A Domain Decomposition Approch in the Electrocardiography Inverse Problem

Nejib Zemzemi[1]

Abstract The mostly used mathematical formulation of the inverse problem in electrocardiography is based on a least method using a transfer matrix that maps the electrical potential on the heart to the body surface potential (BSP). This mathematical model is ill based and a lot of works have been concentrating on the regularization term without thinking of reformulating the problem itself. We propose in this study to solve the inverse problem based on a domain decomposition technique on a fictitious mirror-like boundary conditions. We conduct BSP simulations to produce synthetic data and use it to evaluate the accuracy of the inverse problem solution.

Key words: Cardiac electrophysiology, Inverse problem, Bidomain equations, Heart-torso coupling, Finite element method , domain decomposition

1 Introduction

The inverse problem in cardiac electrophysiology also known as electrocardiography imaging (ECGI) is a new and a powerful diagnosis technique. It allows the reconstruction of the electrical potential on the heart surface from electrical potentials measured on the body surface. This non-invasive technology and other similar techniques like the electroencephalography imaging interest more and more medical industries. The success of these technology would be considered as a breakthrough in the cardiac and brain diagnosis. However, in many cases the quality of reconstructed electrical potential is not sufficiently accurate. The difficulty comes from the fact that the inverse problem in cardiac electrophysiology is well known as a mathematically ill-posed problem. Different methods based on Thikhnov regularization Ghosh and Rudy [2009] have been used in order to regularize the problem, but still the reconstructed

INRIA Bordeaux Sud-Ouest `nejib.zemzemi@inria.fr`

electrical potential is not sufficiently satisfactory. In this study we present a domain decomposition approach to solve the inverse problem.

2 Methods

The domain decomposition method that will be presented in this paper is tested on synthetical data. This data is generated by solving the forward problem of ECGs. In the following paragraphs we will present first, the forward problem then the domain decomposition method for solving the inverse problem.

2.1 Forward problem

The bidomain equations were used to simulate the electrical activity of the heart and extracellular potentials in the whole body (see *e.g.* Pullan et al. [2005], Sundnes et al. [2006], Tung [1978]). These equations in the heart domain $\Omega_{\rm H}$ are given by:

$$\begin{cases} A_{\rm m}\big(C_{\rm m}\dot{V}_{\rm m} + I_{\rm ion}(V_{\rm m},\boldsymbol{w})\big) - \operatorname{div}\big(\boldsymbol{\sigma}_{\rm i}\boldsymbol{\nabla} V_{\rm m}\big) & \\ \qquad\qquad = \operatorname{div}\big(\boldsymbol{\sigma}_{\rm i}\boldsymbol{\nabla} u_{\rm e}\big) + I_{\rm stim}, & \text{in } \Omega_{\rm H}, \\ -\operatorname{div}\big((\boldsymbol{\sigma}_{\rm i}+\boldsymbol{\sigma}_{\rm e})\boldsymbol{\nabla} u_{\rm e}\big) = \operatorname{div}(\boldsymbol{\sigma}_{\rm i}\boldsymbol{\nabla} V_{\rm m}), & \text{in } \Omega_{\rm H}, \\ \dot{\boldsymbol{w}} + \boldsymbol{g}(V_{\rm m},\boldsymbol{w}) = 0, & \text{in } \Omega_{\rm H}, \\ \boldsymbol{\sigma}_{\rm i}\boldsymbol{\nabla} V_{\rm m}\cdot\boldsymbol{n} = -\boldsymbol{\sigma}_{\rm i}\boldsymbol{\nabla} u_{\rm e}\cdot\boldsymbol{n}, & \text{on } \Sigma. \end{cases} \tag{1}$$

The state variables $V_{\rm m}$ and $u_{\rm e}$ stand for the transmembrane and the extracellular potentials. Constants $A_{\rm m}$ and $C_{\rm m}$ represent the rate of membrane surface per unit of volume and the membrane capacitance, respectively. $I_{\rm stim}$ and $I_{\rm ion}$ are the stimulation and the transmembrane ionic currents. The heart-torso interface is denoted by Σ. The intra- and extracellular (anisotropic) conductivity tensors, $\boldsymbol{\sigma}_{\rm i}$ and $\boldsymbol{\sigma}_{\rm e}$, are given by $\boldsymbol{\sigma}_{{\rm i},e} = \sigma^{\rm t}_{{\rm i},e}\boldsymbol{I} + (\sigma^{\rm l}_{{\rm i},e} - \sigma^{\rm t}_{{\rm i},e})\boldsymbol{a}\otimes\boldsymbol{a}$, where $\boldsymbol{a}$ is a unit vector parallel to the local fibre direction and $\sigma^{\rm l}_{{\rm i},e}$ and $\sigma^{\rm t}_{{\rm i},e}$ are, respectively, the longitudinal and transverse conductivities of the intra- and extra-cellular media. The field of variables $\boldsymbol{w}$ is a vector containing different chemical concentrations and various gate variables. Its time derivative is given by the vector of functions $\boldsymbol{g}$.

The precise definition of $\boldsymbol{g}$ and $I_{\rm ion}$ depend on the electrophysiological transmembrane ionic model. In the present work we make use of one of the biophysically detailed human ventricular myocyte model Ten Tusscher and Panfilov [2006]. The ion channels and transporters have been modeled on the basis of the most recent experimental data from human ventricular myocytes.

Figure 1 provides a geometrical representation of the domains considered to compute extracellular potentials in the human body. In the torso domain Ω_{T}, the electrical potential u_{T} is described by the Laplace equation.

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} u_{\mathrm{T}}) = 0, & \text{in} \quad \Omega_{\mathrm{T}}, \\ \boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} u_{\mathrm{T}} \cdot \boldsymbol{n}_{\mathrm{T}} = 0, & \text{on} \quad \Gamma_{\mathrm{ext}}. \end{cases} \tag{2}$$

where $\boldsymbol{\sigma}_{\mathrm{T}}$ stands for the torso conductivity tensor and $\boldsymbol{n}_{\mathrm{T}}$ is the outward unit normal to the torso external boundary Γ_{ext}.

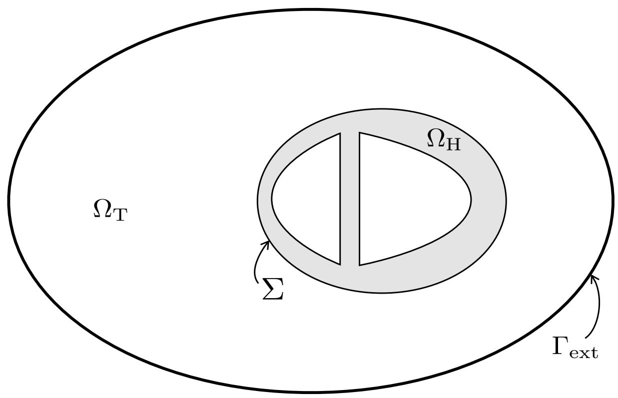

Fig. 1 Two-dimensional geometrical description: heart domain Ω_{H}, torso domain Ω_{T} (extramyocardial regions), heart-torso interface Σ and torso external boundary Γ_{ext}.

The heart-torso interface Σ is supposed to be a perfect conductor. Then we have a continuity of current and potential between the extra-cellular myocardial region and the torso region.

$$\begin{cases} u_{\mathrm{e}} = u_{\mathrm{T}}, & \text{on} \quad \Sigma, \\ (\boldsymbol{\sigma}_{\mathrm{e}} + \boldsymbol{\sigma}_{\mathrm{i}}) \boldsymbol{\nabla} u_{\mathrm{e}} \cdot \boldsymbol{n}_{\mathrm{T}} = \boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} u_{\mathrm{T}} \cdot \boldsymbol{n}_{\mathrm{T}}, & \text{on} \quad \Gamma_{\mathrm{ext}}. \end{cases} \tag{3}$$

Other works Clements et al. [2004], Potse et al. [2003], Lines et al. [2003] consider that the electrical current does not flow from the heart to the torso by assuming that the heart is isolated from torso. This approximation is appealing in terms of computational cost because it uncouples the Laplace equation (2) in the torso from the bidomain equations in the heart (1), which allows to reduce the size of the linear system to solve. It is even more appealing when the interest is only on the ECG computation, in that case the ECG solution could be an "off line" matrix vector multiplication after solving the bidomain equation, details about computing the transfer matrix could be found in Zemzemi [2009]. Although this approach is very appealing in terms of computational cost, numerical evidence has shown that it can compromise the accuracy of the ECG signals (see e.g. Lines et al. [2003], Pullan et al. [2005], Boulakia et al. [2010]). Thus, in order to accurately compute ECGs we consider the state-of-the-art heart-torso full coupled electrophysiological problem (1)-(3) representing the cardiac electrical activity from the cell to the human body surface.

2.2 Inverse problem

The inverse problem in electrocadiography imaging (ECGI) is a technique that allows to construct the electrical potential on the heart surface Σ from data measured on the body surface Γ_{ext}. We assume that the electrical potential is governed by the diffusion equation in the torso as shown in the previous paragraph. For a given potential data d measured on the body surface Γ_{ext}, the goal is to find the extracellular heart potential u_{e} on the heart boundary Σ such that the electrical potential in the torso domain u_{T} satisfies

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma}_{\text{T}} \boldsymbol{\nabla} u_{\text{T}}) = 0, \text{ in } \Omega_{\text{T}}, \\ u_{\text{T}} = d \text{ and } \boldsymbol{\sigma}_{\text{T}} \boldsymbol{\nabla} u_{\text{T}}.\boldsymbol{n} = 0, \text{ on } \Gamma_{\text{ext}}, \\ u_{\text{T}} = u_{\text{e}}, \text{ on } \Sigma. \end{cases} \tag{4}$$

In order to find u_{T} and the flux $\boldsymbol{\sigma}_{\text{T}} \boldsymbol{\nabla} u_{\text{T}}.\boldsymbol{n}$ on the boundary Σ we propose to decompose the problem into two auxiliary problems based on a mirror like boundary conditions in a fictitious domain as shown in Figure 2. We consider $-\Omega_{\text{T}}$ (respectively, $-\Gamma_{\text{ext}}$) as the image of Ω_{T} (respectively, Γ_{ext}) through the interface boundary Σ. Then, we propose to define two functions u and v

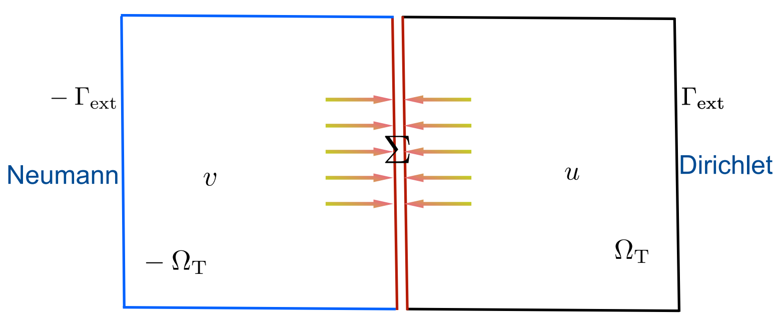

Fig. 2 Two-dimensional geometrical description of the physical torso domain Ω_{T} and its image $(-\Omega_{\text{T}})$ through the boundary Σ. The arrows on the interface Σ show the opposite fluxes $\boldsymbol{\sigma}_{\text{T}} \boldsymbol{\nabla} u.\boldsymbol{n} = -\boldsymbol{\sigma}_{\text{T}} \boldsymbol{\nabla} v.\boldsymbol{n}$.

as follows,

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} u = 0, \text{ in } \Omega_{\mathrm{T}}, \\ u = d, \text{ on } \Gamma_{\mathrm{ext}}, \\ \boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} u . \boldsymbol{n} = -\boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} v . \boldsymbol{n}, \text{ on } \Sigma. \end{cases} \tag{5}$$

The function v is defined in the fictitious domain that we denoted $-\Omega_{\mathrm{T}}$ as shown in Figure 2.

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} v) = 0, \text{ in } -\Omega_{\mathrm{T}}, \\ \boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} v . \boldsymbol{n} = 0, \text{ on } -\Gamma_{\mathrm{ext}}, \\ v = u, \text{ on } \Sigma. \end{cases} \tag{6}$$

In order to solve the coupled problem (5)-(6), we use the domain decomposition technique. For a given initial guess $u^0 = 0$, we compute the incomplete boundary condition following the algorithm 1.

Algorithm 1

for $i = \textit{first time step}$ to *last time step* **do**
- $u^0 = 0$, on Γ_{ext}.
- load the known boundary data $u_{\mathrm{T}}(t_i)_{/\Gamma_{\mathrm{ext}}}$
- load the known boundary flux $\boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} u_{\mathrm{T}}(t_i)_{/\Gamma_{\mathrm{ext}}} . \boldsymbol{n}$
- error = 2 x tolerence.
- **while** error > tolerence) **do**
  - Set $v^{k+1} = u^k$ on Σ
  - compute v^{k+1} in $-\Omega_{\mathrm{T}}$
  - Set $\boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} u^{k+1} . \boldsymbol{n} = -\boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} v^{k+1} . \boldsymbol{n}$ on Σ
  - compute u^{k+1} in Ω_{T}
  - compute error and relative error: norm($u^{k+1} - u^k$)
- **end while**
- save $u_{\mathrm{T}}(t_i)_{/\Sigma} = u^{k+1}_{/\Sigma}$

end for

The domain decomposition algorithm described in Algorithm 1 is accelerated using Aitken algorithm Irons and Tuck [1969]. At the end of the while loop we have that $v^{k+1} = u^{k+1}$ up to the defined tolerance and $\boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} u^{k+1} . \boldsymbol{n} = -\boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} v^{k+1} . \boldsymbol{n}$ on Σ. If we define $\bar{v}^{k+1}$ as the symmetrical image of v^{k+1} through the interface Σ, we obtain $\boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} u^{k+1} . \boldsymbol{n} = \boldsymbol{\sigma}_{\mathrm{T}} \boldsymbol{\nabla} \bar{v}^{k+1} . \boldsymbol{n}$ and $u^{k+1} = \bar{v}^{k+1}$ on Σ. Following Ben Belgacem and El Fekih [2005], Zemzemi [2013], this allows us to conclude that $u^{k+1} = \bar{v}^{k+1}$ in the whole torso domain Ω_{T}. Consequently, u^{k+1} satisfies both Dirichlet and Neumann boundary conditions on the external boundary Γ_{ext} and the Laplace equation in the torso domain, which solves the inverse problem.

3 Numerical results

In order to test the domain decomposition approach for solving the inverse problem in electrocardiography, we generate synthetical data. We conducted numerical simulations solving the forward problem. We use the finite element LifeV [1] library for the numerical implementation of the method. For the sake of simplicity, we perform simulations on the volumes between three concentric spheres. The volume between the small and the medium sphere represents the heart domain and the volume between the medium and the large spheres represents the torso. We show the torso potential in Figure 3.The epicardium Σ is given by the internal sphere in Figure 3 while the body surface Γ_{ext} is given by the external sphere in the same figure. In Figure 3 (top) we show the

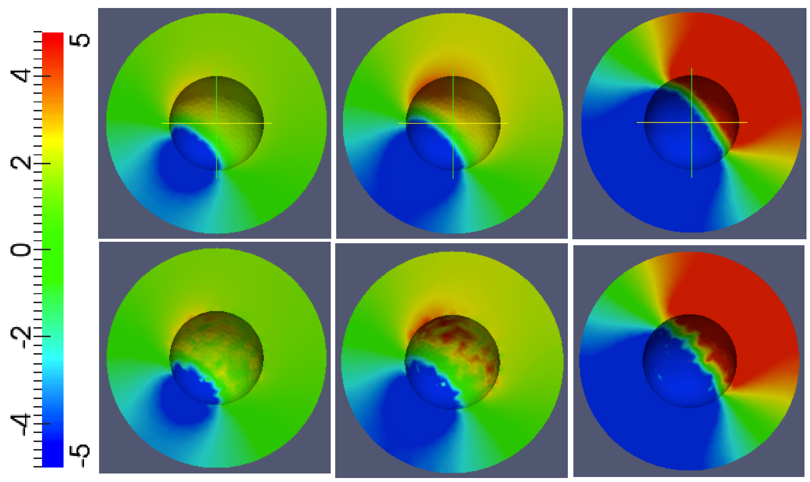

Fig. 3 Top (respectively, bottom): Snapshots of numerical solution of the forward (respectively, inverse) problem at times 5, 10 and 25 ms (from left to right). The unit in the color bar is mV.

forward problem solution at times 5, 10 and 25 ms (from left to right). We extract the electrical potential at the boundary Γ_{ext} and we use it as an input for the inverse problem after adding 10% of white noise. We show the solution of the inverse problem in Figure 3 (bottom). The three panels represent the inverse problem solution at times 5, 10 and 25 ms (from left to right). We observe that the space distribution of the electrical wave is well captured mainly in the internal sphere. This allows to construct the activation times on the heart surface with a good accuracy.

[1] www.lifev.org

4 Conclusion

In this paper we presented a domain decomposition approach to solve the inverse problem in electrocardiography. The problem was formulated using a mirror-like boundary conditions at the heart torso interface, where we have continuity of potential and opposite fluxes. The preliminary results presented in this work show the capability of this method to capture the spatial distribution of the electrical wave. In particular the wave front is well captured even with a relatively high level of noise. In future work we will test this method on clinical data including CT-scans of real geometry of the torso and measurements of the electrical potential on the body surface.

References

F. Ben Belgacem and H. El Fekih. On cauchy's problem: I. a variational steklov–poincaré theory. *Inverse Problems*, 21(6):1915, 2005.

M. Boulakia, S. Cazeau, M.A. Fernández, J.F. Gerbeau, and N. Zemzemi. Mathematical modeling of electrocardiograms: a numerical study. *Annals of biomedical engineering*, 38(3):1071–1097, 2010. ISSN 0090-6964.

J. Clements, J. Nenonen, P.K.J. Li, and B.M. Horacek. Acivation dynamics in anisotropic cardiac tissue via decoupling. *Annals of Biomedical Engineering*, 2(32):984–990, 2004.

S. Ghosh and Y Rudy. Application of l1-norm regularization to epicardial potential solution of the inverse electrocardiography problem. *Annals of Biomedical Engineering*, 37(5):902912, 2009.

B. Irons and R.C. Tuck. A version of the aitken accelerator for computer implementation. *Int. J. Num. Meth. Engng.*, 1:275–277, 1969.

G. T. Lines, M. L. Buist, P. Grottum, A. J. Pullan, J. Sundnes, and A. Tveito. Mathematical models and numerical methods for the forward problem in cardiac electrophysiology. *Comput. Visual. Sci.*, 5(4):215–239, 2003.

M. Potse, B. Dubé, and M. Gulrajani. ECG simulations with realistic human membrane, heart, and torso models. In *Proceedings of the 25th Annual Intemational Conference of the IEEE EMBS*, pages 70–73, 2003.

A.J. Pullan, M.L. Buist, and L.K. Cheng. *Mathematically modelling the electrical activity of the heart. From cell to body surface and back again.* World Scientific, 2005.

J. Sundnes, G.T. Lines, X. Cai, B.F. Nielsen, K.-A. Mardal, and A. Tveito. *Computing the electrical activity in the heart.* Springer-Verlag, 2006.

K. Ten Tusscher and AV Panfilov. Cell model for efficient simulation of wave propagation in human ventricular tissue under normal and pathological conditions. *Physics in medicine and biology*, 51:6141, 2006.

L. Tung. *A bi-domain model for describing ischemic myocardial D–C potentials.* PhD thesis, MIT, 1978.

N. Zemzemi. *Étude théorique et numérique de l'activité électrique du cœur: Applications aux électrocardiogrammes*. PhD thesis, Université Paris XI, 2009. http://tel.archives-ouvertes.fr/tel-00470375/en/.

N. Zemzemi. A steklov-poincaré approach to solve the inverse problem in electrocardiography. In *Computing in Cardiology Conference (CinC), 2013*, pages 703–706. IEEE, 2013.