On the Relation between Optimized Schwarz Methods and Source Transfer

Zhiming Chen[1] Martin J. Gander[2] and Hui Zhang[2]

1 Introduction

Optimized Schwarz methods (OS) use Robin or higher order transmission conditions instead of the classical Dirichlet ones. An optimal Schwarz method for a general second-order elliptic problem and a decomposition into strips was presented in [13]. Here optimality means that the method converges in a finite number of steps, and this was achieved by replacing in the transmission conditions the higher order operator by the subdomain exterior Dirichlet-to-Neumann (DtN) maps. It is even possible to design an optimal Schwarz method that converges in two steps for an arbitrary decomposition and an arbitrary partial differential equation (PDE), see [6], but such algorithms are not practical, because the operators involved are highly non-local. Substantial research was therefore devoted to approximate these optimal transmission conditions, see for example the early reference [11], or the overview [5] which coined the term 'optimized Schwarz method', and references therein. In particular for the Helmholtz equation, [7] presents optimized second-order approximations of the DtN, [17] (improperly) and [14] (properly) tried for the first time using perfectly matched layers (PML, see [1]) to approximate the DtN in OS.

The DtN map arises also naturally in the analytic factorization of partial differential operators. This has been identified by [8] with the Schur complement occurring in the block LU factorization of block tridiagonal matrices, which led to analytic incomplete LU (AILU) preconditioners. The AILU preconditioners consist of one forward and one backward sweep corresponding to block 'L' and 'U' solves. In particular, second-order differential approximations of the DtN were studied by [9] for AILU for the Helmholtz equation.

LSEC, Institute of Computational Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100190, China zmchen@lsec.cc.ac.cn · Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, Case postale 64, 1211 Genève 4, Switzerland {martin.gander, hui.zhang}@unige.ch

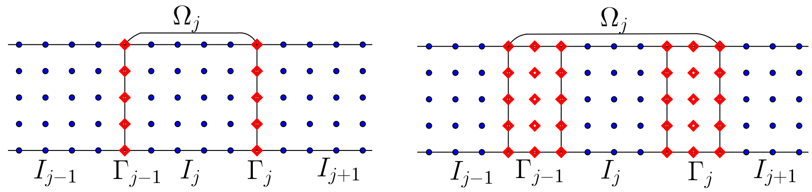

Fig. 1 Non-overlapping and overlapping domain decomposition into strips.

The connection between the DtN and the block LU factorization was rediscovered in [4], where a PML approximation of the DtN was used to improve the AILU preconditioners, and this has quickly inspired more research: [16] showed a "rapidly converging" domain decomposition method (DDM) based on sweeps, [2, 3] presented and analyzed the source transfer DDM (STDDM), and [10] proposed to use the sweeping process to accelerate Jacobi-type optimized Schwarz methods. All these new algorithms use PML but apparently in different formulations. In order to show their tight connection, we present here the relation between STDDM and OS. Such close connections also exist between OS and AILU, the sweeping preconditioner, and the method in [16], but these results, as well as the corresponding discrete formulations will appear elsewhere.

2 Algorithms and Equivalence

We consider a linear second order PDE of the form

$$\mathcal{L}u = f \text{ in } \Omega, \quad \mathcal{B}u = g \text{ on } \partial\Omega, \tag{1}$$

where Ω could either be $\mathbb{R}^d$, or a truncated domain padded with PML, in which case we consider the PML region as part of the domain. We decompose Ω into either overlapping or non-overlapping strips (or slices in higher dimensions) called subdomains Ω_j, $j = 1, \ldots, J$, which are in turn decomposed into boundary layers (overlaps) that are shared with neighboring subdomains, and non-shared interior, i.e. $\Omega_j = \Gamma_{j-1} \cup I_j \cup \Gamma_j$, see Fig. 1 for examples.

We start by introducing the optimized Schwarz method of symmetric Gauss-Seidel type (OS-SGS) for the strip decomposition we consider here, see also [12]. This method is based on subdomain solves that are performed first by sweeping forward across the subdomains, and then backward, a technique often used in the linear algebra community to render a Gauss-Seidel preconditioner symmetric. We then rewrite the OS-SGS method in residual correction form, in order to show how closely related it is to the STDDM from [2, 3]. All our formulations are at the continuous level, but one can also develop the corresponding discrete variants.

In OS-SGS (see below), $\mathcal{S}_j$, $\widetilde{\mathcal{S}}_j$ and $\mathcal{T}_j$, $\widetilde{\mathcal{T}}_j$, are tangential operators on the left and right interfaces of Ω_j which need to ensure well-posedness of the subdomain problems. Note that on Ω_1 and Ω_J we did for simplicity not specify the modification due to the physical boundary there. If $\mathcal{T}_1 = \widetilde{\mathcal{T}}_1$, then $v_1^{(n+1)} = u_1^{(n)}$, because the subdomain problems solved coincide, and so we need only to solve one of them. Even if $\mathcal{T}_1 \neq \widetilde{\mathcal{T}}_1$, $u_1^{(n)}$ is not necessary for iteration $(n+1)$, only to complete iteration (n).

OS-SGS (interface transmission form)

Forward sweep: given $(u_j^{(n-1)})_{j=1}^J$ on $(\Omega_j)_{j=1}^J$ at iteration step $(n-1)$, solve successively for $j = 1, \dots, J-1$ the subdomain problems

$$\begin{aligned}
\mathcal{L}\, v_j^{(n)} &= f \quad \text{in } \Omega_j,\\
\mathcal{B}\, v_j^{(n)} &= g \quad \text{on } \partial\Omega \cap \partial\Omega_j,\\
(\tfrac{\partial}{\partial \mathbf{n}_j} + \mathcal{S}_j)(v_j^{(n)} - v_{j-1}^{(n)}) &= 0 \quad \text{on } \partial\Omega_j \cap \Omega_{j-1},\\
(\tfrac{\partial}{\partial \mathbf{n}_j} + \mathcal{T}_j)(v_j^{(n)} - u_{j+1}^{(n-1)}) &= 0 \quad \text{on } \partial\Omega_j \cap \Omega_{j+1}.
\end{aligned} \tag{2}$$

Backward sweep: solve successively for $j = J, \dots, 1$ the subdomain problems

$$\begin{aligned}
\mathcal{L}\, u_j^{(n)} &= f \quad \text{in } \Omega_j,\\
\mathcal{B}\, u_j^{(n)} &= g \quad \text{on } \partial\Omega \cap \partial\Omega_j,\\
(\tfrac{\partial}{\partial \mathbf{n}_j} + \widetilde{\mathcal{S}}_j)(u_j^{(n)} - v_{j-1}^{(n)}) &= 0 \quad \text{on } \partial\Omega_j \cap \Omega_{j-1},\\
(\tfrac{\partial}{\partial \mathbf{n}_j} + \widetilde{\mathcal{T}}_j)(u_j^{(n)} - u_{j+1}^{(n)}) &= 0 \quad \text{on } \partial\Omega_j \cap \Omega_{j+1}.
\end{aligned}$$

Definition 1. The Dirichlet to Neumann (DtN) map exterior to Ω_j is

$$\mathrm{DtN}_j^c: \; g_D \to g_N = \partial_n v, \text{ s.t.} \qquad \begin{aligned} \mathcal{L}\, v &= 0, && \text{in } \Omega \backslash \Omega_j,\\ \mathcal{B}\, v &= 0, && \text{on } \partial\Omega_j \cap \partial\Omega,\\ v &= g_D, && \text{on } \partial\Omega_j \backslash \partial\Omega. \end{aligned}$$

The optimal choice for the transmission conditions in the optimal Schwarz method is to use the DtN, see [13]. We show here that it suffices to choose for the tangential operators $\mathcal{S}_j$ and $\widetilde{\mathcal{S}}_j$ the DtN operators, independent of what one uses for $\mathcal{T}_j$ and $\widetilde{\mathcal{T}}_j$, to get an optimal result:

Theorem 1. *If $\mathcal{S}_j = \widetilde{\mathcal{S}}_j$, $j = 1, \dots, J-1$ correspond to the DtN maps exterior to Ω_j restricted to $\partial\Omega_j \cap \Omega_{j-1}$, and all the subdomain problems have a unique solution, then OS-SGS converges in one iteration for any initial guess. In particular, convergence is independent of the number of subdomains.*

This result can either be proved following the arguments in [13] using the error equations, or by the approach in [6] at the discrete level or in [2] at the continuous level to substitute exterior source terms with transmission data represented by subdomain solutions. We omit the details here.

In optimized Schwarz methods, one replaces the DtN with an approximation, for example an absorbing boundary condition, or a PML. For the latter, we define the approximation DtN_j^L by

$$\mathrm{DtN}_j^L:\ g_D \to g_N = \partial_n v,\ \text{s.t.}\quad \begin{aligned} \widetilde{\mathcal{L}}\, v &= 0, && \text{in } \Omega_j^L,\\ \mathcal{B}\, v &= 0, && \text{on } \partial\Omega_j \cap \partial\Omega,\\ v &= g_D, && \text{on } \partial\Omega_j \backslash \partial\Omega, \end{aligned}$$

where Ω_j^L is the PML region exterior to Ω_j and $\widetilde{\mathcal{L}}$ is chosen such that DtN_j^L closely approximates DtN_j^c from Definition 1. We notice that if $\mathcal{S}_j = \mathrm{DtN}_j^L$, the subdomain problem (2) is equivalent to solve one PDE in $\Omega_j \cup \Omega_j^L$ with the Dirichlet and Neumann traces continuous across $\partial\Omega_j^L \cap \partial\Omega_j$.

OS-SGS in the interface transmission form requires the evaluation of operators on data, such as $(\partial_{\mathbf{n_j}} + \mathcal{S}_j) v_{j-1}^{(n-1)}$, which can be inconvenient, especially if $\mathcal{S}_j$ is complicated. This can be avoided if we solve for the corrections. To this end, we introduce in the forward sweep $\delta v_j^{(n)} := v_j^{(n)} - \tilde{v}_j^{(n-1)}$ for some $\tilde{v}_j^{(n-1)}$ which has the same Dirichlet and Neumann traces as $v_{j-1}^{(n-1)}$ on $\partial\Omega_j \cap \Omega_{j-1}$ and $u_{j+1}^{(n-1)}$ on $\partial\Omega_j \cap \Omega_{j+1}$. For example, following [2] (see also [15] at the discrete level), we introduce the weighting functions α_j and β_j such that

$$\frac{\partial}{\partial \mathbf{n}_j}\alpha_j = 0, \alpha_j = 1 \text{ on } \partial\Omega_j \cap \Omega_{j-1}, \quad \frac{\partial}{\partial \mathbf{n}_j}\beta_j = 0, \beta_j = 1 \text{ on } \partial\Omega_j \cap \Omega_{j+1}. \tag{3}$$

Then, we can define the auxiliary function $\tilde{v}_j^{(n-1)}$ as

$$\tilde{v}_j^{(n-1)} = \begin{cases} \alpha_j v_{j-1}^{(n)} + (1-\alpha_j) w_j^{(n-1)}, & \text{on } \Gamma_{j-1},\\ w_j^{(n-1)}, & \text{in } I_j,\\ \beta_j u_{j+1}^{(n-1)} + (1-\beta_j) w_j^{(n-1)}, & \text{on } \Gamma_j, \end{cases} \tag{4}$$

where $w_j^{(n-1)}$ is an arbitrary function. One can verify that

$$\begin{aligned} \tfrac{\partial}{\partial \mathbf{n}_j}\tilde{v}_j^{(n-1)} &= \tfrac{\partial}{\partial \mathbf{n}_j} v_{j-1}^{(n)}, & \tilde{v}_j^{(n-1)} &= v_{j-1}^{(n)} \text{ on } \partial\Omega_j \cap \Omega_{j-1},\\ \tfrac{\partial}{\partial \mathbf{n}_j}\tilde{v}_j^{(n-1)} &= \tfrac{\partial}{\partial \mathbf{n}_j} u_{j+1}^{(n-1)}, & \tilde{v}_j^{(n-1)} &= u_{j+1}^{(n-1)} \text{ on } \partial\Omega_j \cap \Omega_{j+1}, \end{aligned}$$

which together with (2) imply

$$\begin{aligned} (\tfrac{\partial}{\partial \mathbf{n}_j} + \mathcal{S}_j)(v_j^{(n)} - \tilde{v}_j^{(n-1)}) &= 0 \ \text{ on } \partial\Omega_j \cap \Omega_{j-1},\\ (\tfrac{\partial}{\partial \mathbf{n}_j} + \mathcal{T}_j)(v_j^{(n)} - \tilde{v}_j^{(n-1)}) &= 0 \ \text{ on } \partial\Omega_j \cap \Omega_{j+1}. \end{aligned}$$

Similar identities also hold for the backward sweep. Therefore, the OS-SGS algorithm in interface transmission form can equivalently be written in the residual–correction form (see below).

Remark 1. Usually one uses the subdomain iterates for defining $\tilde{v}_j^{(n-1)}$ and $\tilde{u}_j^{(n-1)}$, e.g. $w_j^{(n-1)} := u_j^{(n-1)}$ in (4), thus gluing the subdomain solutions together to obtain a global approximation. If the weighting functions $\{\beta_j\}$ for the gluing are the indicator functions of the corresponding non-overlapping partition, we obtain the so called *restricted* Schwarz methods; other choices give the same subdomain iterates but only different global iterates.

OS-SGS (residual–correction form)

Forward sweep: given $(u_j^{(n-1)})_{j=1}^J$ on $(\Omega_j)_{j=1}^J$ at iteration $(n-1)$, solve successively for $j = 1, \ldots, J-1$ the subdomain problems

$$\begin{aligned} \mathcal{L}\,\delta v_j^{(n)} &= f - \mathcal{L}\,\tilde{v}_j^{(n-1)} && \text{in } \Omega_j, \\ \mathcal{B}\,\delta v_j^{(n)} &= g - \mathcal{B}\,\tilde{v}_j^{(n-1)} && \text{on } \partial\Omega \cap \partial\Omega_j, \\ (\tfrac{\partial}{\partial \mathbf{n}_j} + \mathcal{S}_j)\delta v_j^{(n)} &= 0 && \text{on } \partial\Omega_j \cap \Omega_{j-1}, \\ (\tfrac{\partial}{\partial \mathbf{n}_j} + \mathcal{T}_j)\delta v_j^{(n)} &= 0 && \text{on } \partial\Omega_j \cap \Omega_{j+1}, \end{aligned}$$

each followed by letting $v_j^{(n)} \leftarrow \tilde{v}_j^{(n-1)} + \delta v_j^{(n)}$ and setting $\tilde{v}_{j+1}^{(n-1)}$ as in (4).

Backward sweep: solve successively for $j = J, \ldots, 1$ the subdomain problems

$$\begin{aligned} \mathcal{L}\,\delta u_j^{(n)} &= f - \mathcal{L}\tilde{u}_j^{(n)} && \text{in } \Omega_j, \\ \mathcal{B}\,\delta u_j^{(n)} &= g - \mathcal{B}\tilde{u}_j^{(n)} && \text{on } \partial\Omega \cap \partial\Omega_j, \\ (\tfrac{\partial}{\partial \mathbf{n}_j} + \widetilde{\mathcal{S}}_j)\delta u_j^{(n)} &= 0 && \text{on } \partial\Omega_j \cap \Omega_{j-1}, \\ (\tfrac{\partial}{\partial \mathbf{n}_j} + \widetilde{\mathcal{T}}_j)\delta u_j^{(n)} &= 0 && \text{on } \partial\Omega_j \cap \Omega_{j+1}, \end{aligned}$$

each followed by setting $u_j^{(n)} \leftarrow \tilde{u}_j^{(n-1)} + \delta u_j^{(n)}$ and setting $\tilde{u}_{j-1}^{(n-1)}$ as in (4).

Theorem 2. *The source transfer domain decomposition method defined in [2] is an overlapping optimized Schwarz method of symmetric Gauss-Seidel type, with the overlap covering half the subdomains, and using PML transmission conditions on the left and right interfaces in the forward sweep and Dirichlet instead of PML on the right interfaces in the backward sweep. In addition, the source terms are consistently modified in the forward sweep.*

Proof. As we have seen for OS-SGS, the residual–correction form is equivalent to the interface transmission form. The only difference of STDDM from the residual–correction form of OS-SGS is that in the forward sweep the residual for $1 \le j \le J-1$ in the overlap with the right neighbor is set to zero, see ALGORITHM 3.1 in [2]. This modification can also be interpreted as taking the boundary layer Γ_j as part of the PML on the right of the subdomain so the physical subdomains become effectively non-overlapping.

To see the consistency of STDDM, we assume $u_j^{(n-1)}$ is equal to the exact solution of the original problem in Ω_j for $1 \le j \le J$ and check whether $u_j^{(n)} =$

$u_j^{(n-1)}$ holds, i.e. the exact solution is a fixed point of the iteration. We note that STDDM uses $w_j^{(n-1)} = u_j^{(n-1)}$ in (4). In this case, by the assumption on $u_1^{(n-1)}$ and $u_2^{(n-1)}$, we can show $\tilde{v}_1^{(n-1)} = u_1^{(n-1)}$ and so the residual vanishes in Ω_1 both for OS-SGS and STDDM. Therefore, the correction $\delta v_1^{(n)} = v_1^{(n)} - \tilde{v}_1^{(n-1)}$ must be zero because the sub-problem has a unique solution, which gives $v_1^{(n)} = \tilde{v}_1^{(n-1)} = u_1^{(n-1)}$. By induction, we then show that $u_j^{(n)} = u_j^{(n-1)}$ for $1 \le j \le J$. $\square$

3 Numerical Experiments

We solve the Helmholtz equation in rectangles discretized by Q1 finite elements. For the free space and open cavity problems, the wave speed is constant, $c = 1$, and the point source is at (0.5177,0.6177) while the Marmousi model problem has a variable wave speed and the point source at (6100,2200). PML are padded around all the domains except for the open cavity problem, where homogeneous Neumann conditions are imposed at the top and bottom. We use the same depth (counted with mesh elements) of PML for the original domain and the subdomains since already for a PML with two layers the dominating error is around the point source. The PML complex stretching function we use is given by $s(d) = \frac{1}{1 - \mathrm{i}4\pi d^2/(L^3 k)}$ where $k = \omega/c$ is the wavenumber, d is the distance to the physical boundary and L is the geometric depth of the PML. We use the same mesh size and element-wise constant material coefficients in the physical and PML regions. We use a zero initial guess for GMRES with relative residual (preconditioned) tolerance 10^{-6}. The results are shown in Table 1 where 'STDDM2' is the STDDM without changing transmission from PML to Dirichlet in the backward sweep, 'PMLh' represents OS-SGS with two elements overlap and PML on all boundaries, 'TO2h' ('TO0h') is the OS-SGS with the Taylor second- (zero-) order transmission conditions and two elements overlap. The optimized transmission conditions from [7] are also tested with overlap and the results for the optimized condition of second-order are listed (the original boundaries still use Taylor second-order conditions) under the name 'O2h'. The optimized condition of zero-order suffers from too many subdomains and can not converge to the correct solution in all cases. We implemented all the algorithms in the residual–correction form. We also tested the classical Schwarz method of symmetric Gauss-Seidel type with Dirichlet transmission conditions but the preconditioned system is very ill-conditioned so that the obtained solution comprises a significant error even if the preconditioned residual is reduced by the tolerance factor. The same failure happens in Table 1 indicated by middle bars. From the table, we find that, for our particular test problems with open boundaries on both left and right sides, STDDM2 which uses always PML

free space problem on the unit square																
$\frac{\omega}{2\pi}$	$\frac{1}{10h}$	$\frac{J}{10}$	STDDM			STDDM2			PMLh			TO2h	TO0h	O2h		
			it	m	nx	it	m	nx	it	m	nx	it	it	it	m	nx
20	20	2	4	4	26, 22	4	2	22	4	2	16	11	12	14	0	12
20	40	4	4	7	32, 25	4	2	22	4	3	18	21	25	17	0	12
20	80	8	6	7	32, 25	4	4	26	4	7	26	63	45	–	0	12
40	40	4	6	4	26, 22	4	2	22	5	2	16	15	25	49	0	12
80	80	8	6	9	36, 27	4	3	24	6	3	18	25	89	–	0	12
160	160	16	10	22	62, 40	6	4	26	6	5	22	49	> 200	–	0	12

open cavity problem on the unit square																
$\frac{\omega}{2\pi}$	$\frac{1}{10h}$	$\frac{J}{10}$	STDDM			STDDM2			PMLh			TO2h	TO0h	O2h		
			it	m	nx	it	m	nx	it	m	nx	it	it	it	m	nx
20	20	2	11	2	22, 20	6	2	22	8	2	16	41	85	33	0	12
20	40	4	13	3	24, 22	10	2	22	12	5	22	76	216	55	0	12
20	80	8	19	5	28, 23	18	4	26	22	6	24	142	392	–	0	12
40	40	4	19	2	22, 20	10	2	22	14	2	16	123	292	119	0	12
80	80	8	—	–	—	22	2	22	30	2	16	429	—	—	0	12

Marmousi model																
$\frac{\omega}{2\pi}$	h	J	STDDM			STDDM2			PMLh			TO2h	TO0h	O2h		
			it	m	nx	it	m	nx	it	m	nx	it	it	it	m	nx
2	74	10	5	2	26, 24	4	2	26	4	2	18	9	6	7	0	14
2	37	20	6	4	30, 26	4	2	26	4	3	20	16	10	11	0	14
2	18	40	6	5	32, 27	5	2	26	5	3	20	37	17	19	0	14
4	37	20	6	3	28, 25	4	2	26	5	2	18	10	9	8	0	14
8	18	40	8	4	30, 26	5	2	26	6	2	18	12	14	18	0	14
16	9	80	10	4	30, 26	6	2	26	6	2	18	16	25	–	0	14

Table 1 Minimal depth m of the PML layer and the corresponding number nx of mesh elements for each subdomain in the x-direction to reach the given iteration number it (J is the number of subdomains. Two nx are presented for STDDM because in the backward sweep the right interfaces use Dirichlet instead of PML).

on both sides works better than STDDM which changes to Dirichlet on the right side in the backward sweep.

Acknowledgements This work was supported by the Université de Genève. HZ thanks the International Science and Technology Cooperation Program of China (2010DFA14700).

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