Optimal finite element methods for interface problems *

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1 Introduction

There are many physical problems such as multiphase flows and fluid-structure interactions whose solutions are piecewise smooth but may have discontinuity across some curved interfaces. The direct application of standard finite element method may not perform well. In this paper, we study some special finite element methods for this type of problems. For simplicity of exposition, we consider the case that there is only one interface which is smooth. Let $\Omega, \Omega_1 \subset \mathbb{R}^2$ be two bounded domains with $\Omega_1 \subset \Omega$. We assume that $\Gamma = \partial\Omega_1$ is sufficiently smooth, and $\Gamma \cap \partial\Omega = \emptyset$. To be focused on the influence of Γ, we assume $\Omega = (-1,1)^2$.

To be specific, we consider the homogeneous boundary value problems of the diffusion equation $-\mathrm{div}(\alpha\nabla u) = f$, and the Stokes equation $-\mathrm{div}(\alpha\nabla \underset{\sim}{u} - p\underset{\sim}{\mathrm{I}}) = \underset{\sim}{f}$ with the incompressibility condition $\mathrm{div}\underset{\sim}{u} = 0$. In both of the equations, α represents a piecewise smooth function, namely $\alpha \in (\mathcal{C}^\infty(\Omega_1) \oplus \mathcal{C}^\infty(\Omega_2)) \setminus \mathcal{C}(\Omega)$, such that $0 < \alpha_1 \leqslant \alpha \leqslant \alpha_2$ for two constants α_1 and α_2.

Because of the discontinuity of the coefficient α, the solutions lose their smoothness near the interface. Accuracy would be lost if we use general uniform grids for discretisation. A way to remedy the accuracy of approximation is to use interface-fitted/resolved grids. This way, the non-smoothness of the solution can be restricted to a "narrow" subdomain with respect to the grid near the interface, and the approximation error due to the non-smoothness can thus be dominated.

In Xu [1982] (English translation: Xu [2013]), the following error estimate was obtained:

$$\|u - u_I\|_{0,\Omega} + h|u - u_I|_{1,\Omega} \leqslant C|\log h|^{1/2} h^2 |u|_{2,\Omega_1\cup\Omega_2}, \tag{1}$$

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where u_I is the nodal interpolation of u to the linear element space. Here and after, we use $|w|_{m,\Omega_1\cup\Omega_2}$ or $\|w\|_{m,\Omega_1\cup\Omega_2}$ to denote $|w|_{m,\Omega_1}+|w|_{m,\Omega_2}$ or $\|w\|_{m,\Omega_1}+\|w\|_{m,\Omega_2}$, respectively, for $w\in H^m(\Omega_1\cup\Omega_2):=\{v\in L^2(\Omega):v|_{\Omega_i}\in H^m(\Omega_i),\ i=1,2\}$, with $m\in\{0,1,2\}$. See Chen and Zou [1998] for a same result. A sharper estimate was given in Bramble and King [1996]:

$$\|u-u_I\|_{0,\Omega}+h|u-u_I|_{1,\Omega}\leqslant Ch^2|u|_{2,\Omega_1\cup\Omega_2}. \tag{2}$$

The interface-fitting assumption in the works above can be loosened slightly to that the interface Γ is "$\mathcal{O}(h^2)$-resolved by the mesh", see Li et al. [2010], and the shape-regularity restriction of the grid can be loosened to maximal-angle-bounded grids, see Chen et al. [2013]. The optimal approximation accuracy of linear element space can also be proved on these grids.

We refer to Chen et al. [2013] for an algorithm to generate an interface-fitted grid from a shape-regular grid which is not interface-fitted. (c.f. Figure 1.) The algorithm is easy to implement and the generated grid is maximal-angle-bounded. With the linear element functions constructed thereon, the piecewise smooth functions can be approximated optimally and economically.

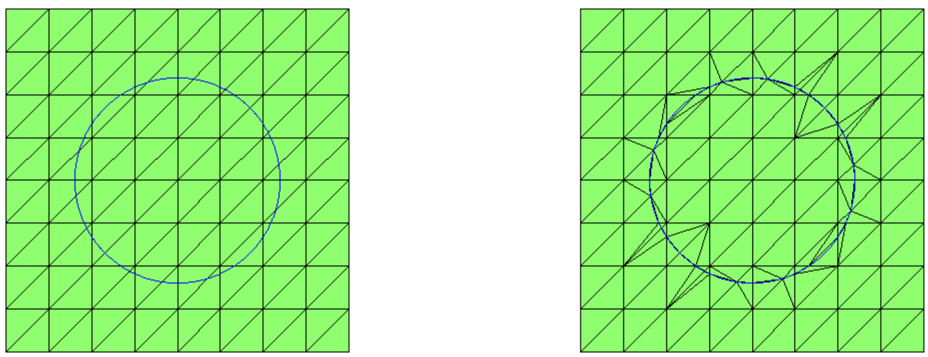

Fig. 1 Left: interface-unfitted mesh; Right: interface-fitted mesh

In this paper, we discuss the linear element schemes for the diffusion equation and the Stokes equation with discontinuous coefficients on interface-fitted maximal-angle-bounded grids. We will consider the conforming (c.f. also Chen et al. [2013]) and nonconforming linear element schemes for the diffusion equation, and the P_1-P_0 element pair for the Stokes equation. Thanks to the above approximation results, the optimal accuracy of conforming linear element discretisation for the diffusion equation is straightforwardly obtained. When the nonconforming element dicretization is considered, the issue of consistency error needs to be addressed. Because of the irregularity of the grid, the traditional technique by trace theorem and scaling argument cannot be applied easily. In this paper, we use the relationship between the nonconforming linear element space and the lowest-order Raviart-Thomas (R-T for short) element space suggested by Acosta and Durán [1999], and obtain the optimal accuracy of the consistency error. As to the incompressible Stokes problem, we have that the P_1-P_0 pair satisfies the inf-sup condition, and prove that it has optimal accuracy.

Then we discuss the optimal multigrid solver for the generated linear system. Particularly, we consider the special grid that is generated from a uniform grid with the algorithm of Chen et al. [2013]. As the underlying grid is obtained by refining an original uniform structured grid, the finite element space thereon is different from the one on the original grid only near the interface. We use the original grid(finite element space) as a coarse grid(subspace, respectively), with some smoothing operations added near the interface, to formulate a nested geometrical multigrid method. We take the conforming linear element system, which is less complicated, for a demonstration, and show the optimality of the formulated multigrid method.

Through the paper, we make use of this notation. Without bringing in ambiguity, we use $|\cdot|$ for the measure of subdomains, especially the area of a 2D manifold or the length of a 1D manifold. We use "$\underset{\sim}{\ }$" for a tensor, and a bold letter for a unit vector (direction). In the paper, "K" will always denote a triangular cell, unless special indication. When the triangulation $\mathcal{T}_h$ is considered, we denote $H^m(\mathcal{T}_h):=\{w\in L^2(\Omega):w|_K\in H^m(K),\ \forall\,K\in\mathcal{T}_h\}$, $m=0,1,2$.

The remaining of the paper is organised as follows. In Section 2, we collect some existing and new estimation results on interpolation operators, especially for piecewise smooth functions on interface-fitted and maximal-angle-bounded grids. In Sections 3 and 4, we discuss the optimal finite element methods for the interface problems of the diffusion equation and of the Stokes equation, respectively. In Section 5, we give an optimal multigrid method for the conforming linear element scheme for diffusion equation. Finally, in Section 6, some concluding remarks are given.

2 Error estimates of interpolation operators

2.1 Element-wise smooth function on interface-fitted grid

As a foundation of the technical analysis, we will show that on interface-fitted grids, functions in $H^m(\Omega_1\cup\Omega_2)$ can be approximated well by functions that are piecewise smooth with respect to the grids. We begin with a sharpened embedding result for the Sobolev space. Here and after, denote $\omega_\eta:=\{x\in\Omega:dist(x,\Gamma)\leqslant\eta\}$.

Lemma 1. *There exists a constant C, depending on Ω and Γ only, such that it holds for $w\in H^1(\Omega_1\cup\Omega_2)$ that*

$$\|w\|_{0,\omega_\eta}^2\leqslant C\eta\|w\|_{1,\Omega_1\cup\Omega_2}^2.$$

The proof of Lemma 1 follows from Theorem 1.1 of Arrieta et al. [2008] directly, and we omit it here. We also refer to Bramble and King [1996] and Li et al. [2010] for similar results.

Lemma 2. *Let $\mathcal{T}_h$ be an interface-fitted grid of Ω, with h the biggest diameter of $K\in\mathcal{T}_h$. Then there exists a constant C depending on Ω and Γ only, such that these inequalities hold:*

1. given $w\in H^1(\Omega_1\cup\Omega_2)$, there exists a $\tilde{w}\in H^1(\mathcal{T}_h)$, such that

$$\sum_{K\in\mathcal{T}_h}\|\tilde{w}\|_{1,K}^2\leqslant C(\|w\|_{1,\Omega_1}^2+\|w\|_{1,\Omega_2}^2),\ \ \|w-\tilde{w}\|_{0,\Omega}^2\leqslant Ch^2(\|w\|_{1,\Omega_1}^2+\|w\|_{1,\Omega_2}^2);$$

2. given $w\in H^2(\Omega_1\cup\Omega_2)$, there exists $\tilde{w}\in H^2(\mathcal{T}_h)$, such that

$$\sum_{K\in\mathcal{T}_h}\|\tilde{w}\|_{2,K}^2\leqslant C(\|w\|_{2,\Omega_1}^2+\|w\|_{2,\Omega_2}^2),\ \sum_{K\in\mathcal{T}_h}(\|w-\tilde{w}\|_{1,K\cap\Omega_1}^2+\|w-\tilde{w}\|_{1,K\cap\Omega_2}^2)\leqslant Ch^2(\|w\|_{2,\Omega_1}^2+\|w\|_{2,\Omega_2}^2);$$

moreover, if $w\in H^1(\Omega)\cap H^2(\Omega_1\cup\Omega_2)$, then $\tilde{w}=w$ on Γ;

3. *given $\underset{\sim}{w}\in(H^1(\Omega_1\cup\Omega_2))^2\cap H(\mathrm{div};\Omega)$, $\exists\tilde{\underset{\sim}{w}}\in H^1(\mathcal{T}_h)\cap H(\mathrm{div};\Omega)$, s.t. $\tilde{\underset{\sim}{w}}\cdot\mathbf{n}=\underset{\sim}{w}\cdot\mathbf{n}$ on Γ,*

$$\sum_{K\in\mathcal{T}_h}\|\tilde{\underset{\sim}{w}}\|_{1,K}^2\leqslant C(\|\underset{\sim}{w}\|_{1,\Omega_1}^2+\|\underset{\sim}{w}\|_{1,\Omega_2}^2),\ \sum_{K\in\mathcal{T}_h}(\|\underset{\sim}{w}-\tilde{\underset{\sim}{w}}\|_{0,K}^2\leqslant Ch^2(\|w\|_{1,\Omega_1}^2+\|w\|_{1,\Omega_2}^2).$$

Proof. We only prove the third item. The others can be found in Bramble and King [1996].

First of all, given $K\in\mathcal{T}_h$, since $\mathcal{T}_h$ is interface fitted, K does not has vertices in different subdomains simultaneously. Besides, by approximation theory, there exists a constant C_0, depending on Γ and Ω, such that if K has a vertex in Ω_i, then $(K\cap\Omega_{3-i})\subset\omega_{C_0h^2}$.

Now given $\underset{\sim}{w}\in(H^1(\Omega_1\cup\Omega_2))^2$, by extension theorem, there exist $\underset{\sim}{w}_1,\underset{\sim}{w}_2\in H^1(\Omega)^2$, such that $(\underset{\sim}{w}_i-\underset{\sim}{w})|_{\Omega_i}=0$, and $\|\underset{\sim}{w}_i\|_{1,\Omega}\leqslant C\|\underset{\sim}{w}\|_{1,\Omega_i}$, C depending on Ω and Γ only. Then we define $\tilde{\underset{\sim}{w}}$ by $\tilde{\underset{\sim}{w}}|_K=\underset{\sim}{w}_i|_K$, if K has a vertex in Ω_i. Here, without loss of generality, we assume the vertices of K are not all on Γ. By the analysis above, $\underset{\sim}{w}-\tilde{\underset{\sim}{w}}=0$, on $\Omega\backslash\omega_{Ch^2}$. Therefore, $\|\underset{\sim}{w}-\tilde{\underset{\sim}{w}}\|_{0,\Omega}^2=\|\underset{\sim}{w}-\tilde{\underset{\sim}{w}}\|_{0,\omega_{Ch^2}}^2\leqslant 3(\|\underset{\sim}{w}\|_{0,\omega_{Ch^2}}^2+\|\underset{\sim}{w}_1\|_{0,\omega_{Ch^2}}^2+\|\underset{\sim}{w}_2\|_{0,\omega_{Ch^2}}^2)$. Further, by Lemma 1, $\|\underset{\sim}{w}-\tilde{\underset{\sim}{w}}\|_{0,\Omega}^2\leqslant Ch^2\|\underset{\sim}{w}\|_{1,\Omega_1\cup\Omega_2}^2$, with C depending on Γ and Ω.

Besides, that $\underset{\sim}{w}\in(H^1(\Omega_1\cup\Omega))^2\cap H(\mathrm{div};\Omega)$ implies $[\![\underset{\sim}{w}\cdot\mathbf{n}]\!]$ vanishes along Γ, this further implies that $\underset{\sim}{w}\cdot\mathbf{n}$, $\underset{\sim}{w}_1\cdot\mathbf{n}$ and $\underset{\sim}{w}_2\cdot\mathbf{n}$ are the same along the interface, thus $\tilde{\underset{\sim}{w}}\cdot\mathbf{n}=\underset{\sim}{w}\cdot\mathbf{n}$ along Γ. Here and after, we use $[\![\cdot]\!]$ to denote the jump between different sides. This finishes the proof.

2.2 Interpolation error for piecewise smooth functions

Let $\mathcal{T}_h$ be a grid on Ω. Denote Q_h the piecewise constant space on $\mathcal{T}_h$, V_h^{CR} the linear Crouzeix-Raviart element space, namely $V_h^{\mathrm{CR}}:=\{w_h\in L^2(\Omega):w_h|_K\in P_1(K),\ \forall K\in\mathcal{T}_h,\ \int_e[\![w_h]\!]=0,\ \text{on any interior edge } e\}$, V_h the continuous piecewise linear function space, and $\mathbb{V}_h^{\mathrm{RT}}$ the lowest order Raviart-Thomas element space, namely $\mathbb{V}_h^{\mathrm{RT}}:=\{\underset{\sim}{w}_h\in(L^2(\Omega))^2:\underset{\sim}{w}_h|_K\in(P_0)^2\oplus\underset{\sim}{x}P_0,\ \int_e[\![\underset{\sim}{w}_h]\!]\cdot\mathbf{n}=0,\ \text{on any interior edge } e\}$. Associated with the local interpolations, we have these globally defined interpolations. Denote by P_h^0 the L^2 projection operator to Q_h, by I_h the interpolation operator to V_h, by Π_h^{CR} the interpolation operator to V_h^{CR}, and by Π_h^{RT} the interpolation operator to $\mathbb{V}_h^{\mathrm{RT}}$. It is evident that $\nabla_h\Pi_h^{\mathrm{CR}}w=P_h^0\nabla w$, and $\mathrm{div}\Pi_h^{\mathrm{RT}}\underset{\sim}{w}=P_h^0\mathrm{div}\underset{\sim}{w}$.

Lemma 3. *Let $\mathcal{T}_h$ be an interface-fitted grid of Ω, with h the biggest size of the elements. With constants C_2 and C_3 depending on the maximal angle of the grid, while C_1 not, we have:*

1. *Let $w\in H^1(\Omega_1\cup\Omega_2)$, then $\|w-P_h^0w\|_{0,\Omega}\leqslant C_1h\|w\|_{1,\Omega_1\cup\Omega_2}$.*
2. *Let $u\in H^1(\Omega)\cap H^2(\Omega_1\cup\Omega_2)$, then $\inf_{v_h\in V_h^{\mathrm{CR}}}|u-v_h|_{1,h}\leqslant|u-I_hu|_{1,\Omega}\leqslant C_2h\|u\|_{2,\Omega_1\cup\Omega_2}$.*

3. Let $\underset{\sim}{w} \in (H^1(\Omega_1 \cup \Omega_2))^2 \cap H(\mathrm{div};\Omega)$*, then* $\|\underset{\sim}{w} - \Pi_h^{\mathrm{RT}}\underset{\sim}{w}\|_{0,\Omega} \leqslant C_3 h \|\underset{\sim}{w}\|_{1,\Omega_1\cup\Omega_2}$.

Proof. We only prove the third item, and the first one is similar. We refer to e.g., Chen et al. [2013] for the second item.

We begin with a stability result. Let K be a triangle, with e_1, e_2 and e_3 its edges. On K, it holds for $i = 1, 2, 3$ that $\int_{e_i} \Pi_h^{\mathrm{RT}}\underset{\sim}{w} \cdot \mathbf{n}_{e_i} = \int_{e_i} \underset{\sim}{w} \cdot \mathbf{n}_{e_i}$. Direct calculation leads to that $\|\Pi_h^{\mathrm{RT}}\underset{\sim}{w}\|_{0,K}^2 \leqslant \frac{3}{16|K|}\sum_i \left[(\int_{e_i} \underset{\sim}{w}\cdot\mathbf{n}_{e_i})^2 \sum_{j\neq i} |e_j|^2 \right]$. Now, given $\underset{\sim}{w} \in (H^1(\Omega_1\cup\Omega_2))^2 \cap H(\mathrm{div};\Omega)$, by Lemma 2, there exists $\tilde{\underset{\sim}{w}} \in H^1(\mathcal{T}_h)^2 \cap H(\mathrm{div};\Omega)$, such that $\tilde{\underset{\sim}{w}}\cdot\mathbf{n} = \underset{\sim}{w}\cdot\mathbf{n}$ on Γ, $\sum_{K\in\mathcal{T}_h} \|\tilde{\underset{\sim}{w}}\|_{1,K}^2 \leqslant C(\|\underset{\sim}{w}\|_{1,\Omega_1}^2 + \|\underset{\sim}{w}\|_{1,\Omega_2}^2)$, and $\|\underset{\sim}{w} - \tilde{\underset{\sim}{w}}\|_{0,\Omega}^2 \leqslant Ch^2(\|w\|_{1,\Omega_1}^2 + \|w\|_{1,\Omega_2}^2)$. By triangle inequality,

$$\|\underset{\sim}{w} - \Pi_h^{\mathrm{RT}}\underset{\sim}{w}\|_{0,\Omega} \leqslant \|\underset{\sim}{w} - \tilde{\underset{\sim}{w}}\|_{0,\Omega} + \|\tilde{\underset{\sim}{w}} - \Pi_h^{\mathrm{RT}}\tilde{\underset{\sim}{w}}\|_{0,\Omega} + \|\Pi_h^{\mathrm{RT}}\tilde{\underset{\sim}{w}} - \Pi_h^{\mathrm{RT}}\underset{\sim}{w}\|_{0,\Omega} := I_1 + I_2 + I_3. \quad (3)$$

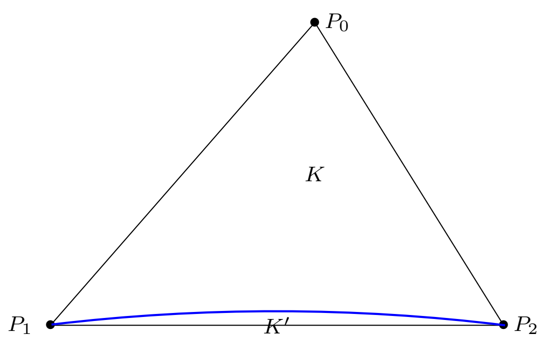

Fig. 2 Illustration of a cell K, and the edge $e = P_1P_2$.

For I_3, we only have to estimate $\Pi_h^{\mathrm{RT}}\tilde{\underset{\sim}{w}} - \Pi_h^{\mathrm{RT}}\underset{\sim}{w}$ on such K that $K \cap \Gamma \neq \emptyset$. Without loss of generality, we choose $K = [P_0, P_1, P_2]$, such that $P_0 \in \Omega_1$, and $K \cap \Omega_2 \neq \emptyset$; particularly, Γ goes through K from P_1 to P_2, c.f. Figure 2. Denote $e = [P_1, P_2]$ and $K' = K \setminus \Omega_1$. Then $\int_e (\tilde{\underset{\sim}{w}} - \underset{\sim}{w})\cdot\mathbf{n}_e = \int_{K'} \nabla\cdot(\tilde{\underset{\sim}{w}} - \underset{\sim}{w}) - \int_{e'} (\tilde{\underset{\sim}{w}} - \underset{\sim}{w})\cdot\mathbf{n}_{e'}$, where $e' = \partial K' \setminus e$ thus $e' \subset \Gamma$. Note that $(\tilde{\underset{\sim}{w}} - \underset{\sim}{w})\cdot\mathbf{n}_{e'} = 0$ on e', and thus $\int_e (\tilde{\underset{\sim}{w}} - \underset{\sim}{w})\cdot\mathbf{n}_e = \int_{K'} \nabla\cdot(\tilde{\underset{\sim}{w}} - \underset{\sim}{w}) \leqslant |K'|^{1/2}\|\nabla\cdot(\tilde{\underset{\sim}{w}} - \underset{\sim}{w})\|_{0,K'}$. Thus,

$$\|\Pi_h^{\mathrm{RT}}(\tilde{\underset{\sim}{w}} - \underset{\sim}{w})\|_{0,K}^2 \leqslant \frac{3}{8}h_K^2\frac{|K'|}{|K|}\|\nabla\cdot(\tilde{\underset{\sim}{w}} - \underset{\sim}{w})\|_{0,K'}^2 \leqslant h_K^2\|\nabla\cdot(\tilde{\underset{\sim}{w}} - \underset{\sim}{w})\|_{0,K'}^2 \leqslant h_K^2\|\nabla\cdot(\tilde{\underset{\sim}{w}} - \underset{\sim}{w})\|_{0,K\cap\omega_{Ch^2}}^2.$$

Further, $\|\Pi_h^{\mathrm{RT}}(\tilde{\underset{\sim}{w}} - \underset{\sim}{w})\|_{0,\Omega}^2 \leqslant \sum_{K\in\mathcal{T}_h} Ch_K^2\|\nabla\cdot(\tilde{\underset{\sim}{w}} - \underset{\sim}{w})\|_{0,K\cap\omega_{Ch^2}}^2 \leqslant Ch^2\|\nabla\cdot(\tilde{\underset{\sim}{w}} - \underset{\sim}{w})\|_{0,\omega_{Ch^2}}^2 \leqslant Ch^2(\|\underset{\sim}{w}\|_{1,\Omega_1\cup\Omega_2}^2)$. Then by (3), we have $\|\underset{\sim}{w} - \Pi_h^{\mathrm{RT}}\underset{\sim}{w}\|_{0,\Omega} \leqslant C_1h\|\underset{\sim}{w}\|_{1,\Omega_1\cup\Omega_2} + C_2h\|\underset{\sim}{w}\|_{1,\Omega_1\cup\Omega_2} +$

$C_3h\|\underset{\sim}{w}\|_{1,\Omega_1\cup\Omega_2} \leqslant Ch\|\underset{\sim}{w}\|_{1,\Omega_1\cup\Omega_2}$, where C_2 depends on the maximal angle of the triangulation (c.f. Acosta and Durán [1999]). This finishes the proof.

3 Optimal linear element methods for diffusion equation

We consider the boundary-interface value problem:

$$\begin{cases} -\nabla\cdot(\alpha(x)\nabla u) = f, \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \\ [\![u]\!] = 0, \ [\![\alpha\nabla u\cdot\mathbf{n}]\!] = 0, \text{ on } \Gamma, \end{cases} \tag{4}$$

where $\mathbf{n}$ is the normal direction of Γ. The variational formulation of the above problem is: Find $u\in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \forall v\in H_0^1(\Omega), \tag{5}$$

where $a(u,v) = \int_\Omega \alpha(x)\nabla u\cdot\nabla v,$ and $(f,v) = \int_\Omega fv.$

Evidently, given the coefficient α, the energy norm of the boundary value problem is equivalent to the H^1 norm (or piecewise H^1 norm for nonconforming element space). In the sequel, we focus ourselves on the analysis of the H^1 norm.

In this section and Section 4, we assume $\mathcal{T}_h$ is an interface-fitted triangulation of Ω, with h the biggest diameter of all $K\in\mathcal{T}_h$. We consider the case $\mathcal{T}_h$ is one in a maximal-angle-bounded family.

3.1 A conforming linear element method

Let $V_{h0} = V_h\cap H_0^1(\Omega)$. The finite element problem is to find $u_h\in V_{h0}$, such that

$$a(u_h,v_h) = (f,v_h), \quad \forall\, v_h\in V_{h0}. \tag{6}$$

Let u be the solution of (5), then by Cea lemma, it is straightforward that

$$|u-u_h|_{1,\Omega} \leqslant C\inf_{v_h\in V_{h0}}|u-v_h|_{1,\Omega} \leqslant Ch\|u\|_{2,\Omega_1\cup\Omega_2}.$$

We also refer to Bramble and King [1996], Li et al. [2010], Chen et al. [2013] for related discussions.

3.2 A nonconforming linear element method

Let $V_{h0}^{\rm CR}\subset V_h^{\rm CR}$ consist of the C-R element functions that vanish at the midpoints of the boundary edges. Then the C-R element scheme of the boundary value problem is to find $u_h\in V_{h0}^{\rm CR}$, such that

$$(\alpha\nabla_h u_h, \nabla_h v_h) = (f,v_h), \quad \forall\, v_h\in V_{h0}^{\rm CR}. \tag{7}$$

Here ∇_h denotes the piecewise gradient.

Theorem 1. *Let u and u_h be the solutions of (5) and (7), respectively. Assume $u \in H^2(\Omega_1 \cup \Omega_2) \cap H^1(\Omega)$. Then it holds with a constant C independent of h that*

$$\|\nabla_h(u - u_h)\|_{0,\Omega} \leqslant Ch(\|u\|_{2,\Omega_1\cup\Omega_2} + \|f\|_{0,\Omega}). \tag{8}$$

Proof. Firstly, recall the Strang lemma and we have, with $|\cdot|_{1,h} = \|\nabla_h \cdot\|_{0,\Omega}$,

$$|u - u_h|_{1,h} \lesssim \inf_{v_h\in V_{h0}^{\mathrm{CR}}} |u - v_h|_{1,h} + \sup_{w_h\in V_{h0}^{\mathrm{CR}}} \frac{(\alpha\nabla u, \nabla w_h) - (f, w_h)}{|w_h|_{1,h}}. \tag{9}$$

By Lemma 3, we have to estimate the consistency error, which is (c.f. also Acosta and Durán [1999])

$$(\alpha\nabla u, \nabla_h w_h) - (f, w_h) = (\alpha\nabla u - \Pi_h^{\mathrm{RT}}\alpha\nabla u, \nabla w_h) - (-\mathrm{div}\alpha\nabla u + \mathrm{div}\Pi_h^{\mathrm{RT}}\alpha\nabla u, w_h) := I - II. \tag{10}$$

By Lemma 3, $|II| = |(f + P_h^0\mathrm{div}\alpha\nabla u, w_h)| = |(f - P_h^0 f, w_h)| = |(f, w_h - P_h^0 w_h)| \leqslant Ch\|f\|_{0,\Omega}|w_h|_{1,h}$. Besides, as $\alpha\nabla u \in (H^1(\Omega_1 \cup \Omega_2))^2 \cap H(\mathrm{div};\Omega)$, $|I| \leqslant \|\alpha\nabla u - \Pi_h^{\mathrm{RT}}(\alpha\nabla u)\|_{0,\Omega}\|\nabla_h w_h\|_{0,\Omega} \leqslant Ch\|\alpha\nabla u\|_{1,\Omega_1\cup\Omega_2}|w_h|_{1,h}$. Substituting all above into (9) finishes the proof.

4 The $P^1 - P^0$ element method for Stokes interface problem

4.1 Model problem and finite element discretization

Now we consider the system of Stokes equation,

$$\begin{cases} -\mathrm{div}(\alpha\nabla \underset{\sim}{u} - p\underset{\sim}{\mathrm{Id}}) = \underset{\sim}{f}, \text{ in } \Omega, \\ \mathrm{div}\underset{\sim}{u} = 0, \text{ in } \Omega, \\ \underset{\sim}{u} = \underset{\sim}{0}, \text{ on } \partial\Omega, \\ [\![\underset{\sim}{u}]\!] = 0, \ [\![(\alpha\nabla\underset{\sim}{u} - p\underset{\sim}{\mathrm{Id}})\cdot\mathbf{n}]\!] = 0, \text{ on } \Gamma. \end{cases} \tag{11}$$

Here $\underset{\sim}{\mathrm{Id}} \in \mathbb{R}^{2\times 2}$ is the identity. The variational formulation is to find $(\underset{\sim}{u}, p) \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$, such that

$$\begin{cases} (\alpha\nabla\underset{\sim}{u}, \nabla\underset{\sim}{v}) - (p\underset{\sim}{\mathrm{Id}}, \nabla\underset{\sim}{v}) = (\underset{\sim}{f}, \underset{\sim}{v}), \forall\, \underset{\sim}{v} \in (H_0^1(\Omega))^2, \\ (q, \mathrm{div}\,\underset{\sim}{u}) = 0, \qquad \forall\, q \in L_0^2(\Omega). \end{cases} \tag{12}$$

Let $\tilde{Q}_h$ be the space of piecewise constant with zero average, then the finite element problem is to find $(\underset{\sim}{u}_h, p_h) \in (V_{h0}^{\mathrm{CR}})^2 \times \tilde{Q}_h$, such that

$$\begin{cases} (\alpha\nabla_h\underset{\sim}{u}_h, \nabla_h\underset{\sim}{v}_h) - (p_h\underset{\sim}{\mathrm{Id}}, \nabla_h\underset{\sim}{v}_h) = (\underset{\sim}{f}, \underset{\sim}{v}_h), \forall\, \underset{\sim}{v}_h \in (V_{h0}^{\mathrm{CR}})^2, \\ (q_h, \nabla_h\cdot\underset{\sim}{u}_h) = 0, \qquad \forall\, q_h \in \tilde{Q}_h. \end{cases} \tag{13}$$

It is well known that, by the commutative property and the inf-sup condition of the model problem (12), the discrete inf-sup condition follows as $\sup_{\underset{\sim}{v}_h\in (V_{h0}^{\rm CR})^2}\frac{(q_h,{\rm div}_h\underset{\sim}{v}_h)}{\|q_h\|_{0,\Omega}\|\underset{\sim}{v}_h\|_{1,h}}\geqslant C,\ \ \text{for}\, q_h\in\tilde{Q}_h.$ Note that the constant does not depend on the triangulation.

4.2 Accuracy analysis

Theorem 2. *Let $(\underset{\sim}{u},p)$ and $(\underset{\sim}{u}_h,p_h)$ be the solutions of (12) and (13), respectively. Assume $\underset{\sim}{u}\in (H^2(\Omega_1\cup\Omega_2)\cap H^1_0(\Omega))^2$, and $p\in H^1(\Omega_1\cup\Omega_2)\cap L^2_0(\Omega)$. Then it holds with a constant C independent of h that*

$$|\underset{\sim}{u}-\underset{\sim}{u}_h|_{1,h}+\|p-p_h\|_{0,\Omega}\leqslant Ch(\|\underset{\sim}{u}\|_{2,\Omega_1\cup\Omega_2}+\|p\|_{1,\Omega_1\cup\Omega_2}+\|\underset{\sim}{f}\|_{0,\Omega}). \tag{14}$$

Proof. We start with this fundamental estimate:(Brezzi and Fortin [1991])

$$|\underset{\sim}{u}-\underset{\sim}{u}_h|_{1,h}+\|p-p_h\|_{0,\Omega}\lesssim \inf_{\underset{\sim}{v}_h\in (V_{h0}^{\rm CR})^2}|\underset{\sim}{u}-\underset{\sim}{v}_h|_{1,h}+\inf_{q_h\in\tilde{Q}_h}\|p-q_h\|_{0,\Omega}+\sup_{\underset{\sim}{w}_h\in (V_{h0}^{\rm CR})^2}\frac{(\alpha\nabla\underset{\sim}{u}-p\underset{\sim}{\rm Id},\nabla\underset{\sim}{w}_h)-(\underset{\sim}{f},\underset{\sim}{w}_h)}{|\underset{\sim}{w}_h|_{1,h}}.$$

By Lemma 3, we only have to estimate the consistency error. Since $\alpha\nabla\underset{\sim}{u}-p\underset{\sim}{\rm Id}\in (H({\rm div};\Omega)\cap (H^1(\Omega_1\cup\Omega_2))^2)^2$, we can use the same technique as that of Theorem 1 and obtain

$$|(\alpha\nabla\underset{\sim}{u}-p\underset{\sim}{\rm Id},\nabla\underset{\sim}{w}_h)-(\underset{\sim}{f},\underset{\sim}{w}_h)|\leqslant Ch(\|\alpha\nabla\underset{\sim}{u}-p\underset{\sim}{\rm Id}\|_{1,\Omega_1\cup\Omega_2}+\|\underset{\sim}{f}\|_{0,\Omega})|\underset{\sim}{w}_h|_{1,h}. \tag{15}$$

Summing all above finishes the proof.

5 A two-level geometric multigrid method

In this section, we consider the optimal solver of the finite element problem (6). Define $A_h:V_{h0}\to V_{h0}$ by $(A_hw_h,v_h)=a_h(w_h,v_h)$, for any $w_h,v_h\in V_{h0}$. In this section, $\widetilde{\mathcal{T}}_h$ is a uniform grid with multilevel structure, and $\mathcal{T}_h$ is an interface-fitted grid generated from $\widetilde{\mathcal{T}}_h$ by local operations near the interface by the algorithm in Chen et al. [2013]. (See Figure 1 for $\widetilde{\mathcal{T}}_h$(left) and $\mathcal{T}_h$(right).) Particularly, $\widetilde{\mathcal{T}}_h$ is shape regular, and $\mathcal{T}_h$ is maximal-angle-bounded. Let $\mathcal{N}_h$ and $\widetilde{\mathcal{N}}_h$ be the sets of vertices of $\mathcal{T}_h$ and $\widetilde{\mathcal{T}}_h$, respectively. Denote $\widetilde{\mathcal{N}}_h^c:=\mathcal{N}_h\setminus\widetilde{\mathcal{N}}_h$.

5.1 Theory of successive subspace correction method

In this section we give some general result of the successive subspace correction method of solving on a linear vector space V with inner product $(\cdot,\cdot)$ the equation $(Au,v)=(f,v)$, where $A:V\to V$ is a symmetric positive definite operator. The presentation follows closely to Xu [1992], Xu and Zikatanov [2002], Xu et al. [2009] and Chen et al. [2012].

We decompose the space $V=\sum_{i=0}^J V_i$ as the summation of subspaces $V_i\subset V$. We do not assume the summation is a direct sum. The original problem associates sub-problems in each V_i with smaller size which are relatively easier to solve. We use the following operators, for $i=0,1,...,J$:

- $Q_i:V\to V_i$ the projection in the inner product $(\cdot,\cdot)$;
- $I_i:V_i\to V$ the natural inclusion which is often called prolongation;
- $P_i:V\to V_i$ the projection in the inner product $(\cdot,\cdot)_A=(A\cdot,\cdot)$;
- $A_i:V_i\to V_i$ the restriction of A to the subspace V_i;
- $R_i:V_i\to V_i$ an approximation of A_i^{-1} (often known as smoother);
- $T_i:V\to V_i$, $T_i=R_iQ_iA=R_iA_iP_i$.

It is easy to verify $Q_iA=A_iP_i$ and $Q_i=I_i^t$ with $(I_i^tu,v_i):=(u,I_iv_i)$. The operator I_i^t is often called restriction. If $R_i=A_i^{-1}$, then we have an exact local solver and $T_i=P_i$. With slightly abused notation, we still use T_i to denote the restriction $T_i|_{V_i}:V_i\to V_i$ and $T_i^{-1}=(T_i|_{V_i})^{-1}:V_i\to V_i$.

The Successive Subspace Correction (SSC) method performs the correction in every subspace in a successive way. In operator form, it reads, given some approximation solution u_k,

$$v^0=u^k, v^{i+1}=v^i+I_iR_iI_i^t(f-Av^i), i=0,\dots,J, u^{k+1}=v^{J+1}, \tag{16}$$

and the corresponding error equation is

$$u-u^{k+1}=\left[\prod_{i=0}^J(I-I_iR_iI_i^tA)\right](u-u^k)=\left[\prod_{i=0}^J(I-T_i)\right](u-u^k). \tag{17}$$

Here we assume there is a built-in ordering from $i=0$ to J. The multiplicative multigrid method for finite element systems is a special SSC method with subspaces constructed by finite element functions on multilevel grids. For the convergence, we have this fundamental estimate.

Lemma 4 (X-Z identity for SSC). *If there is a $\rho<1$, such that $\|I-T_i\|_{A_i}\leqslant\rho,\quad i=0,\dots,J$, then it holds that*

$$\left\|\prod_{i=0}^J(I-T_i)\right\|_A^2=1-\frac{1}{c_1}, \tag{18}$$

where

$$c_1=\sup_{\|v\|_A=1}\inf_{\sum_{i=0}^J v_i=v}\sum_{i=0}^J(\bar{T}_i^{-1}(v_i+T_i^*w_i),v_i+T_i^*w_i)_A, \tag{19}$$

with $w_i=\sum_{j>i}v_j$, and $\bar{T}_i=T_i+T_i^*-T_i^*T_i$, T_i^* the adjoint operator of T_i with respect to $(\cdot,\cdot)_A$.

Remark 1. If we perform a two-level method, and particularly, we perform an exact solver on a subspace V_0, then we have $c_1=\sup_{\|v\|_A=1}(\|P_0v\|_A^2+\|v-\Pi_hv\|_{\bar{R}_1^{-1}})$ where $P_0:V\to V_0$ and

$\Pi_h : V \to V_1$ are the projection operators with respect to $(\cdot,\cdot)_A$ and $(\cdot,\cdot)_{\bar{R}_1^{-1}}$, respectively, and $\bar{R}_1 = R_1^t + R_i - R_i^t A R_i$.

5.2 An optimal multigrid method for (6)

Let $\widetilde{V}_h^c \subset V_h$ be space of nodal basis functions that vanish on $\widetilde{\mathcal{N}}_h$. Then $V_h = \widetilde{V}_h \oplus \widetilde{V}_h^c$, where $\widetilde{V}_h$ is the linear element space on $\widetilde{\mathcal{T}}_h$. Let $\tilde{I}_h$ be the nodal interpolation on $\widetilde{V}_h$. Then $(I-\tilde{I}_h)V_h = \widetilde{V}_h^c$ and $\tilde{I}_h V_h = \widetilde{V}_h$. Let $\widetilde{A}_h$ and $\widetilde{A}_h^c$ be the restrictions of A_h on $\widetilde{V}_{h0} := \widetilde{V}_{h0} \cap H_0^1(\Omega)$ and $\widetilde{V}_h^c$, respectively.

Lemma 5. *It holds for $w_h \in V_{h0}$ that $\|\tilde{I}_h w_h\|_{\widetilde{A}_h} \leqslant \Lambda \|w_h\|_{A_h}$, with Λ a constant independent of h.*

Proof. When h is sufficiently small, for any $p \in \widetilde{N}_h$, there exists a segment e with p being one of its ends, such that e is an edge of $\widetilde{\mathcal{T}}_h$ and $\mathcal{T}_h$ simultaneously, and thus $\tilde{I}_h w_h = w_h$ on e. Therefore, by the standard technique alike to the stability of Scott-Zhang operator (Scott and Zhang [1990]) and a Scott-Zhang type operator (Chen et al. [2012]), we have $|\tilde{I}_h w_h|_{1,\Omega} \leqslant C|w_h|_{1,\Omega}$ with C depending on the shape regularity of $\widetilde{T}_h$ only. This finishes the proof.

Let $\tilde{R}_h : \widetilde{V}_{h0} \to \widetilde{V}_{h0}$ be approximately an inverse of $\tilde{A}_h$. We have this two-level successive subspace correction method. (Algorithm 1)

Algorithm 1 *Implement this iterative procedure until converge:*

1. *do subspace correction on $\widetilde{V}_h$ with an inexact solver $\widetilde{R}_h$;*
2. *do subspace correction on $\widetilde{V}_h^c$ with an exact solver $(\widetilde{A}_h^c)^{-1}$.*

Obviously, Algorithm 1 defines an iterative method for solving $A_h u_h = f_h$. Let $\widetilde{P}_h^c$ and $\widetilde{Q}_h$ be the projection operator onto $\widetilde{V}_h^c$ and $\widetilde{V}_{h0}$ with respect to $a_h(\cdot,\cdot)$ and $(\cdot,\cdot)$, respectively. Denote by B_h the iterator of the method. Then the error contract operator on V_{h0} is $I - B_h A_h = (I - \widetilde{P}_h^c)(I - \widetilde{R}_h \widetilde{Q}_h A_h)$.

Theorem 3. *Assume that $\|I - \tilde{R}_h \tilde{A}_h\|_{\tilde{A}_h} \leqslant \rho < 1$. Then Algorithm **1** is uniformly convergent with respect to the mesh size with*

$$\|I - B_h A_h\|_{A_h}^2 \leqslant \frac{\Lambda}{1-\rho^2+\Lambda}.$$

Proof. By the X-Z identity for the successive subspace correction method, (c.f., e.g., Xu and Zikatanov [2002]) we have

$$\|I - B_h A_h\|_{A_h}^2 = 1 - \frac{1}{c_1},$$

with

$$c_1 = \sup_{v_h \in V_{h0}, \|v_h\|_{A_h}=1} \left(\|\widetilde{P}_h^c v_h\|_{A_h}^2 + \inf_{\tilde{v}_h \in \widetilde{V}_h^c, v_h - \tilde{v}_h \in \widetilde{V}_h} \left((\tilde{R}_h^t + \tilde{R}_h - \tilde{R}_h^t \tilde{A}_h \tilde{R}_h)^{-1}(v - \tilde{v}_h), (v - \tilde{v}_h) \right) \right).$$

Since $\|I - \tilde{R}_h \tilde{A}_h\|_{\tilde{A}_h} \leqslant \rho < 1$, we have $\|I - (\tilde{R}_h^t + \tilde{R}_h - \tilde{R}_h^t \tilde{A}_h \tilde{R}_h)\|_{\tilde{A}_h} \leqslant \|I - \tilde{R}^t \tilde{A}_h\|_{\tilde{A}_h} \|I - \tilde{R}\tilde{A}_h\|_{\tilde{A}_h} \leqslant \rho^2$, and thus $\lambda_{\max}((\tilde{R}_h^t \tilde{A}_h + \tilde{R}_h \tilde{A}_h - \tilde{R}_h^t \tilde{A}_h \tilde{R}_h \tilde{A}_h)^{-1}) \leqslant \frac{1}{1-\rho^2}$. Therefore

$$\Big((\tilde{R}_h^t+\tilde{R}_h-\tilde{R}_h^t\tilde{A}_h\tilde{R}_h)^{-1}(v_h-\tilde{v}_h),(v_h-\tilde{v}_h)\Big)$$
$$=\Big((\tilde{R}_h^t\tilde{A}_h+\tilde{R}_h\tilde{A}_h-\tilde{R}_h^t\tilde{A}_h\tilde{R}_h\tilde{A}_h)^{-1}(v_h-\tilde{v}_h),\tilde{A}_h(v_h-\tilde{v}_h)\Big)\leqslant\frac{1}{1-\rho^2}(v_h-\tilde{v}_h,\tilde{A}_h(v_h-\tilde{v}_h)).$$

Since evidently $\|\tilde{P}_h^c v_h\|_{A_h}\leqslant\|v_h\|_{A_h}$, we have $c_1\leqslant\sup\limits_{v_h\in V_h,\|v_h\|_{A_h}=1}(1+\frac{1}{1-\rho^2}\inf\limits_{\tilde{v}_h\in\widetilde{V}_h^c,v_h-\tilde{v}_h\in\widetilde{V}_h}\|v_h-\tilde{v}_h\|_{A_h}^2)$. Then by Lemma 5, we have $c_1\leqslant 1+\frac{\Lambda}{1-\rho^2}$, and finally obtain $\|I-B_hA_h\|_{A_h}^2\leqslant\frac{\Lambda}{1-\rho^2+\Lambda}$.

When $\widetilde{\mathcal{T}}_h$ is a shape-regular grid with a geometrical multilevel structure, then a geometric multigrid process can be implemented on $\widetilde{V}_{h0}$, and the approximate inverse $\widetilde{R}_h$ of $\widetilde{A}_h$ can be chosen to be the iterator of V-cycle multigrid method. The assumption of Theorem 3 holds(see Xu [1992], Xu and Zhu [2008], Xu and Zikatanov [2002]).

5.3 Numerical examples

To test the numerical methods, we consider the following example. Let the interface Γ be a circle centered at the origin with radius r_0. Let the exact solution be $u(x)=u(\mathbf{r})=2\mathbf{r}^4+|\mathbf{r}^4-r_0^4|$, where $\mathbf{r}=dist(x,\mathbf{0})$. Moreover, we choose $\alpha(x)=1$ if $\mathbf{r}>r_0$ and $\alpha(x)=3$ if $\mathbf{r}<r_0$, and the right hand side can be computed accordingly. Hereafter we set $r_0=0.6$.

We implement Algorithm 1, with $V(1,1)$ cycle geometric multigrid based on the original unfitted grid playing as the coarse grid corrector. We record the numerical results in Table 1. In these examples, the initial guess is $\mathbf{0}$, and the stopping criterion is the l^2 norm of the relative residual being smaller than 10^{-10}. From Table 1, we can see that the multigrid method converges uniformly with respect to the mesh size, which confirms our theoretical results.

Table 1 Numerical performance of Algorithm 1.

h	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
#iter	14	13	13	13	13	13	13

6 Concluding remarks

In this paper, we discussed the optimal finite element method for the interface boundary value problem of the diffusion equation and the Stokes equation. We proved that the linear Crouzeix-Raviart element schemes provide optimal accuracy with respect to the mesh size for the two interface boundary value problems on grids that are interface-fitted and maximal-angle-bounded.

Given a uniform grid, an interface-fitted and maximal-angle-bounded grid can be generated by some local operation close to the interface. On the grids generated that way, we discussed the optimal multigrid method of the discrete linear systems. We took the conforming linear element system, the theory of which is less complicated, for a demonstration, and show that by the methodology of using the original grid as a coarse grid and reinforcing the smoothing effect near the interface, we obtain an optimal multigrid method.

Some other optimal finite element methods and their optimal multigrid solvers for interface boundary value problems will be discussed in the future works.

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