Smoothers for efficient multigrid methods in IGA

Clemens Hofreither, Stefan Takacs, Walter Zulehner

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JKU JOHANNES KEPLER UNIVERSITY LINZ

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Outline

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Poisson model problem

Domain $\Omega \in \mathbb{R}^d$

Given function $f \in L^2(\Omega)$

Find $u \in H^1(\Omega)$ such that

$$
\begin{aligned}
-\Delta u &= f && \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Variational formulation

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- **Maximum smoothness:** $S_{p,\ell}(\Omega) := S_{p,p-1,\ell}(\Omega)$
- **Standard B-spline basis:** $\phi_{p,\ell}^{(1)}(x), \ldots, \phi_{p,\ell}^{(m_\ell)}(x)$

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- For regular geometry mappings: multigrid for parameter domain can be used as **preconditioner**

Discretization

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Matrix-vector notation:

$$K_\ell \underline{u}_\ell = \overline{f}_\ell$$

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Multigrid method

One step of the multigrid method on grid level k applied to iterate $\underline{u}_\ell^{(0,0)} = \underline{u}_\ell^{(0)}$ and right-hand-side $\underline{f}_\ell$ to obtain $\underline{x}_\ell^{(1)}$ is given by:

- Apply ν **smoothing steps**

$$\underline{u}_\ell^{(0,m)} = \underline{u}_\ell^{(0,m-1)} + \tau \widehat{K}_\ell^{-1}(\overline{f}_\ell - K_\ell \underline{u}_\ell^{(0,m-1)})$$

for $m = 1, \ldots, \nu$.

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  If realized exactly (two-grid method):

$$\underline{u}_\ell^{(1)} = \underline{u}_\ell^{(0,\nu)} + I_{\ell-1}^{\ell} K_{\ell-1}^{-1} I_\ell^{\ell-1}(\overline{f}_\ell - K_k \underline{u}_\ell^{(0,\nu)})$$

- Standard arguments: convergence of two-grid method $\Rightarrow$ convergence of the multigrid method (W-cycle)

Intergrid transfer

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- The restriction is its transpose: $I_{\ell}^{\ell-1} = (I_{\ell-1}^{\ell})^T$

Hackbusch-like convergence analysis

Convergence of two-grid method with rate q , i.e.,

$$\|\underline{u}_\ell^* - \underline{u}_\ell^{(1)}\|_{L_\ell} \leq q \|\underline{u}_\ell^* - \underline{u}_\ell^{(0)}\|_{L_\ell},$$

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Convergence of two-grid method with rate $q = \frac{C_A C_S}{\nu}$, i.e.,

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is guaranteed by

$$\begin{aligned} \|L_\ell^{1/2} T_\ell K_\ell^{-1} L_\ell^{1/2}\| &\leq C_A \qquad \textbf{(approximation property)} \\ \|L_\ell^{-1/2} K_\ell S_\ell^\nu L_\ell^{-1/2}\| &\leq C_S \nu^{-1} \qquad \textbf{(smoothing property)} \end{aligned}$$

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Basis-(in)dependent smoother

- **Richardson smoother:**

$$\underline{u}_\ell^{(0,m)} = \underline{u}_\ell^{(0,m-1)} + \tau \qquad (\bar{f}_\ell - K_\ell \underline{u}_\ell^{(0,m-1)}),$$

i.e., $\widehat{K}_\ell := I$

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Lemma

Assume that $\widehat{K}_\ell := L_\ell$ and that τ is chosen such that

$$0 < \tau \leq \frac{1}{\|L_\ell^{-1/2} K_\ell L_\ell^{-1/2}\|}.$$

Then the smoothing property is satisfied for $C_S = \tau^{-1}$.

Proof: Standard eigenvalue analysis.

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Convergence rate: $q = \frac{C_A C_S}{\nu} = \frac{C_A \tau^{-1}}{\nu}$

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- Use $\widehat{K}_\ell := L_\ell := h_\ell^{-2} M_\ell$ **(mass-Richardson smoother)**

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  $\Rightarrow \sup \ldots \geq p \qquad \Rightarrow \tau \leq p^{-2} \qquad \Rightarrow \nu \geq p^2$

Inverse inequality

Theorem ([T., Takacs 2015])

For all $\ell \in \mathbb{N}_0$ *and* $p \in \mathbb{N}$,

$$|u_\ell|_{H^1} \leq 2\sqrt{3} h_\ell^{-1} \|u_\ell\|_{L^2}$$

is satisfied for all $u_\ell \in \widetilde{S}_{p,\ell}(\Omega)$, *where* $\widetilde{S}_{p,\ell}(0,1)$ *is the space of all* $u_\ell \in S_{p,\ell}(0,1)$ *whose* ***odd derivatives vanish at the boundary****:*

$$\frac{\partial^{2l+1}}{\partial x^{2l+1}} u_\ell(0) = \frac{\partial^{2l+1}}{\partial x^{2l+1}} u_\ell(1) = 0 \text{ for all } l \in \mathbb{N}_0 \text{ with } 2l+1 < p.$$

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Note: $\widetilde{S}_{p,\ell}^{\perp}(\Omega)$ has only p (for p even) or $p-1$ (for p odd) dimensions.

Idea: boundary correction

Theorem ([Hofreither, T., Zulehner 2015])

For all $\ell \in \mathbb{N}_0$ and $p \in \mathbb{N}$,

$$|u_\ell|_{H^1} \leq \sqrt{2}(1 + 4\sqrt{6}) h_\ell^{-1} (\|u_\ell\|_{L^2}^2 + |\mathcal{H}_{\Gamma,\ell} u_{\ell,\Gamma}|_{H^1}^2)^{1/2}$$

is satisfied for all $u_\ell = u_{\Gamma,\ell} + u_{I,\ell} \in S_{p,\ell}(\Omega)$, where $u_{I,\ell} \in S_{p,\ell}^{(I)}(\Omega) \subseteq \widetilde{S}_{p,\ell}(\Omega)$, $u_{\Gamma,\ell} \in (S_{p,\ell}^{(I)}(\Omega))^\perp$ and $\mathcal{H}_{\Gamma,\ell}$ is the ***discrete harmonic extension****.*

Proof: need inverse inequality on $\widetilde{S}_{p,\ell}(\Omega)$ **and** robust approximation error estimate also on $\widetilde{S}_{p,\ell}(\Omega)$ (cf. [T., Takacs 2015]).

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Reorder variables such that:

$$K_\ell = \begin{pmatrix} K_{\Gamma\Gamma,\ell} & K_{I\Gamma,\ell}^T \\ K_{I\Gamma,\ell} & K_{II,\ell} \end{pmatrix}, \qquad \underline{u}_\ell = \begin{pmatrix} \underline{u}_{\Gamma,\ell} \\ \underline{u}_{I,\ell} \end{pmatrix}$$

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$$|\mathcal{H}_{\Gamma,\ell} u_{\Gamma,\ell}|_{H^1(\Omega)} = \left\| \begin{pmatrix} \underline{u}_{\Gamma,\ell} \\ -K_{II,\ell}^{-1} K_{I\Gamma,\ell} \underline{u}_{\Gamma,\ell} \end{pmatrix} \right\|_{K_\ell} = \|\underline{u}_{\Gamma,\ell}\|_{K_{\Gamma\Gamma,\ell} - K_{I\Gamma,\ell}^T K_{II,\ell}^{-1} K_{I\Gamma,\ell}}.$$

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$\Rightarrow$ **Smoothing property for** $0 < \tau \leq 2^{-1}(1+4\sqrt{6})^{-2}$

Robust approximation error estimate

Theorem ([T., Takacs 2015])

For each $u \in H^1(\Omega)$, each $\ell \in \mathbb{N}_0$ and $p \in \mathbb{N}$:

$$\|(I - \Pi_{p,\ell})u\|_{L^2(\Omega)} \leq 2\sqrt{2}\, h_\ell |u|_{H^1(\Omega)}$$

is satisfied, where $\Pi_{p,\ell}$ is the H^1-orthogonal projection to $S_{p,\ell}(\Omega)$.

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The **approximation property** follows using

- standard arguments (for the $h_\ell^{-2} M_\ell$ part)
- $\|\underline{u}_\ell\|_{\widetilde{K}_\ell} = |\mathcal{H}_{\Gamma,\ell} u_{\Gamma,\ell}|_{H^1(\Omega)} \le |u_\ell|_{H^1(\Omega)} = \|\underline{u}_\ell\|_{K_\ell}$ (for the $\widetilde{K}_\ell$ part)

Convergence theorem

Theorem ([Hofreither, T., Zulehner 2015])

The two-grid method converges with rate $\frac{C_A C_S}{\nu}$ if $\nu > C_A C_S$ smoothing steps are applied. The constants C_A and C_S do not depend on the grid size h_ℓ and the spline degree p.

The extension to the W-cycle multigrid method is standard.

Outline

Matrices from tensor-product splines

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$$\mathcal{M}_\ell = M_\ell \otimes M_\ell$$

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$$\mathcal{M}_\ell = M_\ell \otimes M_\ell$$

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- Tensor-product structure can be used for inverting $\mathcal{M}_\ell$, but **not** for inverting $\mathcal{K}_\ell$

Smoother for two dimensions

$$
\begin{aligned}
\widehat{\mathcal{K}}_{\ell,orig} &= \widehat{K}_\ell \otimes M_\ell + M_\ell \otimes \widehat{K}_\ell \\
&= (h_\ell^{-2} M_\ell + \widetilde{K}_\ell) \otimes M_\ell + M_\ell \otimes (h_\ell^{-2} M_\ell + \widetilde{K}_\ell) \\
&= 2h_\ell^{-2} M_\ell \otimes M_\ell + \widetilde{K}_\ell \otimes M_\ell + M_\ell \otimes \widetilde{K}_\ell \\
&\eqsim \underbrace{h_\ell^2 \widehat{K}_\ell \otimes \widehat{K}_\ell}_{\text{tensor-product}} - \underbrace{h_\ell^2 \widetilde{K}_\ell \otimes \widetilde{K}_\ell}_{\text{low-rank correction}} \\
&=: \widehat{\mathcal{K}}_\ell,
\end{aligned}
$$

where $\widehat{K}_\ell = h_\ell^{-1} M_\ell + \widetilde{K}_\ell$.

Shermann Morrisson Woodbury formula

Let $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{n\times n}$, $P \in \mathbb{R}^{n\times N}$ with full rank. Then:

$$(A - PBP^T)^{-1} = A^{-1} + A^{-1}P(B^{-1} - P^TA^{-1}P)^{-1}P^TA^{-1}$$

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- The rest lives only in the boundary layer
- **Optimal order:** multigrid solver has the **same order of complexity** as the **multiplication** with $\widehat{K}_\ell$

Convergence theorem

Theorem ([Hofreither, T., Zulehner 2015])

The two-grid method converges with rate $\frac{C_A C_S}{\nu}$ if $\nu > C_A C_S$ smoothing steps are applied. The constants C_A and C_S do not depend on the grid size h_ℓ and the spline degree p.

The extension to the W-cycle multigrid method is standard.

Outline

One dimension

p	1	2	3	4	5	6	7	8
$\ell = 10$	22	20	20	21	21	20	20	20
$\ell = 11$	23	20	20	21	20	20	20	20
$\ell = 12$	23	20	20	20	20	20	20	20

p	9	10	12	14	16	18	20
$\ell = 10$	20	20	20	18	18	17	17
$\ell = 11$	20	19	19	18	19	18	17
$\ell = 12$	20	20	19	18	18	18	18

$\nu_{pre} + \nu_{post} = 1 + 1$, $\tau = 0.14$
Stopping criterion: Euclidean norm of the initial residual is reduced by a factor of $\epsilon = 10^{-8}$.

Two dimensions

p	2	3	4	5	6	7	8	9
$\ell = 5$	82	80	75	76	76	73	72	72
$\ell = 6$	83	87	76	74	75	73	72	72

p	10	11	12	13	14	15	16
$\ell = 5$	70	71	70	68	69	69	66
$\ell = 6$	70	70	69	68	68	67	65

$\nu_{pre} + \nu_{post} = 1 + 1$, $\tau = 0.10$ (for $p \leq 3$) and $\tau = 0.11$ (for $p > 3$)
Stopping criterion: Euclidean norm of the initial residual is reduced by a factor of $\epsilon = 10^{-8}$.

Outline

Conclusions

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Thanks for your attention!