Optimizing Transmission Conditions for Multiple Subdomains in the Magnetotelluric Approximation of Maxwell's Equations

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1 Optimized Schwarz for the Magnetotelluric Approximation

Wave propagation phenomena are ubiquitous in science and engineering. In Geophysics, the magnetotelluric approximation of Maxwell's equations is an important tool to extract information about the spatial variation of electrical conductivity in the Earth's subsurface. This approximation results in a complex diffusion equation [4],

$$\Delta u - (\sigma - i\varepsilon)u = f, \quad \text{in a domain } \Omega, \tag{1}$$

where f is the source function, and σ and ε are strictly positive constants[1].

To study Optimized Schwarz Methods (OSMs) for (1), we use a rectangular domain Ω given by the union of rectangular subdomains $\Omega_j := (a_j, b_j) \times (0, \hat{L})$, $j = 1, 2, \ldots, J$, where $a_j = (j-1)L - \frac{\delta}{2}$ and $b_j = jL + \frac{\delta}{2}$, and δ is the overlap, like in [2]. Our OSM computes for iteration index $n = 1, 2, \ldots$

$$\begin{aligned} \Delta u_j^n - (\sigma - i\varepsilon)u_j^n &= f & &\text{in } \Omega_j, \\ -\partial_x u_j^n + p_j^- u_j^n &= -\partial_x u_{j-1}^{n-1} + p_j^- u_{j-1}^{n-1} & &\text{at } x = a_j, \\ \partial_x u_j^n + p_j^+ u_j^n &= \partial_x u_{j+1}^{n-1} + p_j^+ u_{j+1}^{n-1} & &\text{at } x = b_j, \end{aligned} \tag{2}$$

where p_j^- and p_j^+ are strictly positive parameters in the so called 2-sided OSM, see e.g. [6], and we have at the top and bottom homogeneous Dirichlet boundary

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[1] In the magnetotelluric approximation we have $\sigma = 0$, but we consider the slightly more general case here. Note also that the zeroth order term in (1) is much more benign than the zeroth order term of opposite sign in the Helmholtz equation, see e.g. [5].

conditions, and on the left and right homogeneous Robin boundary conditions, i.e we put for simplicity of notation $u_0^{n-1} = u_{J+1}^{n-1} = 0$ in (2). Note that the parameters p_j^-,p_j^+ are real and not complex (as one would expect in the case of a complex problem) for the sake of simplicity in our analysis. The Robin parameters are fixed at the domain boundaries $x = a_1$ and $x = b_J$ to $p_1^- = p_a$ and $p_J^- = p_b$. As p_a, p_b tend to infinity, this is equivalent to imposing Dirichlet conditions. By linearity, it suffices to study the homogeneous equations, $f = 0$, and analyze convergence to zero of the OSM (2). Expanding the homogeneous iterates in a Fourier series $u_j^n(x, y) = \sum_{m=1}^{\infty} v_j^n(x, \tilde{k}) \sin(\tilde{k} y)$ where $\tilde{k} = \frac{m\pi}{\hat{L}}$ to satisfy the homogeneous Dirichlet boundary conditions at the top and bottom, we obtain for the Fourier coefficients the equations

$$\begin{aligned} \partial_{xx} v_j^n - (\tilde{k}^2 + \sigma - i\varepsilon) v_j^n &= 0 & x \in (a_j, b_j), \\ -\partial_x v_j^n + p_j^- v_j^n &= -\partial_x v_{j-1}^{n-1} + p_j^- v_{j-1}^{n-1} & \text{at } x = a_j, \\ \partial_x v_j^n + p_j^+ v_j^n &= \partial_x v_{j+1}^{n-1} + p_j^+ v_{j+1}^{n-1} & \text{at } x = b_j. \end{aligned} \tag{3}$$

The general solution of the differential equation is $v_j^n(x, \tilde{k}) = \tilde{c}_j e^{-\lambda(\tilde{k})x} + \tilde{d}_j e^{\lambda(\tilde{k})x}$, where $\lambda = \lambda(\tilde{k}) = \sqrt{\tilde{k}^2 + \sigma - i\varepsilon}$. We next define the Robin traces, $\mathcal{R}_-^{n-1}(a_j, \tilde{k}) := -\partial_x v_{j-1}^{n-1}(a_j, \tilde{k}) + p_j^- v_{j-1}^{n-1}(a_j, \tilde{k})$ and $\mathcal{R}_+^{n-1}(b_j, \tilde{k}) := \partial_x v_{j+1}^{n-1}(b_j, \tilde{k}) + p_j^+ v_{j+1}^{n-1}(b_j, \tilde{k})$. Inserting the solution into the transmission conditions in (3), a linear system arises where the unknowns are $\tilde{c}_j$ and $\tilde{d}_j$, whose solution is

$$\begin{aligned} \tilde{c}_j &= \frac{1}{D_j} (e^{\lambda b_j} (p_j^+ + \lambda) \mathcal{R}_-^{n-1}(a_j, \tilde{k}) - e^{\lambda a_j} (p_j^- - \lambda) \mathcal{R}_+^{n-1}(b_j, \tilde{k})), \\ \tilde{d}_j &= \frac{1}{D_j} (-e^{-\lambda b_j} (p_j^+ - \lambda) \mathcal{R}_-^{n-1}(a_j, \tilde{k}) + e^{-\lambda a_j} (p_j^- + \lambda) \mathcal{R}_+^{n-1}(b_j, \tilde{k})), \end{aligned}$$

where $D_j := (\lambda + p_j^+)(\lambda + p_j^-) e^{\lambda(L+\delta)} - (\lambda - p_j^+)(\lambda - p_j^-) e^{-\lambda(L+\delta)}$. We thus arrive for the Robin traces in the OSM at the iteration formula

$$\begin{aligned} \mathcal{R}_-^n(a_j, \tilde{k}) &= \alpha_j^- \mathcal{R}_-^{n-1}(a_{j-1}, \tilde{k}) + \beta_j^- \mathcal{R}_+^{n-1}(b_{j-1}, \tilde{k}),\ j = 2, \ldots, J, \\ \mathcal{R}_+^n(b_j, \tilde{k}) &= \beta_j^+ \mathcal{R}_-^{n-1}(a_{j+1}, \tilde{k}) + \alpha_j^+ \mathcal{R}_+^{n-1}(b_{j+1}, \tilde{k}),\ j = 1, \ldots, J-1, \end{aligned}$$

where

$$\alpha_j^- := \frac{(\lambda + p_{j-1}^+)(\lambda + p_j^-) e^{\lambda\delta} - (\lambda - p_{j-1}^+)(\lambda - p_j^-) e^{-\lambda\delta}}{(\lambda + p_{j-1}^+)(\lambda + p_{j-1}^-) e^{\lambda(L+\delta)} - (\lambda - p_{j-1}^+)(\lambda - p_{j-1}^-) e^{-\lambda(L+\delta)}},\ j = 2, \ldots, J,$$

$$\alpha_j^+ := \frac{(\lambda + p_{j+1}^-)(\lambda + p_j^+) e^{\lambda\delta} - (\lambda - p_{j+1}^-)(\lambda - p_j^+) e^{-\lambda\delta}}{(\lambda + p_{j+1}^+)(\lambda + p_{j+1}^-) e^{\lambda(L+\delta)} - (\lambda - p_{j+1}^+)(\lambda - p_{j+1}^-) e^{-\lambda(L+\delta)}},\ j = 1, \ldots, J-1,$$

$$\beta_j^- := \frac{(\lambda + p_j^-)(\lambda - p_{j-1}^-) e^{-\lambda L} - (\lambda - p_j^-)(\lambda + p_{j-1}^-) e^{\lambda L}}{(\lambda + p_{j-1}^+)(\lambda + p_{j-1}^-) e^{\lambda(L+\delta)} - (\lambda - p_{j-1}^+)(\lambda - p_{j-1}^-) e^{-\lambda(L+\delta)}},\ j = 2, \ldots, J,$$

$$\beta_j^+ := \frac{(\lambda+p_j^+)(\lambda-p_{j+1}^+)e^{-\lambda L}-(\lambda-p_j^+)(\lambda+p_{j+1}^+)e^{\lambda L}}{(\lambda+p_{j+1}^+)(\lambda+p_{j+1}^-)e^{\lambda(L+\delta)}-(\lambda-p_{j+1}^+)(\lambda-p_{j+1}^-)e^{-\lambda(L+\delta)}},\ j=1,\ldots,J-1.$$

Defining the matrices

$$T_j^1 := \begin{bmatrix} \alpha_j^- & \beta_j^- \\ 0 & 0 \end{bmatrix},\ j=2,..,J \quad \text{and} \quad T_j^2 := \begin{bmatrix} 0 & 0 \\ \beta_j^+ & \alpha_j^+ \end{bmatrix},\ j=1,..,J-1,$$

we can write the OSM in substructured form (keeping the first and last rows and columns to make the block structure appear), namely

$$\underbrace{\begin{bmatrix} 0 \\ \mathcal{R}_+^n(b_1,\tilde{k}) \\ \mathcal{R}_-^n(a_2,\tilde{k}) \\ \mathcal{R}_+^n(b_2,\tilde{k}) \\ \vdots \\ \mathcal{R}_-^n(a_j,\tilde{k}) \\ \mathcal{R}_+^n(b_j,\tilde{k}) \\ \vdots \\ \mathcal{R}_-^n(a_{N-1},\tilde{k}) \\ \mathcal{R}_+^n(b_{N-1},\tilde{k}) \\ \mathcal{R}_-^n(a_N,\tilde{k}) \\ 0 \end{bmatrix}}_{\mathcal{R}^n} = \underbrace{\begin{bmatrix} & T_1^2 & & & & \\ T_2^1 & & T_2^2 & & & \\ & \ddots & & \ddots & & \\ & & T_j^1 & & T_j^2 & \\ & & & \ddots & & \ddots \\ & & & T_{N-1}^1 & & T_{N-1}^2 \\ & & & & T_N^1 & \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} 0 \\ \mathcal{R}_+^{n-1}(b_1,\tilde{k}) \\ \mathcal{R}_-^{n-1}(a_2,\tilde{k}) \\ \mathcal{R}_+^{n-1}(b_2,\tilde{k}) \\ \vdots \\ \mathcal{R}_-^{n-1}(a_j,\tilde{k}) \\ \mathcal{R}_+^{n-1}(b_j,\tilde{k}) \\ \vdots \\ \mathcal{R}_-^{n-1}(a_{N-1},\tilde{k}) \\ \mathcal{R}_+^{n-1}(b_{N-1},\tilde{k}) \\ \mathcal{R}_-^{n-1}(a_N,\tilde{k}) \\ 0 \end{bmatrix}}_{\mathcal{R}^{n-1}}. \tag{4}$$

If the parameters $p_j^\pm$ are constant over all the interfaces, and we eliminate the first and the last row and column of T, T becomes a block Toeplitz matrix. The best choice of the parameters minimizes the spectral radius $\rho(T)$ over a numerically relevant range of frequencies $K := [\tilde{k}_{\min}, \tilde{k}_{\max}]$ with $\tilde{k}_{\min} := \frac{\pi}{\hat{L}}$ (or 0 for simplicity) and $\tilde{k}_{\max} := \frac{M\pi}{\hat{L}}$, $M \sim \frac{1}{h}$, where h is the mesh size, and is thus solution of the min-max problem $\min_{p_j^\pm} \max_{\tilde{k}\in K} |\rho(T(\tilde{k}, p_j^\pm))|$.

The traditional approach to obtain optimized transmission conditions for optimized Schwarz methods is to optimize performance for a simple two subdomain model problem, and then to use the result also in the case of many subdomains. We want to study here if this approach is justified, by directly optimizing the performance for two and more subdomains, and then comparing the results. We obtain our results from insight by numerical optimisation for small overlap, in order to find asymptotic formulas for the convergence factor and the parameters involved. The constants in the asymptotic results are then obtained by rigorous analytical computations of asymptotic series. We thus do not obtain existence and uniqueness results, but our asymptotically optimized convergence factors equioscillate as one would expect. For Robin conditions with complex parameters for two subdomains, existence and uniqueness results can be found in B. Delourme and L. Halpern [3].

2 Optimization for 2, 3, 4, 5 and 6 subdomains

For two subdomains, the general substructured iteration matrix becomes

$$T = \begin{bmatrix} 0 & \beta_1^+ \\ \beta_2^- & 0 \end{bmatrix}.$$

The eigenvalues of this matrix are $\pm\sqrt{\beta_1^+\beta_2^-}$ and thus the square of the convergence factor is $\rho^2 = \left|\beta_1^+\beta_2^-\right|$.

Theorem 1 (Two Subdomain Optimization) *Let $s{:=}\sqrt{\sigma - i\varepsilon}$, where the complex square root is taken with a positive real part, and let C be the real constant*

$$C{:=}\Re\frac{s((p_b+s)(p_a+s)-(s-p_b)(s-p_a)e^{-4sL})}{((s-p_a)e^{-2sL}+s+p_a)((s-p_b)e^{-2sL}+s+p_b)}. \tag{5}$$

where p_a and p_b are the Robin parameters at the outer boundaries. Then for two subdomains with $p_1^+ = p_2^- =: p$ and $\tilde{k}_{\min} = 0$, the asymptotically optimized parameter p for small overlap δ and associated convergence factor are

$$p = 2^{-1/3}C^{2/3}\delta^{-1/3}, \quad \rho = 1 - 2\cdot 2^{1/3}C^{1/3}\delta^{1/3} + O(\delta^{2/3}). \tag{6}$$

If $p_1^+ \neq p_2^-$ and $\tilde{k}_{min} = 0$, the asymptotically optimized parameters for small overlap δ and associated convergence factor are

$$p_1^+ = 2^{-2/5}C^{2/5}\delta^{-3/5},\ p_2^- = 2^{-4/5}C^{4/5}\delta^{-1/5},\ \rho = 1-2\cdot 2^{-1/5}C^{1/5}\delta^{1/5}+O(\delta^{2/5}). \tag{7}$$

Proof From numerical experiments, we obtain that the solution of the min-max problem equioscillates, $\rho(0) = \rho(\tilde{k}^*)$, where $\tilde{k}^*$ is an interior maximum point, and asymptotically $p = C_p\delta^{-1/3}$, $\rho = 1 - C_R\delta^{1/3} + O(\delta^{2/3})$, and $\tilde{k}^* = C_k\delta^{-2/3}$. By expanding for δ small, and setting the leading term in the derivative $\frac{\partial\rho}{\partial\tilde{k}}(\tilde{k}^*)$ to zero, we get $C_p = \frac{C_k^2}{2}$. Expanding the maximum leads to $\rho(\tilde{k}^*) = \rho(C_k\delta^{-2/3}) = 1-2C_k\delta^{1/3}+O(\delta^{2/3})$, therefore $C_R = 2C_k$. Finally the solution of the equioscillation equation $\rho(0) = \rho(\tilde{k}^*)$ determines uniquely $C_k = 2^{1/3}C^{1/3}$.

In the case with two parameters, we have two equioscillations, $\rho(0) = \rho(\tilde{k}_1^*) = \rho(\tilde{k}_2^*)$, where $\tilde{k}_j^*$ are two interior local maxima, and asymptotically $p_1 = C_{p1}\delta^{-3/5}$, $p_1 = C_{p1}\delta^{-1/5}$, $\rho = 1 - C_R\delta^{1/5} + O(\delta^{2/5})$, $\tilde{k}_1^* = C_{k1}\delta^{-2/5}$ and $\tilde{k}_2^* = C_{k2}\delta^{-4/5}$. By expanding for δ small, and setting the leading terms in the derivatives $\frac{\partial\rho}{\partial k}(\tilde{k}_{1,2}^*)$ to zero, and we get $C_{p1} = C_{k2}^2$, $C_{p2} = \frac{C_{k1}^2}{C_{k2}^2}$. Expanding the maxima leads to $\rho(\tilde{k}_1^*) = \rho(C_k\delta^{-2/5}) = 1 - 2\frac{C_{k1}}{C_{k2}^2}\delta^{1/5} + O(\delta^{2/5})$ and $\rho(\tilde{k}_2^*) = \rho(C_k\delta^{-4/5}) = 1 - 2C_{k_2}\delta^{1/5} + O(\delta^{2/5})$ and equating $\rho(\tilde{k}_1^*) = \rho(\tilde{k}_2^*)$ we get $C_{k1} = C_{k2}^3$ and $C_R = 2C_{k2}$. Finally equating $\rho(0) = \rho(\tilde{k}_2^*)$ asymptotically determines uniquely $C_{k2} = 2^{-1/5}C^{1/5}$ and then $C_{k1} = C_{k2}^3$ and $C_{p1} = C_{k2}^2$, $C_{p2} = C_{k2}^4$.

Corollary 1 (Two Subdomains with Dirichlet outer boundary conditions) *The case of Dirichlet outer boundary conditions can be obtained by letting p_a and p_b go to infinity, which simplifies* (5) *to*

$$C = \Re \frac{s(1+e^{2sL})}{(e^{2sL}-1)} \tag{8}$$

and the asymptotic results in Theorem 1 simplify accordingly.

For three subdomains, the general substructured iteration matrix becomes

$$T = \begin{bmatrix} 0 & \beta_1^+ & \alpha_1^+ & 0 \\ \beta_2^- & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_2^+ \\ 0 & \alpha_3^- & \beta_3^- & 0 \end{bmatrix},$$

and we obtain for the first time an optimization result for three subdomains:

Theorem 2 (Three Subdomain Optimization) *For three subdomains with equal parameters $p_1^+ = p_2^- = p_2^+ = p_3^- = p$, the asymptotically optimized parameter p for small overlap δ and associated convergence factor are*

$$p = 2^{-1/3} C^{2/3} \delta^{-1/3}, \quad \rho = 1 - 2 \cdot 2^{1/3} C^{1/3} \delta^{1/3} + O(\delta^{2/3}), \tag{9}$$

where C is a real constant that can be obtained in closed form. If the parameters are different, their asymptotically optimized values for small overlap δ are such that

$$p_1^+,\, p_2^+,\, p_2^-,\, p_3^- \in \{2^{-2/5} C^{2/5} \delta^{-3/5},\, 2^{-4/5} C^{4/5} \delta^{-1/5}\},\; p_1^+ \neq p_2^-,\; p_2^+ \neq p_3^-, \tag{10}$$

and the associated convergence factor is

$$\rho = 1 - 2 \cdot 2^{-1/5} C^{1/5} \delta^{1/5} + O(\delta^{2/5}). \tag{11}$$

Proof The characteristic polynomial of the iteration matrix is

$$G(\mu) = \mu^4 - (\beta_2^- \beta_1^+ + \beta_3^- \beta_2^+)\mu^2 - \alpha_3^- \beta_2^- \alpha_1^+ \beta_2^+ + \beta_3^- \beta_2^+ \beta_2^- \beta_1^+.$$

This biquadratic equation has the roots $\mu_1 = \pm\sqrt{\frac{m_1+\sqrt{m_2}}{2}}, \mu_2 = \pm\sqrt{\frac{m_1-\sqrt{m_2}}{2}}$ where

$$m_1 = \beta_2^- \beta_1^+ + \beta_3^- \beta_2^+,\; m_2 = 4\alpha_3^- \beta_2^- \alpha_1^+ \beta_2^+ + (\beta_2^- \beta_1^+ - \beta_3^- \beta_2^+)^2.$$

Therefore $\rho(T) = \max\{|\mu_1|, |\mu_2|\}$. Following the same reasoning as in the proof of Theorem 1, we observe that the solution equioscillates, and minimizing the maximum asymptotically for δ small then leads to the desired result, for more details, see [7].□

Notice that the optimized parameters and the relation between them is the same as in the two-subdomain case, the only difference is the equation whose solution gives the exact value of the constant C. The only difference between a two subdomain optimization and a three subdomain optimization is therefore the constant.

Table 1: Asymptotic results for four subdomains: $\sigma = \varepsilon = 1$, $L = 1$, $p_a = p_b = 1$

	Many parameters							One parameter	
δ	ρ	p_1^+	p_2^-	p_2^+	p_3^-	p_3^+	p_4^-	ρ	p
$1/10^2$	0.5206	13.1269	1.2705	10.1871	0.7748	16.5975	2.1327	0.6202	2.8396
$1/10^3$	0.6708	37.9717	1.4208	42.9379	1.6005	68.1923	2.4896	0.8022	6.0657
$1/10^4$	0.7789	152.9323	2.3266	152.0873	3.1841	161.0389	2.4919	0.9029	13.0412
$1/10^5$	0.8510	651.7536	4.1945	645.0605	4.1519	649.8928	4.1828	0.9537	28.0834

Table 2: Asymptotic results for five subdomains : $\sigma = \varepsilon = 1$, $L = 1$, $p_a = p_b = 1$

	Many parameters									One parameter	
δ	ρ	p_1^+	p_2^-	p_2^+	p_3^-	p_3^+	p_4^-	p_4^+	p_5^-	ρ	p
$1/10^2$	0.5273	8.5648	1.4619	9.1763	0.8030	9.1398	0.8426	15.5121	2.2499	0.6290	2.6747
$1/10^3$	0.7333	24.6097	0.9209	23.4189	0.4499	37.2200	0.8433	34.8142	0.9181	0.8072	5.7261
$1/10^4$	0.7769	156.0648	2.4223	156.0502	2.4221	161.2036	2.5009	166.3478	2.5941	0.9055	12.3166
$1/10^5$	0.8547	704.4063	4.3378	611.3217	3.7296	611.3217	3.7296	690.8837	4.2116	0.9550	26.5260

Table 3: Asymptotic results for six subdomains: $\sigma = \varepsilon = 1$, $L = 1$, $p_a = p_b = 1$

δ	ρ	p_1^+	p_2^-	p_2^+	p_3^-	p_3^+	p_4^-	p_4^+	p_5^-	p_5^+	p_6^-
$1/10^2$	0.5460	10.5283	1.4526	7.7653	1.2124	8.2834	0.6573	7.6445	1.3410	8.0029	0.9586
$1/10^3$	0.7011	30.3314	0.9049	30.3452	1.1096	30.3010	0.9363	30.3458	0.8901	30.1139	1.1307
$1/10^4$	0.7837	145.7147	2.1126	146.4533	2.1231	145.7147	2.1126	149.1802	2.1743	146.7200	2.1909
$1/10^5$	0.8553	660.5326	3.9932	611.9401	3.7012	606.1453	3.6661	606.1144	3.6659	606.0914	3.8534

Corollary 2 (Three subdomains with Dirichlet outer boundary conditions) *When Dirichlet boundary conditions are used at the end of the computational domain, we obtain for the constant*

$$C = \Re \frac{s(e^{2sL} - e^{sL} + 1)}{e^{2sL} - 1}, \tag{12}$$

which is different from the two subdomain constant in (8).

For four subdomains, we show in Table 1 the numerically optimized parameter values when the overlap δ becomes small. We observe that again the optimized parameters behave like in Theorem 1 and Theorem 2 when the overlap δ becomes small. It is in principle possible to continue the asymptotic analysis from two and three subdomains, but this is beyond the scope of the present paper. Continuing the numerical optimization for five and six subdomains, we get the results in Table 2 and Table 3, which show again the same asymptotic behavior. We therefore conjecture the following two results for an arbitrary fixed number of subdomains:

1. When all parameters are equal to p, then the asymptotically optimized parameter p for small overlap δ and the associated convergence factor have the same form as for two-subdomains (6) in Theorem 1, only the constant is different.
2. If all parameters are allowed to be different, the optimized parameters behave for small overlap δ like

$$p_j^+,\ p_{j+1}^- \in \{2^{-2/5}C^{2/5}\delta^{-3/5},\ 2^{-4/5}C^{4/5}\delta^{-1/5}\} \text{ and } p_j^+ \neq p_{j+1}^- \,\forall j = 1..,J-1,$$

as we have seen in the three subdomain case in Theorem 2, and we have again the same asymptotic convergence factor as for two and three subdomains, only the constant is different.

3 Optimization for many subdomains

In order to obtain a theoretical result for many subdomains, we use the technique of limiting spectra [1] to derive a bound on the spectral radius which we can then minimize. The technique of limiting spectra allows us to get an estimate of the spectral radius when the matrix size goes to infinity. To do so, we must however assume that the outer Robin boundary conditions use the same optimized parameter as at the interfaces, in order to have the Toeplitz structure needed for the limiting spectrum approach.

Theorem 3 (Many Subdomain Optimization) *With all Robin parameters equal, $p_j^- = p_j^+ = p$, the convergence factor of the OSM satisfies the bound*

$$\rho = \lim_{N\to+\infty} \rho(T_{2d}^{OS}) \leq \max\left\{ |\alpha - \beta|, |\alpha + \beta| \right\} < 1,$$

where $\alpha = \frac{(\lambda+p)^2 e^{\lambda\delta} - (\lambda-p)^2 e^{-\lambda\delta}}{(\lambda+p)^2 e^{\lambda(L+\delta)} - (\lambda-p)^2 e^{-\lambda(L+\delta)}}$, $\beta = \frac{(\lambda-p)(\lambda+p)(e^{-\lambda L} - e^{\lambda L})}{(\lambda+p)(\lambda+p)e^{\lambda(L+\delta)} - (\lambda-p)(\lambda-p)e^{-\lambda(L+\delta)}}$. The asymptotically optimized parameter and associated convergence factor are

$$p = 2^{-1/3}C^{2/3}\delta^{-1/3}, \quad \rho = 1 - 2\cdot 2^{1/3}C^{1/3}\delta^{1/3} + O(\delta^{2/3}) \tag{13}$$

with the constant $C := \Re \frac{s(1-e^{-sL})}{1+e^{-sL}}$. If we allow two-sided Robin parameters, $p_j^- = p^-$ and $p_j^+ = p^+$, the OSM convergence factor satisfies the bound

$$\rho = \lim_{N\to+\infty} \rho(T_{2d}^{OS}) \leq \max\left\{ \left|\alpha - \sqrt{\beta_-\beta_+}\right|, \left|\alpha + \sqrt{\beta_-\beta_+}\right| \right\} < 1,$$

where $\alpha = \frac{(\lambda+p^+)(\lambda+p^-)e^{\lambda\delta} - (\lambda-p^+)(\lambda-p^-)e^{-\lambda\delta}}{D}$, $\beta^\pm = \frac{(\lambda^2-(p^\mp)^2)(e^{-\lambda L} - e^{\lambda L})}{D}$, with $D = (\lambda+p^+)(\lambda+p^-)e^{\lambda(L+\delta)} - (\lambda-p^+)(\lambda-p^-)e^{-\lambda(L+\delta)}$. The asymptotically optimized parameter choice $p^- \neq p^+$ and the associated convergence factor are

$$p^-, p^+ \in \left\{C^{2/5}\delta^{-3/5},\ C^{4/5}\delta^{-1/5}\right\}, \quad \rho = 1 - 2C^{1/5}\delta^{1/5} + O(\delta^{2/5}),$$

with the same constant $C := \Re \frac{s(1-e^{-sL})}{1+e^{-sL}}$ *as for one parameter.*

Proof As in the case of two and three subdomains, we observe equioscillation by numerical optimization, and asymptotically that $p = C_p\delta^{-1/3}$, $\rho = 1 - C_R\delta^{1/3} + O(\delta^{2/3})$ and the convergence factor has a local maximum at the point $\tilde{k}^* = C_k\delta^{-2/3}$. By expanding for small δ, the derivative $\frac{\partial\rho}{\partial k}(\tilde{k}^*)$ needs to have a vanishing leading order term, which leads to $C_p = \frac{C_k^2}{2}$. Expanding the convergence factor at the maximum point $\tilde{k}^*$ gives $\rho(\tilde{k}^*) = \rho(C_k\delta^{-2/3}) = 1 - 2C_k\delta^{1/3} + O(\delta^{2/3})$, and hence $C_R = 2C_k$. Equating now $\rho(0) = \rho(\tilde{k}^*)$ determines uniquely C_k and then $C_p = \sqrt{C_k/2}$ giving (13). By following the same lines as for two and three subdomains, we also get the asymptotic result in the case of two different parameters. □

We can therefore safely conclude that for the magnetotelluric approximation of Maxwell's equations, which contains the important Laplace and screened Laplace equation as special cases, it is sufficient to optimize transmission conditions for a simple two subdomain decomposition in order to obtain good transmission conditions also for the case of many subdomains, a new result that was not known so far.

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