Asynchronous Multi-Subdomain Methods With Overlap for a Class of Parabolic Problems

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1 Introduction

In previous work [3] and [5], we presented asynchronous iterations for solving second order elliptical partial differential equations based on an overlapping domain decompositions. Asynchronous iterations are not only a family of algorithms suitable for asynchronous computations on multiprocessors, but also a general framework in order to formulate general iteration methods associated with a fixed point mapping on a product space, including the most standard ones such as the successive approximation method (Jacobi, Gauss-Seidel and their block versions). in this chapter, we will associate with the alternate method of Schwarz for the parabolic problems of the second order, an affine fixed point map of which we show that the linear part is a contraction in uniform norm. In this context we will develop a method of analyzing the multi-subdomain case, as well as asynchronous iterations for parabolic problems. We will give a new technical result to update a Hopf maximum principle and construct a new exponential weighted norm. The work is devoted to the framework of a class of parabolic problems of the second order with Dirichlet condition. We associate with a method based on the resolution of sub-problems on subdomains with overlap, a fixed point map defined by the restrictions on subdomains without overlap. We examine a mathematical property, the contraction with respect to a new norm of this fixed point map. One important feature of the results presented here is the use of exponential weighted norms, which allows us to obtain a stronger convergence property than the usual uniform norm. One thus obtains a result of convergence of the asynchronous iterations for a norm finer than the usual one, and this

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including for the basic situation of the very traditional alternate method of Schwarz. At the level of subdomains having a common border portion with the boundary of the domain, this requires the implementation of the principle of the maximum of Hopf. The formalism used is particularly effective, compared to that used previously described, to examine the influence of the size of the overlaps and the comparison of the contraction constant of the application of fixed point. After the introduction, we present in the second section the problem formulation, introduce the notation used in the sequel and give our new technical result. In the third section we define the linear mapping $\mathcal{T}$ which defines the substructured solution process. Then we define the linear fixed point mapping T which is the composition of $\mathcal{T}$ with a suitable restriction operator R. We prove that T is a linear mapping in a suitable function space context. We also study the contraction property of T . We finally introduce an affine mapping whose linear part is T and whose fixed point is the solution of the parabolic partial differential equation. We state in the closing proposition the convergence of asynchronous iterations applied to the approximation of this affine fixed point mapping.

2 Notation and Assumptions

Let Ω be an open bounded domain of $\mathbb{R}^n$ with boundary $\partial\Omega$ and m an integer such that $m \geq 2$. In order to formulate our algorithm, we need an overlapping decomposition of Ω with certain overlap properties. We build such a decomposition by decomposing Ω into m non-overlapping open subdomains $\widetilde{\Omega}_i$ as

$$\widetilde{\Omega}_i \cap \widetilde{\Omega}_j = \emptyset \text{ if } i \neq j \text{ and } \cup_{i=1}^{m} \overline{\widetilde{\Omega}}_i = \overline{\Omega} \tag{1}$$

and $\partial\Omega$ (resp. $\partial\widetilde{\Omega}_i$) the boundary of Ω (resp. $\widetilde{\Omega}_i$) and $\widetilde{\Gamma}_i = \partial\widetilde{\Omega}_i \cap \Omega$; $\widetilde{\Gamma}'_i = \partial\widetilde{\Omega}_i \cap \partial\Omega$ such that $\Omega = \cup_{i=1}^{m} \left(\widetilde{\Gamma}_i \cup \widetilde{\Omega}_i\right)$. From this non-overlapping decomposition of Ω the desired overlapping decomposition which will be used by our algorithm. To $\widetilde{\Omega}_i$, we associate Ω_i, the overlapping multi-subdomain decomposition: $\widetilde{\Omega}_i \subset \Omega_i \subset \Omega$, $\Omega = \cup_{i=1}^{m} \Omega_i$, and

$$\Gamma_i = \partial\Omega_i \cap \Omega \text{ ; } \Gamma'_i = \partial\Omega_i \cap \partial\Omega \tag{2}$$

such that :

$$\overline{\widetilde{\Omega}}_i \cap \overline{\Gamma}_i = \emptyset, \ i = 1, \ldots, m \tag{3}$$

For the exchange of information between subdomains, we will also employ the index notation

$$\Gamma_{i,j} = \Gamma_i \cap \widetilde{\Omega}_j, \ j \in J(i) \tag{4}$$

where the index set J is defined by

$$J(i) = \left\{ j : \Gamma_i \cap \widetilde{\Omega}_j \neq \emptyset, j \neq i \right\} \tag{5}$$

2.1 Technical result

Consider a bounded domain D of $\mathbb{R}^n$, the boundary $\partial D = \Gamma \cup \Gamma'$ and an other domain $\widetilde{D} \subset D$ such that $\partial D \cap \partial \widetilde{D} \subset \Gamma'$ and $\overline{\Gamma} \cap \overline{\widetilde{D}} = \emptyset$

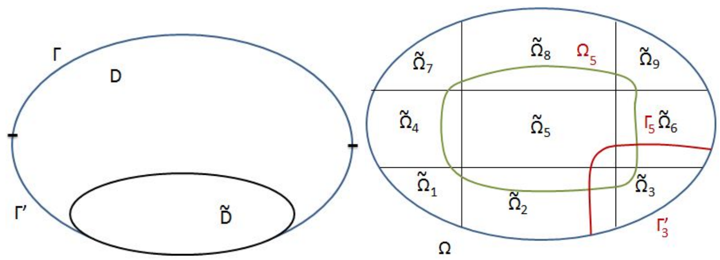

Fig. 1: Illustrative example of decomposition

Lemma 1 *Consider the kernel* $k(x, y)$ *defined on* $\overline{\widetilde{D}} \times \overline{\Gamma}$. *Suppose that* $k(x, y)$ *is continuously differentiable* $\left(\frac{\partial k}{\partial x_j}(x, y)\right.$ *exist and continuous on* $\left.\overline{\widetilde{D}} \times \overline{\Gamma}\right)$ *then for all integrable function* $g : y \to g(y),\ g \in L^1(\overline{\Gamma})$, *the function* $r(x) = \int_{\overline{\Gamma}} k(x, y)g(y)dy$ *admits continuous partial derivatives with respect to the components* x_j *of* x *on* $\overline{\widetilde{D}}$, *which can be expressed by :*

$$\frac{\partial r}{\partial x_j}(x) = \int_{\overline{\Gamma}} \frac{\partial k}{\partial x_j}(x, y)g(y)dy.$$

2.2 Problem statement

We introduce the time interval $[0; \bar{t}]$. Let define :

$$\begin{cases} Q = \Omega \times \left[0; \bar{t}\right] ; \widetilde{Q}_i = \widetilde{\Omega}_i \times \left[0; \bar{t}\right] ; Q_i = \Omega_i \times \left[0; \bar{t}\right] \\ \partial \widetilde{Q}_i = \partial \widetilde{\Omega}_i \times \left[0; \bar{t}\right] ; \partial Q_i = \partial \Omega_i \times \left[0; \bar{t}\right] ; \partial Q = \partial \Omega \times \left[0; \bar{t}\right] \end{cases} \tag{6}$$

and

$$\begin{cases} \Sigma_i = \Gamma_i \times \left[0; \bar{t}\right] ; \Sigma'_i = \Gamma'_i \times \left[0; \bar{t}\right] \\ \Sigma_{i,j} = \Gamma_{i,j} \times \left[0; \bar{t}\right] \end{cases} \tag{7}$$

For $0 < t \leq \bar{t}$, we denote :

$$\begin{cases} \widetilde{Q}_i^a = \overline{\widetilde{\Omega}}_i \times \left[0; \bar{t}\right] ; Q_i^a = \overline{\Omega}_i \times \left[0; \bar{t}\right] ; Q^a = \overline{\Omega} \times \left[0; \bar{t}\right] \\ \Sigma_i^a = \overline{\Gamma}_i \times \left[0; \bar{t}\right] ; \Sigma_i'^a = \overline{\Gamma}'_i \times \left[0; \bar{t}\right] ; \Sigma_{i,j}^a = \overline{\Gamma}_{i,j} \times \left[0; \bar{t}\right] \end{cases} \tag{8}$$

and

$$\Gamma = \cup_{i=1}^{m} \cup_{j \in J(i)} \Gamma_{i,j} = \cup_{i=1}^{m} \Gamma_i,\ \Sigma = \cup_{i=1}^{m} \cup_{j \in J(i)} \Sigma_{i,j} = \cup_{i=1}^{m} \Sigma_i \tag{9}$$

Suppose that

$$L^1(\Sigma) = \prod_{i=1}^{m} \prod_{j \in J(i)} L^1\left(\Sigma_{i,j}\right) \tag{10}$$

and

$$\begin{cases} A \text{ a second order elliptic operator with regular coefficients on } \Omega \\ \text{and suppose that there exist } e \in C(\overline{\Omega}),\ e > 0 \text{ such that : } Ae = \lambda e,\ \lambda \in R,\ \lambda > 0 \end{cases} \tag{11}$$

and $p,\ q,\ r$ integers

$$f \in L^p(Q)\ ;\ g \in L^q(\Sigma)\ ;\ u^0 \in L^r(\Omega),\ p, q > 1 \text{ and } r \geq 1. \tag{12}$$

We consider the linear parabolic problem with Cauchy conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = f_{/Q} \\ u = g_{/\partial Q} \\ u(x,0) = u^0_{/\Omega} \end{cases} \tag{13}$$

Assume that the problem (13) has a unique solution u^* in a suitable function space. On $\overline{Q} = \overline{\Omega} \times \left[0, \bar{t}\right]$, we define the weighted norm :

$$|u|_{e,\infty}^{\bar{t}} = \max_{(x,t) \in \overline{Q}} \frac{|u(x,t)|}{e(x)} \tag{14}$$

We can notice that if the initial condition $u^0_{/\Omega} = 0$, then $|u|_{e,\infty}^{\bar{t}} = \max_{(x,t) \in Q^a} \frac{|u(x,t)|}{e(x)}$

3 Fixed point mappings

3.1 The linear mapping $\mathcal{T}$

Consider the function space $C^{\bar{t}} = \prod_{i=1}^{m} C\left(C(\overline{\overline{\Omega}}_i); \left]0, \bar{t}\right]\right)$ and define the linear mapping

$$\mathcal{T} : L^1(\Sigma) \to C^{\bar{t}}, \mathcal{T} : \widetilde{w} \to \widetilde{v}$$

Note that $C^{\bar{t}} \neq C\left(C(\overline{\Omega}; \left]0, \bar{t}\right]\right)$. For each given function $\widetilde{w} \in L^1(\Sigma)$,

$$\widetilde{w} = \{\ldots, \widetilde{w}_i, \ldots\}_{i=1,\ldots,m},\ \widetilde{w}_i = \left\{\ldots, \widetilde{w}_{i,j}, \ldots\right\}_{j \in J(i)} \in L^1(\Sigma_i)$$

we compute using the solutions v_i of the subproblems

$$\frac{\partial v_i}{\partial t} + Av_i = 0 \text{ in } Q_i, v_{i/\Sigma_{i,j}} = \widetilde{w}_{i,j},\ j \in J(i), v_{i/\Sigma'_i} = 0, v_i(x,0) = 0 \text{ on } \Omega_i \quad (15)$$

where, we suppose the following regularity of subdomain solutions

$$v_i \in \begin{cases} C^\infty(\widetilde{Q}_i^a), & if\ \Gamma'_i \neq \emptyset \\ C^1(\widetilde{Q}_i^a), & otherwise \end{cases} \quad (16)$$

Now we take the restriction $\widetilde{v}_i = v_{i/\overline{\overline{Q}}_i}$ and we define the linear operator $\mathcal{T}_i$ by $\mathcal{T}_i(\widetilde{w}) = \widetilde{v}_i$. Finally we set

$$\widetilde{v} = \{\ldots, \widetilde{v}_i, \ldots\} = \{\ldots, \mathcal{T}_i(\widetilde{w}), \ldots\} = \mathcal{T}(\widetilde{w})$$

Proposition 1 $\mathcal{T} \in \mathcal{L}(L^1(\Sigma); C^{\bar{t}})$ *is a linear isotone mapping with respect to the natural order.*

3.2 The linear mapping T

Let $C_e(\overline{\overline{Q}}_j)$ be the space formed by all elements of $C(\overline{\overline{Q}}_j)$ endowed with the norm $\left|w_j\right|^{\bar{t}}_{e,\infty,j} = \max_{(x,t)\in\overline{\overline{Q}}_j} \frac{|w_j(x,t)|}{e(x)}$ where $e(x)$ denotes the eigenfunction in (11). We define $C_e^{\bar{t}} = \prod_{j=1}^{m} C_e(\overline{\overline{Q}}_j)$ equipped with the norm $|w|^{\bar{t}}_{e,\infty} = \max_{j=1,\ldots,m} \left|w_j\right|^{\bar{t}}_{e,\infty,j}$ where $w = \{\ldots, w_i, \ldots\} \in C_e$.

Define R'_i the restriction operator from $C^{\bar{t}}$ to $\prod_{j\in J(i)} C(\overline{\overline{Q}}_j)$

by $R'_i(w) = \overline{w}_i = \left\{\ldots, w_j, \ldots\right\}_{j\in J(i)}$ and R''_i the restriction operator from $\prod_{j\in J(i)} C(\overline{\overline{Q}}_j)$ to $\prod_{j\in J(i)} C(\Sigma^a_{i,j})$ which at each $\overline{w}_i$ we associate $\left\{\ldots, \overline{w}_{i,j}, \ldots\right\}_{j\in J(i)}$ where $\overline{w}_{i,j}$ are defined by :

$$\overline{w}_{i,j} = w_{j/\Sigma^a_{i,j}}$$

and

$$R_i = R''_i \circ R'_i$$

Then

$$R(w) = \{R_1(w), \ldots, R_m(w)\} \text{ and } R \in \mathcal{L}(\prod_i C(\overline{\overline{Q}}_i); \prod_{i=1}^{m} \prod_{j\in J(i)} C(\Sigma^a_{i,j}))$$

We define the mapping $T = \{\ldots, T_i, \ldots\}$ at each $i \in \{1, \ldots, m\}$ by :

$$\widetilde{v}_i = T_i(w) = \mathcal{T}_i \circ R_i(w) = \mathcal{T}_i \circ R(w)$$

then

$$\widetilde{v} = \{\ldots, \widetilde{v}_i, \ldots\}_{i=1,\ldots,m} = T(w)$$

Proposition 2 *$T \in \mathcal{L}(C_e^{\bar{t}})$ is a linear isotone mapping with respect to the natural order.*

3.3 Contraction property of *T*

Using (11), we take the restriction $\widetilde{\Psi}^{\bar{t}}_{e,i} = \Psi^{\bar{t},e}_{i/\overline{\overline{Q}}_i}$ where $\Psi_i^{\bar{t},e}$ is solution of :

$$\begin{cases} \frac{\partial \Psi_i^{\bar{t},e}}{\partial t} + A\Psi_i^{\bar{t},e} = 0_{/Q_i} \\ \Psi^{\bar{t},e}_{i/\Sigma_i \cup \Sigma_i'} = e_{/\Sigma_i \cup \Sigma_i'} \\ \Psi_i^{\bar{t},e}(.,0) = 0_{/\Omega_i} \end{cases} \tag{17}$$

Lemma 2 *$\frac{\widetilde{\Psi}^{\bar{t}}_{e,i}(t,x)}{e(x)}$ is well defined and continuous on $\overline{\overline{Q}}_i$, and*

$$\max_{(x,t)\in\overline{\overline{Q}}_i} \frac{\widetilde{\Psi}^{\bar{t}}_{e,i}(t,x)}{e(x)} \leq \mu_i < 1 \tag{18}$$

Proposition 3 *$T \in \mathcal{L}(C_e^{\bar{t}})$ is a contraction with contraction constant*

$$\mu = \max_{i=1,\ldots,m} \mu_i, \text{ where } \mu_i = \max_{(x,t)\in\overline{\overline{Q}}_i} \frac{\widetilde{\Psi}^{\bar{t}}_{i,e}(x,t)}{e(x)} \tag{19}$$

First, we resolve the subproblems for $i = 1, \cdots, m$: $\frac{\partial u_i}{\partial t} + Au_i = f_{i/Q_i}$, $u_{i/\Sigma_i} = 0_{/\Sigma_i}$, $u_{i/\Sigma_i'} = g_{i/\Sigma_i'}$, $u_i(x,0) = u^0$ on Ω_i. Restricting u_i to $\overline{u}_i = u_{i/\overline{\overline{Q}}_i}$ and consider the new subproblems $\frac{\partial v_i}{\partial t} + Av_i = f_{i/Q_i}$, $v_{i/\Sigma_i} = w_{/\Sigma_i}$, $v_{i/\Sigma_i'} = g_{i/\Sigma_i'}$, $v_i(x,0) = u^0$, we get the restricted values $\widetilde{v}_i = v_{i/\overline{\overline{Q}}_i}$ so that the fixed point is given by

$$\widetilde{v}_i = \overline{u}_i + T_i(w). \tag{20}$$

Proposition 4 *The asynchronous iterations initialized by u^0, applied to the affine fixed point mapping : $F(w) = T(w) + \overline{u}$ give rise to a sequence of iterates which converges, with respect to the uniform weighted norm $|\,|^{\bar{t}}_{e,\infty}$ towards u^* the solution of problem (13).*

Proof The proof of all proposed will be given in the extended version paper. □

4 Constants of contraction comparison with respect to the weighted exponential norm

Let A_0 an operator verifying the previously conditions and $\alpha \in \mathbb{R}$, we define the operator A_α by $A_\alpha = A_0 + \alpha I$.
We consider two open subdomains $\Omega_i^k,\ k = 1,2$ such that : $\Omega_i^1 \subset \Omega_i^2$ then $Q_i^1 = \Omega_i^1 \times \left]0;\bar{t}\right]$; $Q_i^2 = \Omega_i^2 \times \left]0;\bar{t}\right]$; $Q_i^1 \subset Q_i^2$. Denote :

$$\begin{cases} \Gamma_i^k = \partial\Omega_i^k \cap \Omega \ ; \ \Gamma_i'^k = \partial\Omega_i^k \cap \partial\Omega \\ \Sigma_i^k = \Gamma_i^k \times \left]0;\bar{t}\right] \ ; \ \Sigma_i'^k = \Gamma_i'^k \times \left]0;\bar{t}\right] \end{cases}$$

and assume that for $k = 1,2$: Γ_i^k, $\Gamma_i'^k$ satisfy (3), $\Psi_i^{\bar{t},e,k}$, $\Psi_{e,i}^{\bar{t},k}$ are obtained by (17) and (18) with respect to Ω_i^k. Lets T_1 (resp T_2) the fixed point mapping associate to Q^1 (resp Q^2), and μ_1 (resp μ_2) the contraction constant of T_1 (resp T_2) defined by (19).

Proposition 5 *Under previous notations,* $\widetilde{\Psi}_{e,i}^{\bar{t},2} < \widetilde{\Psi}_{e,i}^{\bar{t},1}$ *and* $\mu_2 < \mu_1 < 1$

Let us to solve the problem, with $A = A_\beta$ for $\beta \in \mathbb{R}$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = f_{/Q} \\ u = g_{/\partial Q} \\ u(x,0) = u^0_{/\Omega} \end{cases} \tag{21}$$

Let $\mathcal{D}$ bounded domain of $\mathbb{R}^n$, $\bar{t} \in \mathbb{R}_+$. Let $\alpha \in \mathbb{R}$, $e_{\mathcal{D}}$ a positive function on $\overline{\mathcal{D}}$.
We define on $\overline{\mathcal{D}} \times \left[0,\bar{t}\right]$, the weighted exponential norm $|\ |^{\bar{t}}_{e_{\mathcal{D}},\infty,\alpha}$ by :

$$|\ u|^{\bar{t}}_{e_{\mathcal{D}},\infty,\alpha} = \max_{(x,t)\in\overline{\mathcal{D}}\times\left[0,\bar{t}\right]} \left| \frac{\exp(-\alpha t)u(x,t)}{e_{\mathcal{D}}(x)} \right| \tag{22}$$

Replacing $\overline{\mathcal{D}}$ (resp. $e_{\mathcal{D}}$) by $\widetilde{\Omega}_i$ (resp. $e_{\overline{\widetilde{\Omega}}_i}$), we can define on $C_e(\overline{\widetilde{Q}}_i)$, the norm $|\ |^{\bar{t}}_{e,\infty,\alpha,i}$ by : $|u_i|^{\bar{t}}_{e,\infty,\alpha,i} = |u_i|^{\bar{t}}_{e_{\widetilde{\Omega}_i},\infty,\alpha} = \max_{(x,t)\in\overline{\widetilde{\Omega}}_i\times\left[0,\bar{t}\right]} \left|\frac{\exp(-\alpha t)u(x,t)}{e(x)}\right|$
Then, we define on $C_e^{\bar{t}}$ the norm $|\ |^{\bar{t}}_{e,\infty,\alpha}$ by : $|u|^{\bar{t}}_{e,\infty,\alpha} = \max_{i=1,\ldots,m} |u_i|^{\bar{t}}_{e,\infty,\alpha,i}$
Taking $v = \exp(-\alpha t)u$, then v verify :

$$\begin{cases} \exp(\alpha t)\dfrac{\partial v}{\partial t} + \exp(\alpha t)Av + \exp(\alpha t)\alpha v = f_{/Q} \\ \exp(\alpha t)v = g_{/\partial Q} \\ v(x,0) = u^0_{/\Omega} \end{cases} \tag{23}$$

If $\overline{A}_\alpha = A + \alpha I$, then the problem become as :

$$\begin{cases} \dfrac{\partial v}{\partial t} + \overline{A}_\alpha v = \exp(-\alpha t) f_{/Q} \\ v = \exp(-\alpha t) g_{/\partial Q} \\ v(x,0) = u^0_{/\Omega} \end{cases} \tag{24}$$

where $\overline{A}_\alpha = A_0 + (\alpha + \beta) I = A_{\alpha+\beta}$ and if we choose $\alpha = -\beta$ then $\overline{A}_\alpha = A_0$.

Proposition 6 *Lets $\alpha \geq 0$ and $w \in C_e^{\bar{t}}$. For the subproblems $\frac{\partial u_i}{\partial t} + Au_i + \alpha u_i = f_{i/Q_i}$, $u_{i/\Sigma_i} = w_{/\Sigma_i}$, $u_{i/\Sigma_i'} = 0_{/\Sigma_i'}$, $u_i(x,0) = 0_{/\Omega_i}$. we correspond T_α the affine fixed point application and μ_α its constant contraction, then μ_α is strictly decreasing as a function of α.*

Let $w \in C_e^{\bar{t}}$, and suppose that u_i solution of the subproblems

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + Au_i = f_{/Q_i} \\ u_{i/\Sigma_i'} = g_{i/\Sigma_i'} \\ u_{i/\Sigma_i} = w_{/\Sigma_i'} \\ u_i(x,0) = u^0_{/\Omega_i} \end{cases} \tag{25}$$

We can define the affine fixed point application F by $u_i = F_i(w) = T_i(w) + \overline{u}$

Proposition 7 *F is a fixed point mapping with respect to the weighted exponential norm with contraction constant μ where μ is a contraction constant of the fixed point mapping $\mathbb{F}$ associated to A_0 with respect to the weighted norm $|\ |^{\bar{t}}_{e,\infty,\alpha}$ defined by (22).*

Proof The proof of all proposed will be given in the extended version paper. □

Proposition 8 *The asynchronous iterations initialized by u^0, applied to the affine fixed point mapping : $F(w) = T(w) + \overline{u}$ give rise to a sequence of iterates which converges, with respect to the uniform weighted norm $|\ |^{\bar{t}}_{e,\infty,\alpha}$ towards u^* the solution of problem (25).*

The proof is based on the use of El Tarazi's theorem [2].

Remark 1

1. Si $\beta > 0$, the norm $|\ |^{\bar{t}}_{e,\infty,\alpha}$, for $\alpha = -\beta$ is more fine than $|\ |^{\bar{t}}_{e,\infty}$ (which means it converges faster).
2. If $\beta < 0$, we obtain the convergence for the norm $|\ |^{\bar{t}}_{e,\infty,\alpha}$ with $\alpha = -\beta$ which is less fine than $|\ |^{\bar{t}}_{e,\infty}$. For this last norm and any $\beta < 0$ there is no convergence in general.
3. Let λ the smallest positive eigenvalue of A_0. For $\beta \in]-\lambda, 0]$, $A = A_\beta$ satisfies the previous conditions with however the contraction constant which satisfy $\mu \leq \mu_\beta < 1$ and when $\beta \searrow -\lambda$, $\mu_\beta \nearrow 1$.
4. It is possible to make the change $v = s^{(-\alpha t)} u,\ s \in R^{+*}$, instead $v = \exp(-\alpha t) u$ and all the results still valid.

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