

A Discontinuous Coarse Space (DCS) Algorithm for Cell Centered Finite Volume based Domain Decomposition Methods: the DCS-RJMin algorithm

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Abstract In this paper, we introduce a new coarse space algorithm, the “Discontinuous Coarse Space Robin Jump Minimizer” (DCS-RJMin), to be used in conjunction with one-level domain decomposition methods (DDMs). This new algorithm makes use of Discontinuous Coarse Spaces (DCS), and is designed for DDM that naturally produce discontinuous iterates such as Optimized Schwarz Methods (OSM). This algorithm is suitable both at the continuous level and for cell-centered finite volume discretizations. At the continuous level, we prove, under some conditions on the parameters of the algorithm, that the difference between two consecutive iterates goes to 0. We also provide numerical results illustrating the convergence behavior of the DCS-RJMin algorithm.

Key words: discontinuous coarse space, optimized Schwarz method

1 Introduction

Due to the ever increasing parallelism in modern computers, and the ever increasing affordability of massively parallel calculators, it is of utmost importance to develop algorithms that are not only parallel but scalable. In this paper, we are interested in Domain Decomposition Methods (DDMs), which is one way to parallelize the numerical resolution of Partial Differential Equations(PDE).

In Domain Decomposition Methods, the whole domain is subdivided in several subdomains and a computation unit is assigned to each subdomain. In this paper, we only consider non-overlapping domain decompositions. The numerical solution is then computed in parallel inside each subdomain with artificial boundary conditions. Then, subdomains exchange information between each other. This process is reapplied until convergence. In practice, such a scheme, called iterative DDM,

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should be accelerated using Krylov methods. However, for the purpose of analyzing an algorithm, it can be interesting to work directly with the iterative algorithm itself as Krylov acceleration is so efficient it can hide small design problems in the algorithm.

In one-level DDMs, only neighboring subdomains exchange information. Most classical DDM are one-level. While one-level DDMs can be very efficient and can converge in few iterations, they are not scalable: convergence can never occur before information has propagated between the two furthest apart subdomains, *i.e.*, a one level DDM must iterate at least as many times as the diameter of the connectivity graph of the domain decomposition. Typically, if N is the number of subdomains, this means at least $O(N)$ iterations for one-dimensional problems, $O(\sqrt{N})$ for two-dimensional ones and $O(\sqrt[3]{N})$ for three-dimensional ones. For DDMs to be scalable, some kind of global information exchange is needed. The traditional approach to achieve such global information exchange is adding a coarse space to a pre-existing one-level DDM.

To the author's knowledge, the first use of coarse spaces in Domain Decomposition Methods can be found in [17]. Because coarse spaces enable global information exchange, scalability becomes possible. Well known methods with coarse spaces are the two-level Additive Schwarz method [3], the FETI method [14], and the balancing Neumann-Neumann method [13, 4, 15]. Coarse spaces are also an active area of research, see for example [2, 16, 19, 8] for high contrast problems. It is not trivial to add an effective coarse space to one-level DDMs that produce discontinuous iterates such as Optimized Schwarz Methods, see [6, 7], and [5, chap.5].

In [10], the authors introduced the idea of using discontinuous coarse spaces. Since many DDM algorithms produce discontinuous iterates, the use of discontinuous coarse corrections is needed to correct the discontinuities between subdomains, where the iterates of the one-level OSM are discontinuous. In [10], one possible algorithm, the DCS-DMNV (Discontinuous Coarse Space Dirichlet Minimizer Neumann Variational), was described at the continuous level and at the discrete level for Finite Element Methods on a non-overlapping Domain Decomposition. In [18], a similar method, the DCS-DGLC algorithm was proposed. Both the DCS-DMNV and the DCS-DGLC are well suited to finite element discretizations. Also, a similar approach was proposed in [9] for Restricted Additive Schwarz (RAS), an overlapping DDM,

It was proven recently that the proof of convergence for Schwarz found in [12, 1] can be extended to the Discrete Optimized Schwarz algorithm with cell centered finite volume methods, see [11]. It would be interesting to have a discontinuous coarse space algorithm that is suited to cell centered finite volumes. Unfortunately, neither the DCS-DMNV algorithm nor the DCS-DGLC algorithm are practical for cell centered-finite volume methods: the stiffness matrix necessary to compute the coarse correction is not as sparse as one would intuitively believe. In this paper, our main goal is to describe one family of algorithms making use of discontinuous coarse spaces suitable for cell centered finite volumes discretizations.

In §2, we briefly recall the motivations behind the use of discontinuous coarse spaces. In §3, we present the DCS-RJMin algorithm. In §4, we prove that under

some conditions on the algorithm parameter, the L^2 -norm of the difference between two consecutive iterates goes to zero. Finally, we present numerical results in §5.

2 Optimized Schwarz and Discontinuous Coarse Spaces

Let us consider a polygonal domain Ω in \mathbb{R}^2 . As a simple test case, we wish to solve

$$\begin{aligned}\eta u - \Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega.\end{aligned}$$

Without a coarse space, the Optimized Schwarz Method is defined as

Algorithm 1 (One-level OSM).

1. Set u_i^0 to either the null function or to a first approximation.
2. Until convergence
 - a. Set u_i^{n+1} as the unique solution to

$$\begin{aligned}\eta u_i^{n+1} - \Delta u_i^{n+1} &= f \text{ in } \Omega_i, \\ \frac{\partial u_i^{n+1}}{\partial \mathbf{n}_i} + pu_i^{n+1} &= \frac{\partial u_j^n}{\partial \mathbf{n}_i} + pu_j^n \text{ on } \partial\Omega_i \cap \partial\Omega_j, \\ u^{n+1} &= 0 \text{ on } \partial\Omega_i \cap \partial\Omega.\end{aligned}$$

The main shortcoming of the one-level Optimized Schwarz method is the absence of direct communication between distant subdomains. To get a scalable algorithm, one can use a coarse space. A general version of a coarse space method for the OSM is

Algorithm 2 (Generic Two-level OSM).

1. Set u_i^0 to either the null function or to the coarse solution.
2. Until convergence
 - a. Set u_i^{n+1} as the unique solution to

$$\begin{aligned}\eta u_i^{n+1/2} - \Delta u_i^{n+1} &= f \text{ in } \Omega_i, \\ \frac{\partial u_i^{n+1/2}}{\partial \mathbf{n}_i} + pu_i^{n+1/2} &= \frac{\partial u_j^n}{\partial \mathbf{n}_i} + pu_j^n \text{ on } \partial\Omega_i \cap \partial\Omega_j, \\ u^{n+1/2} &= 0 \text{ on } \partial\Omega_i \cap \partial\Omega.\end{aligned}$$

- b. Compute in some way a coarse corrector U^{n+1} belonging to the coarse space X , then set

$$u^{n+1} = u^{n+1/2} + U^{n+1}.$$

More important than the algorithm used to compute the coarse correction U^{n+1} is the choice of an adequate coarse space itself. The ideas presented in [10] still apply. In particular, the coarse space should contain discontinuous functions and the discontinuities of the coarse corrector should be located at the interfaces between subdomains. For these reasons, we suppose the whole domain Ω is meshed by either a coarse triangular mesh or a cartesian mesh \mathcal{T}_H and we use each coarse cell of \mathcal{T}_H as a subdomain Ω_i of Ω . The optimal theoretical coarse space \mathcal{A} is the set of all functions that are solutions to the homogenous equation inside each subdomain: for linear problems, the errors made by any iterate are guaranteed to belong to that space. With an adequate algorithm to compute U^{n+1} , the coarse space \mathcal{A} gives a convergence in a single coarse iteration. Unfortunately this complete coarse space is only practical for one dimensional problems as it is of infinite dimension in higher dimensions. One should therefore choose a finite dimensional subspace X_d of \mathcal{A} .

The choice of the coarse space X_d is primordial. It should have a dimension that is a small multiple of the number of subdomains. To choose X_d , one only needs to choose boundary conditions on every subdomain, then fill the interior of each subdomain by solving the homogenous equation in each subdomain. In this paper, we have not tried to optimize X_d and for the sake of simplicity have chosen X_d as the set of all functions in \mathcal{A} with linear Dirichlet boundary conditions on each interface between any two adjacent subdomains.

3 The DCS-RJMin Algorithm

We now describe the DCS-Robin Jump Minimizer algorithm:

Algorithm 3 (DCS-RJMin).

1. Set $p > 0$ and $q > 0$ and X_d a finite dimensional subspace of \mathcal{A} .
2. Set u^0 to either 0 or to the coarse space solution.
3. Until Convergence

- a. Set $u^{n+\frac{1}{2}}$ as the unique solution to

$$\begin{aligned} \eta u^{n+\frac{1}{2}} - \Delta u^{n+\frac{1}{2}} &= f \text{ in } \Omega_i, \\ \frac{\partial u_i^{n+\frac{1}{2}}}{\partial \mathbf{v}_{ij}} + p u_i^{n+\frac{1}{2}} &= \frac{\partial u_j^n}{\partial \mathbf{v}_{ij}} + p u_j^n \text{ on } \partial \Omega_i \cap \partial \Omega_j, \\ u_i &= 0 \text{ on } \partial \Omega_i \cap \partial \Omega_j. \end{aligned}$$

- b. Set U^{n+1} in X_d as the unique coarse function that minimizes

$$\sum_{i=1}^N \sum_{j \in \mathcal{N}(i)} \left\| \frac{\partial(u_i^{n+\frac{1}{2}} + U_i^{n+1})}{\partial \mathbf{v}_i} + q(u_i^{n+\frac{1}{2}} + U_i^{n+1}) - \frac{\partial(u_j^{n+\frac{1}{2}} + U_j^{n+1})}{\partial \mathbf{v}_i} - q(u_j^{n+\frac{1}{2}} + U_j^{n+1}) \right\|_{L^2(\partial\Omega_i \cap \partial\Omega_j)}^2,$$

where \mathbf{v}_i is the outward normal to subdomain Ω_i and $\mathcal{N}(i)$ is the set of all j such that Ω_j and Ω_i are adjacent.

c. Set $u^{n+1} := u^{n+1/2} + U^{n+1}$.

4 Partial Convergence results for DCS-RJMin

We do not have a complete convergence theorem for the DCS-RJMin algorithm. However, we can prove the following results concerning the iterates of the DCS-RJMin algorithm when $p = q$:

Proposition 1. *If $q = p$, then the iterates produced by the DCS-RJMin algorithm 3 satisfy $\lim_{n \rightarrow +\infty} \|u_i^{n+1/2} - u_i^n\|_{L^2} = 0$.*

Proof. Let u be the mono-domain solution and set $e_i^n = u_i^n - u_i$. Then, following Lions energy estimates [12], we compute

$$\begin{aligned} & \eta \int_{\Omega_i} |e_i^{n+1/2} - e_i^n|^2 \mathbf{d}\mathbf{x} + \int_{\Omega_i} |\nabla(e_i^{n+1/2} - e_i^n)|^2 \mathbf{d}\mathbf{x} \\ &= \int_{\partial\Omega_i} \frac{\partial(e_i^{n+1/2} - e_i^n)}{\partial \mathbf{v}} \cdot (e_i^{n+1/2} - e_i^n) \\ &= \frac{1}{4p} \left(\int_{\partial\Omega_i} \left| \frac{\partial(e_i^{n+1/2} - e_i^n)}{\partial \mathbf{v}} + p(e_i^{n+1/2} - e_i^n) \right|^2 - \left| \frac{\partial(e_i^{n+1/2} - e_i^n)}{\partial \mathbf{v}} - p(e_i^{n+1/2} - e_i^n) \right|^2 \right) \\ &= \frac{1}{4p} \left(\int_{\partial\Omega_i} \left| \frac{\partial(e_i^{n+1/2} - e_i^n)}{\partial \mathbf{v}} + p(e_i^{n+1/2} - e_i^n) \right|^2 - \int_{\partial\Omega_i} \left| \frac{\partial(e_i^{n+1/2} - e_i^n)}{\partial \mathbf{v}} - p(e_i^{n+1/2} - e_i^n) \right|^2 \right) \\ &= \frac{1}{4p} \left(\sum_j \int_{\partial\Omega_i \cap \partial\Omega_j} \left| \left(\frac{\partial e_j^n}{\partial \mathbf{v}_i} + p e_j^n \right) - \left(\frac{\partial e_i^n}{\partial \mathbf{v}_i} + p e_i^n \right) \right|^2 \right. \\ & \quad \left. - \sum_j \int_{\partial\Omega_i \cap \partial\Omega_j} \left| \left(\frac{\partial e_i^{n+1/2}}{\partial \mathbf{v}_i} - p e_i^{n+1/2} \right) - \left(\frac{\partial e_j^{n+1/2}}{\partial \mathbf{v}_i} - p e_j^{n+1/2} \right) \right|^2 \right). \end{aligned}$$

We sum the above equality over all subdomains Ω_i and get

$$\begin{aligned} \eta \sum_i \int_{\Omega_i} |e_i^{n+1/2} - e_i^n|^2 \mathbf{d}\mathbf{x} + \int_{\Omega_i} |\nabla(e_i^{n+1/2} - e_i^n)|^2 \mathbf{d}\mathbf{x} = \\ = \sum_{(i,j)} \frac{1}{4p} \left(\int_{\Gamma_{ij}} \left| \left[\frac{\partial e^n}{\partial \mathbf{v}_i} + p e^n \right] \right|^2 - \int_{\Gamma_{ij}} \left| \left[\frac{\partial e^{n+1/2}}{\partial \mathbf{v}_i} + p e^{n+1/2} \right] \right|^2 \right), \end{aligned}$$

where $[\cdot]$ represents a jump across the interface. Since the coarse step of the DCS-RJMin algorithm minimizes the Robin Jumps, we have

$$\begin{aligned} \eta \sum_i \int_{\Omega_i} |e_i^{n+1/2} - e_i^n|^2 \mathbf{d}\mathbf{x} + \int_{\Omega_i} |\nabla(e_i^{n+1/2} - e_i^n)|^2 \mathbf{d}\mathbf{x} \leq \\ \leq \sum_{(i,j)} \frac{1}{4p} \left(\int_{\Gamma_{ij}} \left| \left[\frac{\partial e^n}{\partial \mathbf{v}_i} + p e^n \right] \right|^2 - \int_{\Gamma_{ij}} \left| \left[\frac{\partial e^{n+1}}{\partial \mathbf{v}_i} + p e^{n+1} \right] \right|^2 \right). \end{aligned}$$

Summing over $n \geq 0$ yields the stated result.

Remark 1. For $q \neq p$, convergence can be proven in the two subdomain case if each subdomain is obtained by reflection of the other with respect to the common interface.

5 Numerical Results

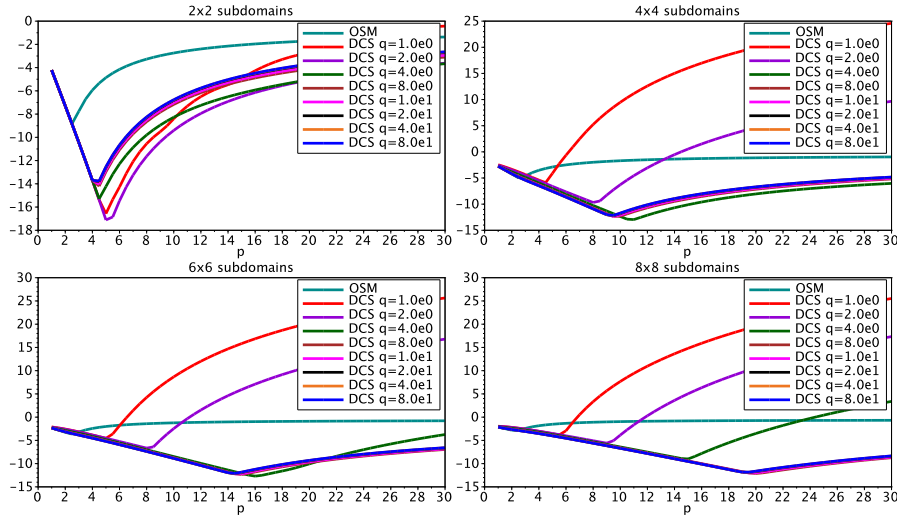


Fig. 1 Convergence for OSM and DCS-RJmin with $\Omega = [0, 4]^2$, $f(x, y) = 0$ and random guess. Plotting $\log(\|e_{50}\|_{\infty} / \|e_0\|_{\infty})$.

We have implemented the DCS-RJMin algorithm in C++ for cell-centered finite volumes on a cartesian grid. We chose $\Omega =]0, 4[\times]0, 4[$, $\eta = 0$ and iterated directly on the errors by choosing $f = 0$. We initialized the Robin transmission conditions at the interfaces between subdomains at random and performed multiple runs of the DCS-RJMin algorithm for various values of p , q and of the number of subdomains. We had p vary from 1.0 to 20.0 with 0.5 increments and q took the values $q_m \times 10^{q_e}$ with q_m in $\{1.0, 2.0, 4.0, 8.0\}$ and q_e in $\{0, 1\}$. We consider 2×2 , 4×4 , 6×6 and 8×8 subdomains. There are always 20×20 cells per subdomains. In Figure 1, we plot $\log(\|e_{50}\|_\infty / \|e_0\|_\infty)$ as a function of p for various values of q . First, we notice that for each value of q , the convergence deteriorates above a certain p_q . In fact, for low values of q and high values of p , the iterates diverge. For two different values of q , the curves are very close when p is smaller than both p_q . We also notice that even though we could only prove Proposition 1 for the case $p = q$, we observe numerical convergence even when $p \neq q$. In fact $p = q$ is not the numerical optimum. This is to be expected intuitively: for a theoretical proof of convergence, we want the algorithm to keep lowering some functional. The existence of such a functional is likely only if all the substeps of the algorithm are optimized for the same kind of errors. If $p = q$, both the coarse step or the local step will either remove low frequency errors (small p and q) or high frequency ones (high p and q). An efficient numerical algorithm should have substeps optimized for completely different kind of errors.

6 Conclusion

In this paper, we have introduced a new discontinuous coarse space algorithm, the DCS-RJMin, which is suitable for cell-centered finite volume discretizations. The coarse space greatly improves numerical convergence. It would be of great interest to study which is the optimal low-dimensional subspace of all piecewise discontinuous piecewise harmonic functions. Future work also includes the development of a possible alternative to a coarse space in order to get scalability: "Piecewise Krylov Methods" where the same minimization problem than the one used in DCS-RJMin is used but where the coarse space is made of piecewise (per subdomain) differences between consecutive one-level iterates.

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