Two new enriched multiscale coarse spaces for the Additive Average Schwarz method

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1 Introduction

We propose additive Schwarz methods with spectrally enriched coarse spaces for the standard finite element discretization of second order elliptic problems with highly varying and discontinuous coefficients. Such discontinuities may occur arbitrarily both inside and across subdomains. The convergence of the proposed methods depend linearly on the mesh parameter ratio H/h, and is independent of the distribution of the coefficient in the model problem when the coarse space is large enough. For similar work on domain decomposition methods addressing such problems, we refer to Galvis and Efendiev [2010], Spillane et al. [2014] and references therein.

The present method is an extension of a classical and an almost twenty years old additive Schwarz method, also known as the additive average Schwarz method, which was first proposed and analyzed in Bjørstad et al. [1997] for problems where the coefficients are constant in each subdomain, and later analyzed for varying coefficients in Dryja and Sarkis [2010]. The condition number bound as shown in the last paper, depends quadratically on the mesh parameter ratio, and linearly on the contrast, that is the ratio between the maximum and the minimum value of the coefficient, in each subdomain boundary layer. Recently, the additive average Schwarz method has been extended to the case of Crouzeix-Raviart finite volume elements where, again, demonstrating that the method is robust with respect to coefficients varying inside the subdomain but not along the subdomain boundary; cf. Loneland et al. [2015a,b]. It is clear that, with standard coarse spaces it is hard to make an additive Schwarz method robust with respect to the contrast, unless some way of enrichment of the coarse spaces has been made.

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Additive Schwarz methods for solving elliptic problems discretized by the finite element method have been studied extensively; see Toselli and Widlund [2005] for an overview. There are now several works on the additive average Schwarz method which exist in the literature, see e.g. Bjørstad et al. [1997], Dryja and Sarkis [2010].In the present work, borrowing some of the main ideas of Bjørstad and Krzyżanowski [2002], Chartier et al. [2003], Spillane et al. [2014], Galvis and Efendiev [2010], Klawonn et al. [2015], we propose to enrich the classical coarse space of the additive average Schwarz method by using a set of eigenfunctions of specially designed generalized eigenvalue problem in each subdomain. Those functions correspond to the eigenvalues that are larger than a given threshold. The analysis shows that the condition number bounds of the enriched method depend only on the threshold and the mesh parameter ratio. So, by enriching the coarse space, we are able to make the condition number to be independent of the contrast, thereby restore the bound which is known to be true for the case of piecewise constant coefficients.

The remainder of the paper is organized as follows: in Section 2, we introduce our model problem, and the finite element discrete formulation. Section 3 describes the classical Additive Average Schwarz method. In Section 4, we propose the two locally generalized eigenvalue problems in each subdomain, and show how we use their eigenfunctions to enrich the average coarse space of the method. In Section 5, we discuss the convergence of the method with the enrichment, and present some of the numerical results in Section 6.

2 Discrete Problem

In this paper we consider the following model elliptic partial differential equation:

$$-\nabla \cdot (\alpha(x)\nabla u) = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1}$$

where Ω is a polygonal domain in $\mathbb{R}^2$ and $f \in L^2(\Omega)$.

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω consisting of closed triangle elements such that $\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$. Let h_K be the diameter of K, and define $h = \max_{K\in\mathcal{T}_h} h_K$ as the largest diameter of the triangles $K \in \mathcal{T}_h$. We assume that there exists a nonoverlapping partitioning of Ω into open and connected Lipschitz polytopes $\{\Omega_i\}$, such that $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i$, which are aligned with the fine triangulation implying that an element of $\mathcal{T}_h$ can only be contained in one of the substructures Ω_i. Each subdomain then inherits a unique local triangulation $\mathcal{T}_h(\Omega_k)$ from $\mathcal{T}_h$. We also assume that the set of these subdomains form a coarse triangulation of the domain, which is shape regular in the sense of Brenner and Sung [1999]. We define the sets of nodal points Ω_h, $\partial\Omega_h$, Ω_{ih} and $\partial\Omega_{ih}$ as the sets of vertices of the elements of $\mathcal{T}_h$ belonging to the regions Ω, $\partial\Omega$, Ω_i and $\partial\Omega_i$, respectively.

Let S_h be the standard continuous piecewise linear finite element space defined on the triangulation $\mathcal{T}_h$,

$$S_h = S_h(\Omega) := \{u \in C(\Omega) \cap H_0^1(\Omega) : v_{|K} \in P_1, \quad K \in \mathcal{T}_h\}.$$

The finite element approximation u_h of (1) is then defined as the solution to the following discrete problem: Find $u_h^* \in S_h$ such that

$$a(u_h^*, v) = (f, v)\,, \qquad \forall v \in S_h, \tag{2}$$

where $a(u,v) = \sum_{K\in\mathcal{T}_h} \int_K \alpha \nabla u \nabla v \; dx$. Through scaling we can assume that $\alpha(x) \geq 1$. Also, since ∇u and ∇v are both piecewise constant on the elements of $\mathcal{T}_h$, $a(u,v)$ restricted to each element K can be written as $\int_K \alpha \nabla u \nabla v \, dx = (\nabla u)_{|K} (\nabla v)_{|K} \int_K \alpha(x)\, dx$, and hence we can assume that α is piecewise constant on each element of $\mathcal{T}_h$.

3 The classical Additive Average Schwarz method

In this section we introduce the Additive Average Schwarz method for the discrete problem (2).

We first introduce the average coarse space. For $u \in S_h(\Omega)$, we define the average operator $I_{av} u \in S_h(\Omega)$ as

$$I_{av} u := \begin{cases} u(x), & x \in \partial\Omega_{ih}, \\ \bar{u}_i, & x \in \Omega_{ih}, \end{cases} \qquad i = 1, \ldots, N, \tag{3}$$

where

$$\bar{u}_i := \frac{1}{n_i} \sum_{x \in \partial\Omega_{i,h}} u(x). \tag{4}$$

Here, n_i is the number of nodal points on $\partial\Omega_i$, i.e., $\bar{u}_i$ is the discrete average of u over the boundary of the subdomain Ω_i.

The coarse space V_0 is defined as the image of the operator I_{av}, i.e.,

$$V_0 := Im(I_{av}). \tag{5}$$

Now, to introduce the local spaces, let $S_{h,k}$ be the restriction to $\overline{\Omega}_k$ of the function space S_h, i.e., $S_{h,k} = \{v \in C(\overline{\Omega}_k) : v_{|\tau} \in P_1, \tau \in \mathcal{T}_h(\Omega_k), v_{|\partial\Omega} = 0\}$,and the corresponding local subspace with zero boundary condition be $S_{h,k}^0 = S_{h,k} \cap H_0^1(\Omega_k)$. Then we let the local spaces V_k to be equal to $S_{h,k}^0$. We decompose the finite element space S_h into $S_h(\Omega) = V_0 + \sum_{k=1}^N V_k$.

Note that this is a direct sum of the subspaces. However, only the local spaces are a-orthogonal to each other.

For $i = 0, \ldots, N$ we define projection like operators $T_i \colon S_h \to V_i$, as

$$a(T_i u, v) = a(u, v) \qquad \forall v \in V_i. \tag{6}$$

Now introducing $T := T_0 + \sum_{k=1}^{N} T_k$, we can replace the original problem by the equation

$$T u_h^* = g, \tag{7}$$

where $g = \sum_{i=0}^{N} g_i$ and $g_i = T_i u$. g_i is computed without knowing the solution u_h^* of (2):

$$a_i(g_i, v) = (f, v) \qquad \forall v \in V_i.$$

The bilinear form $a_i(\cdot,\cdot)$ is the restriction of $a(\cdot,\cdot)$ to Ω_i.

4 Eigenvalue problems

In this section, we introduce the two generalized eigenvalue problems. We propose an extension of the coarse space by including some extensions of selected eigenfunctions of those problems in order to obtain better convergence properties of the method.

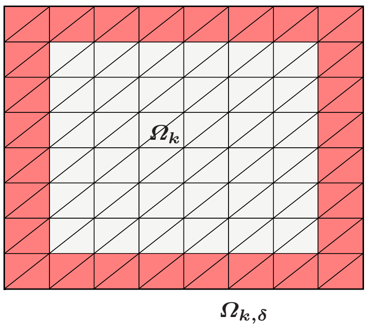

Fig. 1 The layer corresponding to the subdomain Ω_k, consisting of elements (triangles) of $\mathcal{T}_h(\Omega_k)$ touching the subdomain boundary $\partial\Omega_k$.

The layer corresponding to the subdomain Ω_k, consisting of elements of $\mathcal{T}_h(\Omega_k)$ touching the boundary $\partial\Omega_k$, is denoted by $\Omega_{k,\delta}$, cf. Fig.1. For each subdomain and its layer, we define the maximum and the minimum values of the coefficient α as the following:

$$\begin{aligned} \overline{\alpha}_{k,\delta} &:= \sup_{x \in \bar{\Omega}_{k,\delta}} \alpha(x), & \underline{\alpha}_{k,\delta} &:= \inf_{x \in \bar{\Omega}_{k,\delta}} \alpha(x), \\ \overline{\alpha}_{k} &:= \sup_{x \in \bar{\Omega}_{k}} \alpha(x), & \underline{\alpha}_{k} &:= \inf_{x \in \bar{\Omega}_{k}} \alpha(x). \end{aligned} \tag{8}$$

The generalized eigenvalue problem is then defined as follows, with p as a superscript referring to the type of the problem: Find $(\lambda_j^{k,p},\psi_j^{k,p}) \in \mathbb{R}_+ \times S_{h,k}^0$ such that

$$a_k(\psi_j^{k,p},v) = \lambda_j^{k,p} b_k^{(p)}(\psi_j^{k,p},v), \qquad \forall v \in S_{h,k}^0, \qquad p=1,2, \tag{9}$$

where the bilinear forms are defined as

$$a_k(u,v) = a_{|\Omega_k}(u,v) = \int_{\Omega_k} \alpha \nabla u \nabla v \, dx, \tag{10}$$

$$b_k^{(1)}(u,v) = \underline{\alpha}_k (\nabla u, \nabla v)_{L^2(\Omega_k)}, \tag{11}$$

$$b_k^{(2)}(u,v) = \underline{\alpha}_{k,\delta} \int_{\Omega_{k,\delta}} \nabla u \nabla v \, dx + \int_{\Omega_k \setminus \Omega_{k,\delta}} \alpha \nabla u \nabla v \, dx, \tag{12}$$

with $\underline{\alpha}_k$ and $\underline{\alpha}_{k,\delta}$ being defined as in (8). Further, we extend $\psi_j^{k,p}$ to the rest of Ω by zero, and denote it by the same symbol; cf. also (13). We order the eigenvalues in the decreasing order as $\lambda_1^k \geq \lambda_2^k \geq \ldots \lambda_{M_k}^k$ where $M_k = dim(S_{h,k}^0)$. Then those bounds on the eigenvalues are true: $1 \leq \lambda_j^{k,p} \leq C_p$,where $C_1 = \frac{\overline{\alpha}_k}{\underline{\alpha}_k}$ and $C_2 = \frac{\overline{\alpha}_{k,\delta}}{\underline{\alpha}_{k,\delta}}$. Now define the local spectral component of the coarse space by

$$V_{k,0}^p = \mathrm{Span}(\psi_j^{k,p})_{j=1}^{n_k} \qquad k=1,\ldots,N, \qquad p=1,2, \tag{13}$$

where $n_k \leq M_k = \dim(S_{h,k}^0)$ is preset by the user or chosen adaptively for each subdomain. By adding this spectral component to the average coarse space, we propose a new and enriched coarse space defined as $V_0^{(p)} = V_0 + \sum_{k=1}^N V_{k,0}^p$, $p=1,2$.Accordingly, the new coarse operator $T_0^{(p)} : S_h \to V_0^p$ is defined as

$$a(T_0^{(p)}u,v) = a(u,v) \quad \forall v \in V_0^{(p)}, \qquad p=1,2. \tag{14}$$

With the local operators $T_k, k=1,\ldots,N$ from the previous section, the new additive Schwarz operator $T^{(p)}$ becomes $T^{(p)} = T_0^{(p)} + \sum_{k=1}^N T_k$. The problem (2) is then replaced by the following ones:

$$T^{(p)} u_h^* = g^{(p)} \qquad p=1,2, \tag{15}$$

where $g^p = g_0^{(p)} + \sum_k g_k$ with $g_0^{(p)} = T_0^{(p)} u_h^*$ and $g_k = T_k u_h^*$ for $k=1,\ldots,N$.

5 Condition number estimates

In this section, we provide theoretical bounds on the condition number of our method. The bounds are formulated in the following theorem.

Theorem 1. *For $p=1,2$ it holds that*

$$c\left(\min_k \frac{1}{\lambda^{k,p}_{n_k+1}}\right)\frac{h}{H}a(u,u) \le a(T^{(p)}u,u) \le C\,a(u,u), \qquad \forall u \in S_h,$$

where C, c are positive constants independent of the coefficient α, h and $H = \max_{k=1,\dots,N} diam(\Omega_k)$.

The proof is based on the abstract framework for the additive Schwarz method, cf. e.g. Toselli and Widlund [2005].

Remark 1. In the original paper, cf. Bjørstad et al. [1997], where the authors assume that α is constant in each subdomain, the bound obtained for the Additive Average Schwarz method has the form: $\text{cond}(T) \le C\frac{H}{h}$. For the multiscale problem, the bound as given in the paper Dryja and Sarkis [2010] has the following form: $\text{cond}(T) \le C \max_k \frac{\overline{\alpha}_k}{\underline{\alpha}_k}\left(\frac{H}{h}\right)^2$.

Remark 2. If α is piecewise constant in each subdomain Ω_k, both eigenvalue problems become trivial, having only one eigenvalue which is equal to one. If the coefficient is constant in the boundary layers $\Omega_{k,\delta}$, although varying inside, in which case $\frac{\overline{\alpha}_{k,\delta}}{\underline{\alpha}_{k,\delta}} = 1$, the only eigenvalue of the second type of eigenvalue problem ($p = 2$) is also equal to one.

6 Numerical experiments

For the numerical experiment we choose our model elliptic problem to be defined on a unit square, with homogeneous boundary condition and $f(x) = 2\pi^2\sin(\pi x)sin(\pi y)$. For the coefficient α, we chose the following distribution, consisting of a background, channels crossing inside and stretching out of a subdomain, and inclusions along the boundary of a subdomain placed at the corners, where α takes different values. α_b, α_c, and α_i are the values of α respectively in the background, in the channels, and in the inclusions. We have chosen one particular distribution of the coefficient for this paper, cf. Fig. 2.

h \ H	1/3	1/6	1/12	1/3	1/6	1/12
1/24	34 (5.73e1)			16 (1.46e1)		
1/48	56 (1.31e2)	49 (5.32e1)		28 (3.30e1)	25 (1.36e1)	
1/96	76 (2.80e2)	84 (1.20e2)	55 (5.35e1)	37 (7.04e1)	44 (3.03e1)	28 (1.36e1)

Table 1 Number of iterations and a condition number estimate (in parentheses) for each case, for the average Schwarz method, is shown. The left block of results correspond to the additive version, while the right block corresponds to the multiplicative version of the average Schwarz method. $\alpha_b = 1$, $\alpha_c = 1e4$, and $\alpha_i = 1e6$.

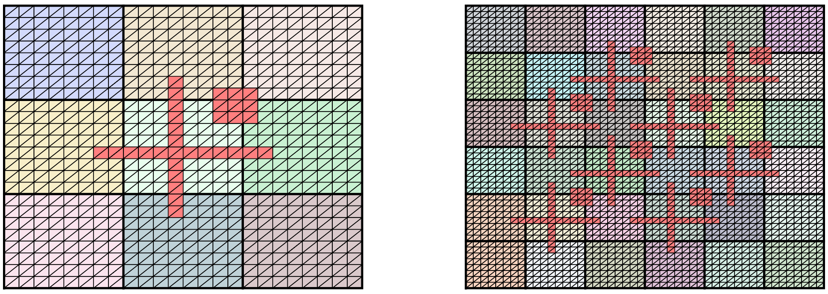

Fig. 2 Discretization and coarse partitioning of the unit square with different mesh sizes. The mesh size ratio $\frac{H}{h}$ are the same in this figure. Coefficient distribution includes both crossing channels and inclusions on the subdomain boundary.

	none	2	4	6	8	10
Add	299 (2.72e6)	321 (7.98e5)	197 (1.36e4)	118 (7.10e3)	46 (4.48e1)	46 (4.44e1)
Mlt	159 (6.79e5)	163 (2.00e5)	99 (3.38e3)	59 (1.78e3)	23 (1.15e1)	23 (1.14e1)

Table 2 Number of iterations and a condition number estimate (in parentheses) for each case is shown. The first line (Add) of results correspond to the additive version, while the second line (Mlt) corresponds to the multiplicative version of the method. $\alpha_b = 1$, $\alpha_c = 1e4$, and $\alpha_i = 1e6$. Each column corresponds to the number of eigenfunctions (preset) used in each subdomain for the test.

The results are presented in tables 1-2 using the average Schwarz method with the type 2 generalized eigenvalue problem. The tables show the number of iterations required to reduce the residual norm by 5e-6, and a condition number estimate (in parentheses), in each test case. Both the additive and the multiplicative version of the average method have been tried, the latter one converges twice as fast as the former one.

As seen from the first table, the proposed method is scalable and the condition number grow as the ratio $\frac{H}{h}$. For this table the eigenfunctions were chosen adaptively in each subdomain, those corresponding to the eigenvalues greater than 100. As we know it from the analysis that there is a minimum number of eigenfunctions (corresponding to the bad eigenvalues) that should be added in the enrichment for the method to be robust with respect to the contrast. For the distribution shown in Fig. 2, this number is eight as seen from the second table. In the adaptive version, cf. the same test case in Table 1, the maximum number of eigenfunctions that were used in this particular case was also eight.

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