where}\, x^o = x/H - \lfloor x/H \rfloor, y^o = y/H - \lfloor y/H \rfloor, \text{and}\, z^o = z/H - \lfloor z/H \rfloor.\}$. Here, $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \,|\, m \le x\}$, where $\mathbb{Z}$ is the set of integers. We use the α_i and β_i specified in Table 1 and 2 as coefficients for $\Omega_i \backslash \Omega_i^o$. For Ω_i^o, we assign $100\alpha_i$ and $100\beta_i$ and with α_i and β_i in a checkerboard pattern. Table 4 and 5 show the results. We see that our method works well even though we have discontinuities inside the subdomains.

Table 4 Condition numbers and iteration counts (in parentheses). Specified values as indicated in Table 1 with jumps inside subdomains and $N = 4$.

	$H/h = 4$	$H/h = 8$	$H/h = 16$
$\alpha_b = 10^{-2}$	2.32 (9)	3.34 (11)	4.41 (13)
$\alpha_b = 10^{-1}$	2.64 (9)	3.83 (12)	5.05 (14)
$\alpha_b = 10^{0}$	2.69 (10)	3.90 (12)	5.16 (14)
$\alpha_b = 10^{1}$	2.69 (10)	3.91 (12)	5.17 (14)
$\alpha_b = 10^{2}$	2.69 (10)	3.91 (12)	5.17 (14)

Finally, for a comparison, we report on some numerical experiments using conventional techniques. We have performed three different types of experiments with the same set of coefficient distributions. The first set of exper-

Table 5 Condition numbers and iteration counts (in parentheses). Specified values as indicated in Table 2 with jumps inside subdomains and $N = 4$.

	$H/h = 4$	$H/h = 8$	$H/h = 16$
$\beta_b = 10^{-2}$	1.05 (4)	1.09 (4)	1.13 (4)
$\beta_b = 10^{-1}$	1.51 (6)	1.90 (8)	2.34 (9)
$\beta_b = 10^{1}$	1.53 (6)	1.95 (8)	2.39 (9)
$\beta_b = 10^{2}$	1.06 (4)	1.09 (4)	1.13 (4)

iments, named "deluxe", is based on the deluxe scaling techniques. In the second, "diag", we use the conventional methods described in [4, 13]. In this case, the scaling is based on the diagonal entries of each subdomain matrix. We use the cardinality in the last set, "card". For Raviart-Thomas elements, only two subdomains share a subdomain face in common. Hence, we use $1/2$ as scaling factors. As we see in Table 6, our weighted averaging technique works well while the others are sensitive to the discontinuities across the interface.

Table 6 Condition numbers and iteration counts (in parentheses). Checkerboard pattern, $N = 4$, and $H/h = 8$.

(α_b, β_b)	(α_w, β_w)	deluxe	diag	card
$(10^{-3}, 10^{3})$	$(1, 1)$	1.05e0 (3)	9.03e2 (47)	2.66e2 (43)
$(10^{-2}, 10^{2})$	$(1, 1)$	1.17e0 (4)	1.88e2 (36)	5.13e1 (31)
$(10^{-1}, 10^{1})$	$(1, 1)$	1.82e0 (7)	7.22e1 (43)	2.19e1 (30)
$(10^{1}, 10^{-1})$	$(1, 1)$	1.89e0 (8)	8.63e1 (48)	2.61e1 (32)
$(10^{2}, 10^{-2})$	$(1, 1)$	1.09e0 (4)	1.01e3 (74)	2.58e2 (66)
$(10^{3}, 10^{-3})$	$(1, 1)$	1.01e0 (3)	1.48e4 (130)	3.71e3 (120)

We remark that the deluxe scaling technique requires additional computational costs for solving local subproblems on each subdomain face. Experimentally, conventional methods are approximately 5 to 6 times faster than deluxe scaling in each iteration. However, deluxe scaling requires much less iteration counts especially for the case where we have large jumps between subdomains. Hence, we can expect a better performance. We note that a more computationally efficient version of deluxe scaling is introduced in [7].

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References

[1] Douglas N. Arnold, Richard S. Falk, and R. Winther. Preconditioning in $H(\mathrm{div})$ and applications. *Math. Comp.*, 66(219):957–984, 1997.

[2] Douglas N. Arnold, Richard S. Falk, and Ragnar Winther. Multigrid in H(div) and H(curl). *Numer. Math.*, 85(2):197–217, 2000.

[3] Zhiqiang Cai, Raytcho D. Lazarov, Thomas A. Manteuffel, and Stephen F. McCormick. First-order system least squares for second-order partial differential equations: Part I. *SIAM J. Numer. Anal.*, 31(6):1785–1799, 1994.

[4] Clark R. Dohrmann. A preconditioner for substructuring based on constrained energy minimization. *SIAM J. Sci. Comput.*, 25(1):246–258, 2003.

[5] Clark R. Dohrmann and Olof B. Widlund. An iterative substructuring algorithm for two-dimensional problems in H(curl). *SIAM J. Numer. Anal.*, 50(3):1004–1028, 2012.

[6] Clark R. Dohrmann and Olof B. Widlund. Some Recent Tools and a BDDC Algorithm for 3D Problems in H(curl). In *Domain decomposition methods in science and engineering XX*, volume 91, pages 15–25. Springer-Verlag, Lecture Notes in Computational Science and Engineering, 2013.

[7] Clark R. Dohrmann and Olof B. Widlund. A BDDC algorithm with deluxe scaling for three-dimensional H(curl) problems. Technical Report TR2014-964, Courant Institue of Mathematical Sciences, Mar. 2014. Department of Computer Science.

[8] R. Hiptmair. Multigrid method for $\mathbf{H}$(div) in three dimensions. *Electron. Trans. Numer. Anal.*, 6(Dec.):133–152, 1997. Special issue on multilevel methods (Copper Mountain, CO, 1997).

[9] Ralf Hiptmair and Andrea Toselli. Overlapping and multilevel Schwarz methods for vector valued elliptic problems in three dimensions. In *Parallel solution of partial differential equations (Minneapolis, MN, 1997)*, volume 120 of *IMA Vol. Math. Appl.*, pages 181–208. Springer, New York, 2000.

[10] Tzanio V. Kolev and Panayot S. Vassilevski. Parallel auxiliary space AMG solver for H(div) problems. *SIAM J. Sci. Comput.*, 34(6):A3079–A3098, 2012.

[11] Jing Li and Olof B. Widlund. FETI-DP, BDDC, and block Cholesky methods. *Internat. J. Numer. Methods Engrg.*, 66(2):250–271, 2006.

[12] Ping Lin. A sequential regularization method for time-dependent incompressible Navier-Stokes equations. *SIAM J. Numer. Anal.*, 34(3):1051–1071, 1997.

[13] Jan Mandel, Clark R. Dohrmann, and Radek Tezaur. An algebraic theory for primal and dual substructuring methods by constraints. *Appl. Numer. Math.*, 54(2):167–193, 2005.

[14] Duk-Soon Oh. *Domain Decomposition Methods for Raviart-Thomas Vector Fields.* PhD thesis, Courant Institue of Mathematical Sciences, 2011. TR2011-942. URL: `http://cs.nyu.edu/web/Research/TechReports/TR2011-942/TR2011-942.pdf`.

[15] Duk-Soon Oh. An Alternative Coarse Space Method for Overlapping Schwarz Preconditioners for Raviart-Thomas Vector Fields. In *Domain decomposition methods in science and engineering XX*, volume 91, pages 361–367. Springer-Verlag, Lecture Notes in Computational Science and Engineering, 2013.

[16] Duk-Soon Oh. An Overlapping Schwarz Algorithm for Raviart–Thomas Vector Fields with Discontinuous Coefficients. *SIAM J. Numer. Anal.*, 51(1):297–321, 2013.

[17] Duk-Soon Oh, Olof B. Widlund, and Clark R. Dohrmann. A BDDC algorithm for Raviart-Thomas vector fields. Technical Report TR2013-951, Courant Institue of Mathematical Sciences, Feb. 2013. Department of Computer Science.

[18] Bedrich Sousedík. Nested BDDC for a saddle-point problem. *Numer. Math.*, 125(4):761–783, 2013.

[19] Andrea Toselli. Neumann-Neumann methods for vector field problems. *Electron. Trans. Numer. Anal.*, 11:1–24, 2000.

[20] Andrea Toselli. Overlapping Schwarz methods for Maxwell's equations in three dimensions. *Numer. Math.*, 86(4):733–752, 2000.

[21] Andrea Toselli. Dual-primal FETI algorithms for edge finite-element approximations in 3D. *IMA J. Numer. Anal.*, 26(1):96–130, 2006.

[22] Andrea Toselli and Axel Klawonn. A FETI domain decomposition method for edge element approximations in two dimensions with discontinuous coefficients. *SIAM J. Numer. Anal.*, 39(3):932–956, 2001.

[23] Xuemin Tu. A BDDC algorithm for a mixed formulation of flow in porous media. *Electron. Trans. Numer. Anal.*, 20:164–179 (electronic), 2005.

[24] Xuemin Tu. A BDDC algorithm for flow in porous media with a hybrid finite element discretization. *Electron. Trans. Numer. Anal.*, 26:146–160, 2007.

[25] Barbara I. Wohlmuth, Andrea Toselli, and Olof B. Widlund. An iterative substructuring method for Raviart-Thomas vector fields in three dimensions. *SIAM J. Numer. Anal.*, 37(5):1657–1676, 2000.