# Convergence Bounds for One-Dimensional ASH and RAS

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## 1 Introduction

The ASH and RAS methods were introduced in [2] and rate of convergence theory is still missing; apparently it does not fall into the abstract theory of Schwarz methods since the nonsymmetric terms are no compact perturbations of $H^1$-norms. As far as we know, the algebraic convergence theory using weighted max norms introduced in [3] is the only theoretical work which establishes convergence however no rate of convergence. Here, we introduce new techniques to analyze RAS and ASH for the one-dimensional case. Some of these techniques can be used to establish rate of convergence in higher dimensions and they will be discussed elsewhere.

Let

$$Au = f \tag{1}$$

be a system of linear algebraic equations corresponding to the finite difference approximations of the Poisson problem $-u^*_{xx} = f$ on the interval $\Omega = (0, 1)$ with homogeneous Dirichlet boundary conditions on a uniform mesh in $\overline{\Omega}_h = \Omega_h \cup x_0 \cup x_{n+1}$, where $\Omega_h = \{x_j\}_{j=1}^n$ is the set of interior nodes of the mesh, and $x_0 = 0$ and $x_{n+1} = 1$ are the boundary nodes. Denote $h = 1/(n + 1)$ as the mesh size. The discretization is obtained by setting $u(x_0) = u(x_{n+1}) = 0$ and

$$(-\Delta_h u)(x_j) = h^{-2} \left(-u(x_{j-1}) + 2u(x_j) - u(x_{j+1})\right) \quad j = 1, \cdots, n.$$

Denote the inner product in $L^2_h(0, 1)$ (which we denote by $V_h$) by

$$(u, v) \equiv (u, v)_h = h \sum_{j=1}^{n} u(x_j)\, v(x_j) \quad \text{and denote} \quad \|v\|^2 = (v, v).$$

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We introduce the matrix $A$

$$(v, Au) = (v, -\Delta_h u).$$

also as an operator defined on $L^2_h(0,1)$ with inner product $(\cdot,\cdot)$ and zero Dirichlet data at $x_0 = 0$ and $x_{n+1} = 1$. Here the matrix and the operator $A$ will be denoted by the same letter. It is known that $(Av, v) = (\nabla I_h v, \nabla I_h v)_{L^2(0,1)}$ for $v \in V_h$, where $I_h v$ is the piecewise linear and continuous function with given $v(x_j)$ for $0 \le j \le n+1$.

In order to avoid proliferation of constants, we will often use the notation $A \preceq B$ ($A \succeq B$) to represent $A \le cB$ ($A \ge cB$) where the positive constant $c$ is independent of $h, H, \delta, \ell$ and $r$.

## 2 ASM, RAS, ASH and RASH methods

Let us decompose the nodes of $\Omega_h$ into $N$ subdomains and without loss of generality assume that $m = n/N$ is an integer; see Fig. 1 with $n = 28$, $N = 4$ and $\ell = 2$. Define the nonoverlapping subdomains nodes of $\Omega_{ih}$

$$\Omega_{ih} = \{x_{j+1}, x_{j+2}, \cdots, x_{j+m}\}, \quad \text{where} \quad j = (i-1)\,m,\ 1 \le i \le N.$$

Let $\ell \ge 0$ be an integer and let $\delta = (1+\ell)h$. We note that $\ell = 0$ is related a block diagonal preconditioner. Let the extended subdomain nodes of $\Omega_{i\delta}$ be obtained by extending by $\ell$ nodes to each side of $\Omega_{ih}$ inside $\Omega_h$, that is,

$$\Omega_{i\delta} = \{x_{j+1-\ell}, x_{j+2-\ell}, \cdots, x_{j+m+\ell}\} \cap \Omega_h, \quad \text{where} \quad j = (i-1)m,\ 1 \le i \le N.$$

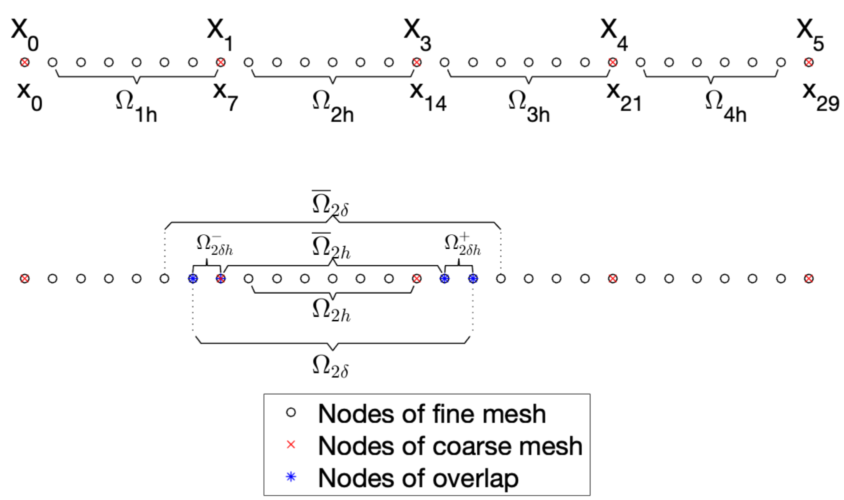


**Fig. 1** (top) $\Omega_h$ with $n = 28$ nodes decomposed into four subdomains $\Omega_{ih}$ with $V_0^1$ coarse nodes. (below) The visualization of $\Omega_{ih}$, $\overline{\Omega}_{ih}$, $\Omega_{i\delta}$, $\overline{\Omega}_{i\delta}$, and $\Omega_{i\delta h} = \Omega^-_{i\delta h} \cup \Omega^+_{i\delta h}$ when $i = 2$ and $\ell = 2$.

The mathematical analysis introduced below can also be extended easily for the case the domain decomposition is obtained by nonoverlapping subdomains elements. We also use the notation $\overline{\Omega}_{i\delta} = \{x_{j-\ell}, x_{j+1-\ell}, \cdots, x_{j+m+\ell+1}\} \cap \overline{\Omega}_h$ and $\overline{\Omega}_{ih} = \{x_j, x_{j+1}, \cdots, x_{j+m+1}\} \cap \overline{\Omega}_h$ to include their boundary nodes $\partial\Omega_{i\delta}$ and $\partial\Omega_{ih}$, respectively. Note that here and below $j$ is a function of $i$ given by $j = (i-1)m$ for $1 \leq j \leq N$.

Associated to each $\Omega_{i\delta}$, we introduce the restriction operator $R_{i\delta}$. In matrix terms, $R_{i\delta}$ is an $m_i \times n$ matrix such that $(R_{i\delta}v)(x_j) = v(x_j)$ for $x_j \in \Omega_{i\delta}$, $\forall v \in V_h$. Here, $m_1 = m + \ell$, $m_i = m + 2\ell$ for $2 \leq i \leq N-1$ and $m_N = m + \ell$. Define $A_{i\delta} = R_{i\delta} A R_{i\delta}^T$.

Associated to each $\Omega_{i\delta}$ and $\Omega_{ih}$, we introduce the restriction operator $\tilde{R}_{ih}$. In matrix terms, $\tilde{R}_{ih}$ is an $m_i \times n$ matrix such that $(\tilde{R}_{ih}v)(x_j) = v(x_j)$ for $x_j \in \Omega_{ih}$ and $(\tilde{R}_{ih}v)(x_j) = 0$ for $x_j \in \Omega_{i\delta} \backslash \Omega_{ih}$, $\forall v \in V_h$. The superscript tilde notation is used to recall $\tilde{R}_{ih}$ maps to $\Omega_{i\delta}$ rather than $\Omega_{ih}$. For analysis, we will also consider $R_{i\delta h} = R_{i\delta} - \tilde{R}_{ih}$ and denote $\Omega_{i\delta h} = \Omega_{i\delta} \backslash \Omega_{ih}$.

We will also consider preconditioners with a coarse problem. In order to mimic the 2D and 3D difficulties, we consider two cases of coarse spaces, the $V_0^1$ and the $V_0^2$ coarse spaces.

$V_0^1$ *case*: The coarse nodes are given by $\Omega_H = \{X_i\}_{i=1}^{N-1}$ and $\overline{\Omega}_H = \{X_i\}_{i=0}^{N}$ where $X_i = imh$ for $0 \leq i \leq N$ and with a zero Dirichlet data at $X_0 = x_0$ and $X_N = x_{n+1}$. In other words, the coarse node $X_i$ is the rightmost node of $\Omega_{ih}$ for $1 \leq i \leq N-1$. In this case, the coarse nodes belong to the overlapping region (if $\ell \geq 1$).

$V_0^2$ *case*: The coarse nodes are given by $\Omega_H = \{X_i\}_{i=1}^{N}$ and $\overline{\Omega}_H = \{X_i\}_{i=0}^{N+1}$ where the coarse nodes are $X_i = (i-1)mh + \lfloor m/2 \rfloor h$ for $1 \leq i \leq N$, and $X_0 = x_0$ and $X_{N+1} = x_{n+1}$. Here, $\lfloor m/2 \rfloor$ is the integer part of $m/2$. In other words, the coarse node $X_i$ is about the mid node of $\Omega_{ih}$. This is the case the coarse nodes belong to just one extended subdomain when $\ell$ is not too large.

In both cases, zero Dirichlet data is imposed at the end nodes. The extrapolation operator $R_0^T$ from $\Omega_H$ to $\Omega_h$ is the embedding piecewise linear and continuous coarse functions on the coarse triangulation $\overline{\Omega}_H$ to the fine mesh $\Omega_h$. Define the coarse matrix by $A_0 = R_0 A R_0^T$.

The Additive Schwarz Method–ASM preconditioner is defined by

$$T_{\text{asm}} = B_{\text{asm}}^{-1} A = \left( \sum_{i=1}^{N} R_{i\delta}^T A_{i\delta}^{-1} R_{i\delta} + R_0^T A_0^{-1} R_0 \right) A.$$

The Restricted Additive Schwarz Method–RAS preconditioner is defined by

$$T_{\text{ras}} = B_{\text{ras}}^{-1} A = \left( \sum_{i=1}^{N} \tilde{R}_{ih}^T A_{i\delta}^{-1} R_{i\delta} + R_0^T A_0^{-1} R_0 \right) A.$$

The Additive Schwarz with Harmonic Overlap Method–ASH preconditioner is given by

$$T_{\mathrm{ash}} = B_{\mathrm{ash}}^{-1} A = \left( \sum_{i=1}^{N} R_{i\delta}^T A_{i\delta}^{-1} \tilde{R}_{ih} + R_0^T A_0^{-1} R_0 \right) A.$$

The symmetrized RAS method, denoted by RASH, is defined by

$$T_{\mathrm{rash}} = B_{\mathrm{rash}}^{-1} A = \left( \sum_{i=1}^{N} \tilde{R}_{ih}^T A_{i\delta}^{-1} \tilde{R}_{ih} + R_0^T A_0^{-1} R_0 \right) A.$$

By construction, the matrices $B_{\mathrm{asm}}^{-1}$, $B_{\mathrm{ras}}^{-1}$, $B_{\mathrm{ash}}^{-1}$ and $B_{\mathrm{rash}}^{-1}$ are well defined. It is well known that $B_{\mathrm{asm}}^{-1}$ is symmetric positive definite. The contributions of this paper proceedings are: 1) to show that $B_{\mathrm{ras}}^{-1}$ and $B_{\mathrm{ash}}^{-1}$ are nonsymmetric and positive definite on subspaces of $V_h$ and, 2) to establish their lower and upper bounds for exact local solvers. Lower and upper bounds for $B_{\mathrm{rash}}^{-1}$ are also established.

The original system (1) is solved by Richardson iterative methods with an optimal relaxation parameter (or GMRES) with a $B^{-1}$ left preconditioner, where $B^{-1}$ will be $B_{\mathrm{asm}}^{-1}$, $B_{\mathrm{ras}}^{-1}$, $B_{\mathrm{ash}}^{-1}$ or $B_{\mathrm{rash}}^{-1}$. We discuss two interpretations (residual and solution vectors) of the methods. Then the analysis of convergence of the discussed method is given. The Richardson iterative method for the solution vector is given by

$$u^{k+1} = u^k - \tau B^{-1}(Au^k - f), \tag{2}$$

where $\tau > 0$ is a relaxation parameter. By multiplying (2) by $A$ and setting the residual vector $r^k = Au^k - f$ we get

$$r^{k+1} = r^k - \tau A B^{-1} r^k. \tag{3}$$

We recall that $(u, v) = h \sum_{i=1,n} u(x_i)v(x_i)$ and denote $\|u\|_C^2 = (u, Cu)$ for any symmetric positive definite matrix $C$. The convergence analysis of $\|u - u^k\|_A$-norm follows from the convergence analysis of (3) with the $\|r^k\|_{A^{-1}}$-norm, and vice-versa, since $r^k = A(u^k - u)$. A bound for the convergence rate for (3) with the optimal parameter $\tau_k$, or for the GMRES on the $A$-norm, is given by the following well known lemma, for example, see Lemma C.11 of [4].

**Lemma 1.** *Assume that for any* $r \in \mathbb{R}^n$

$$\gamma_1 (A^{-1} r, r) \le (B^{-1} r, r) \tag{4}$$

*and*

$$(AB^{-1} r, B^{-1} r) \le \gamma_2 (A^{-1} r, r). \tag{5}$$

*Then the iterative method* (3) *converges with rate*

$$\|r^{k+1}\|_{A^{-1}} \le \rho_*^k \|r^k\|_{A^{-1}} \quad \textit{with optimal} \quad \tau_* = \gamma_1/\gamma_2 \quad \textit{and} \quad \rho_* = (1 - \gamma_1^2/\gamma_2)^{1/2}.$$

# 3 Reduction of the iterative scheme to a subspace

## 3.1 ASH inital correction

We first discuss $B_{\text{ash}}^{-1}$ without the coarse problem. Let $u^0$ be determined by

$$u^0 = B_{\text{ash}}^{-1} A u = B_{\text{ash}}^{-1} f.$$

The problem (1) now reduces to solving $A\hat{u} = \hat{f}$ where $\hat{f} = f - Au^0$ and $\hat{u} = u - u^0$. Denote $\mathbb{R}^n$ as the Euclidean space, and denote $\mathbb{R}^n_{\text{ash}} \subset \mathbb{R}^n$ as the set of residual vectors which are zero at all nodes except at the nodes of $\cup_{i=1}^{N} \partial\Omega_{i\delta} \cap \Omega_h$. It is easy to see, by using that $\sum_{i=1}^{N} R_{i\delta}^T \tilde{R}_{ih} = I_n$ that $\hat{f} \in \mathbb{R}^n_{\text{ash}}$. Let $\mathbb{V}^h_{\text{ash}} = A^{-1}\mathbb{R}^n_{\text{ash}}$ be the space of discrete harmonic vectors on $\Omega_h$ except at the nodes of $\cup_{i=1}^{N} \partial\Omega_{i\delta} \cap \Omega_h$. Note that $\hat{u} \in V^h_{\text{ash}}$. We also note that the subspace $\mathbb{R}^n_{\text{ash}}$ is a natural choice since $A(u^k - u^{k-1}) \in \mathbb{R}^n_{\text{ash}}$ for the preconditioned Richardson with $\tau = 1$ without the initial correction. From now on, we assume this initial correction was performed and the superscript hat is dropped. Consider the Richardson method, with $u^0 = 0$,

$$u^{k+1} = u^k - \tau B_{\text{ash}}^{-1}(Au^k - f) \quad k = 0, 1, \cdots \tag{6}$$

It is not hard to see, by recursion, that $r^k \in \mathbb{R}^n_{\text{ash}}$ and $u^k \in V^h_{\text{ash}}$ for $k = 0, 1, 2, \cdots$.

**Lemma 2.** [1] *For $u \in V^h_{\text{ash}}$*

$$B_{\text{ash}}^{-1} A u = B_{\text{asm}}^{-1} A u.$$

*Proof.* It follows from $\tilde{R}_{ih} A u = R_{i\delta} A u$ for $u \in \mathbb{V}^n_{\text{ash}}$. □

As consequence, the upper and lower bounds for $B_{\text{asm}}^{-1}$ on the space $V_{\text{ash}}$ are also the upper and lower bounds for $B_{\text{ash}}^{-1}$. We note Lemma 2 also holds for the strip case in 2D and 3D since no more than two extended subdomains overlap the same node.

We now consider the ASH method with a coarse space. First note that the image of $AR_0^T$ vanishes at all nodes except the coarse nodes. Therefore if there are no coarse nodes in any of the $\Omega_{i\delta h}$, then Lemma 2 holds and this is the $V_0^2$ case. Therefore, we consider coarse spaces where the coarse nodes are in the overlapping regions, which is the $V_0^1$ coarse space case. It is easy to see after the initial correction $u^0$, $\mathbb{R}^n_{\text{ash}} \subset \mathbb{R}^n$ is now the set of residual vectors which are zero at all nodes except for the nodes of $\cup_{i=1}^{N} \partial\Omega_{i\delta} \cap \Omega_h$ and at the coarse nodes. It easy to see that all the $u^k \in \mathbb{V}^n_{\text{ash}} := A^{-1}\mathbb{R}^n_{\text{ash}}$ and that Lemma 2 does not hold. New techniques are introduced below to treated this case.

### 3.2 RAS and RASH initial corrections

After an initial correction $\hat{u}^0 = B_{\text{ras}}^{-1} f$, $\mathbb{R}^n_{\text{ras}} \subset \mathbb{R}^n$ is now the set of RAS residual vectors which are zero at all nodes except for the nodes on $\cup_{i=1}^N \partial\Omega_{ih} \cap \Omega_h$ and at the coarse nodes. After a correction $\hat{u}^0 = B_{\text{ras}}^{-1} f$ or $\hat{u}^0 = B_{\text{rash}}^{-1} f$, $\mathbb{R}^n_{\text{rash}} = \mathbb{R}^n_{\text{ras}}$.

## 4 Lower and upper bounds for ASH, RAS and RASH methods

Note that $B_{\text{ras}}^{-1} \geq \gamma_1 A^{-1}$ is equivalent to $B_{\text{ash}}^{-1} \geq \gamma_1 A^{-1}$ on the space $\mathbb{R}^n$ since

$$(B_{\text{ras}}^{-1} r, r) = (r, B_{\text{ash}}^{-1} r) = (B_{\text{ash}}^{-1} r, r) \quad r \in \mathbb{R}^n. \tag{7}$$

We note however that the lower bound for $B_{\text{ash}}^{-1}$ for $r \in \mathbb{R}^n_{\text{ash}}$ is not necessarily equivalent to the lower bound for $B_{\text{ras}}^{-1}$ for $r \in \mathbb{R}^n_{\text{ras}}$, therefore, separate analyses are done for the ASH and RAS methods. In order to establish the lower bounds for the ASH and RAS, we introduce the following interesting result:

**Lemma 3.** *For any $r \in \mathbb{R}^n$,*

$$2(B_{\text{ash}}^{-1} r, r) = 2(B_{\text{ras}}^{-1} r, r) = (B_{\text{asm}}^{-1} r, r) + (B_{\text{rash}}^{-1} r, r) - \sum_{i=1}^N (A_{i\delta}^{-1} R_{i\delta h} r, R_{i\delta h} r). \tag{8}$$

*Proof.* First we add and subtract $\tilde{R}_{i\delta h}$ to obtain

$$(B_{\text{ash}}^{-1} r, r) = \sum_{i=1}^N (A_{i\delta}^{-1} R_{i\delta} r, \tilde{R}_{ih} r) + (A_0^{-1} R_0 r, R_0 r) = (B_{\text{asm}}^{-1} r, r) - \sum_{i=1}^N (A_{i\delta}^{-1} R_{i\delta} r, R_{i\delta h} r),$$

and using $R_{i\delta} = R_{i\delta h} + \tilde{R}_{ih}$ we have

$$\begin{aligned}
(B_{\text{ash}}^{-1} r, r) =& (B_{\text{asm}}^{-1} r, r) - \sum_{i=1}^N (A_{i\delta}^{-1} \tilde{R}_{ih} r, R_{i\delta h} r) - \sum_{i=1}^N (A_{i\delta}^{-1} R_{i\delta h} r, R_{i\delta h} r), \quad \text{hence,} \\
(B_{\text{ash}}^{-1} r, r) =& (B_{\text{asm}}^{-1} r, r) - \sum_{i=1}^N (A_{i\delta}^{-1} \tilde{R}_{ih} r, R_{i\delta} r) + \sum_{i=1}^N (A_{i\delta}^{-1} \tilde{R}_{ih} r, \tilde{R}_{ih} r) \\
&- \sum_{i=1}^N (A_{i\delta}^{-1} R_{i\delta h} r, R_{i\delta h} r)
\end{aligned}$$

and the lemma follows by adding and subtracting $(A_0^{-1} R_0 r, R_0 r)$. □

In order to use equation (8) to establish the lower bound of RAS and ASH, we need to understand the lower bound for RASH, which is treated at the end of this section.

We assume from now on that $\Omega_{(i+1)\delta} \cap \Omega_{(i-1)\delta} = \emptyset$, that is, the overlap $\delta = (1+\ell)h$ is not too large. We recall that $\ell = 0$ is the block Jacobi preconditioner and that ASH, RAS and RASH are all equal to the ASM.

**We first consider the ASH lower bound** with $B^{-1} = B^{-1}_{\text{ash}}$. Since the coarse space $V_0^2$ has already been treated in the previous section, in the next lemma we consider only the $V_0^1$ case.

**Lemma 4.** *For any $r \in \mathbb{R}^n_{\text{ash}}$, there exists $\gamma_1 = O(1 + \frac{H}{\delta})^{-1}$ for which (4) holds.*

*Proof.* The strategy of the proof is the following: Consider the equality (8) and use the following three steps:
*Step 1*: Consider the equality (8)
*Step 2*: Find a positive number $c_1$ such that

$$(A^{-1}_{i\delta} R_{i\delta h} r, R_{i\delta h} r) \leq c_1 h^2 \|R_{i\delta h} r\|^2 \quad 1 \leq i \leq N.$$

*Step 3*: Find positive numbers $c_2$ and $c_3$ and let $0 \leq \gamma \leq 1$ such that

$$\sum_{i=1}^{N} \|R_{i\delta h} r\|^2 \leq h^{-2} \sum_{i=1}^{N} \left( \gamma c_2 (A^{-1}_{i\delta} R_{i\delta} r, R_{i\delta} r) + (1-\gamma) c_3 (A^{-1}_{i\delta} \tilde{R}_{ih} r, \tilde{R}_{ih} r) \right).$$

Then using Steps 1 and 2 we obtain

$$\sum_{i=1}^{N} (A^{-1}_{i\delta} R_{i\delta h} r, R_{i\delta h} r)| \leq \gamma c_1 c_2 (B^{-1}_{\text{asm}} r, r) + (1-\gamma) c_1 c_3 (B^{-1}_{\text{rash}} r, r).$$

*Step 3*: Choose a $\gamma$ such that $\max\{\gamma c_1 c_2, (1-\gamma) c_1 c_3\} < 1$, independent of $H, h$ and $\delta$. Then use equality (8), and the RASH lower bound (see Lemma 8) and the ASM lower bound [4] to obtain the lower bound $O(1 + H/\delta)^{-1}$.

**Step 1** Assume that $r \in \mathbb{R}^n_{\text{ash}}$ and let $u_{i\delta h} := A^{-1}_{i\delta} \tilde{R}_{i\delta h} r$. The $\Omega_{i\delta}$ is given by (see Fig. 1)

$$\Omega_{i\delta} = \{x_{j+1-\ell}, \cdots, x_{j+m+\ell}\} \cap \Omega_h, \quad j = j(i) = (i-1)\,m, \quad 1 \leq i \leq N$$

see Fig. 1, and let

$$\overline{\Omega}_{i\delta} = (x_{j-\ell} \cup \Omega_{i\delta} \cup x_{j+m+\ell+1}) \cap \overline{\Omega}_h.$$

Remember that $\Omega_{i\delta h} = \Omega_{i\delta} \backslash \Omega_{ih}$. Decompose $\Omega_{i\delta h} = \Omega^-_{i\delta h} \cup \Omega^+_{i\delta h}$, where

$$\Omega^-_{i\delta h} = \{x_{j+1-\ell}, \cdots x_j\} \cap \Omega_h \quad \text{and} \quad \Omega^+_{i\delta h} = \{x_{j+m+1}, \cdots x_{j+m+\ell}\} \cap \Omega_h.$$

Note that $\Omega^-_{1\delta h}$ and $\Omega^+_{N\delta h}$ are empty sets and $\Omega^-_{i\delta h} \subset \Omega_{(i-1)h}$ for $2 \leq i \leq N$, and $\Omega^+_{i\delta h} \subset \Omega_{(i+1)h}$ for $1 \leq i \leq N-1$.

The only node where $R_{i\delta h} r$ is not necessarily zero is at $x_j \in \Omega^-_{i\delta h}$ since for the coarse nodes of $V_0^1$, it has no coarse nodes in $\Omega^+_{i\delta h}$. We have

$$(A^{-1}_{i\delta} R_{i\delta h} r, R_{i\delta h} r) = (u_{i\delta h}, R_{i\delta h} r) = h u_{i\delta h}(x_j) r(x_j) = \|R_{i\delta h} r\| h^{1/2} |u_{i\delta h}(x_j)|.$$

Note that $u_{i\delta h} = A_{i\delta}^{-1} R_{i\delta h} r$ vanishes at $x_{j-\ell}$ (the node on the boundary of $\overline{\Omega}_{i\delta}$ inside $\Omega_{(i-1)h}$), and it is linear (harmonic) from $x_{j-\ell}$ to $x_j$. We can relate $|u_{i\delta h}(x_j)|$ with its energy on the interval $(x_{j-\ell}, x_j)$ since $u_{i\delta h}(x_{j-\ell}) = 0$ and

$$hu_{i\delta h}^2(x_j) = \ell h^2 \left(\frac{u_{i\delta h}(x_j) - u_{i\delta h}(x_{j-\ell})}{h\ell}\right)^2 \ell h = \ell h^2 |u_{i\delta h}|^2_{H^1(x_{j-\ell}, x_j)},$$

and

$$|u_{i\delta h}|^2_{H^1(x_{j-\ell}, x_j)} \leq (A_{i\delta} u_{i\delta h}, u_{i\delta h}) = (A_{i\delta}^{-1} R_{i\delta h} r, R_{i\delta h} r).$$

Hence, we obtain $c_1 = \ell$.

**Step 2** Denote $R_{i\delta h}^{(i-1)} = R_{(i-1)\delta} R_{i\delta}^T R_{i\delta h}$. Easy to see that

$$\begin{aligned}\|R_{i\delta h} r\|^2 =& r(x_j)^2 \\ =& \frac{\gamma}{2}(R_{(i-1)\delta} r, R_{i\delta h}^{(i-1)} r) + \frac{\gamma}{2}(R_{i\delta} r, R_{i\delta h} r) + (1-\gamma)(\tilde{R}_{(i-1)h} r, R_{i\delta h}^{(i-1)} r).\end{aligned}$$

Let us first bound $(R_{(i-1)\delta} r, R_{i\delta h}^{(i-1)} r)$. Denote $u_{(i-1)\delta} = A_{(i-1)\delta}^{-1} R_{(i-1)\delta} r$. First see that $u_{(i-1)\delta}$ vanishes at $x_{j+1+\ell}$ (the rightmost node of $\overline{\Omega}_{(i-1)\delta}$), is linear from $x_j$ (a coarse node) to $x_{j+1+\ell}$, and is linear from $x_{j-\ell}$ (the leftmost node of $\overline{\Omega}_{i\delta}$) to $x_j$. Hence, we obtain $u_{(i-1)\delta} = A_{(i-1)\delta}^{-1} R_{(i-1)\delta} r$,

$$(R_{(i-1)\delta} r, R_{i\delta h}^{(i-1)} r) = (A_{(i-1)\delta} u_{(i-1)\delta}, R_{i\delta h}^{(i-1)} r) = (A_{(i-1)\delta} u_{(i-1)\delta}, E(R_{i\delta h}^{(i-1)} r)),$$

where $E(R_{i\delta h}^{(i-1)} r) \in V_h(\Omega_{(i-1)\delta})$ is an extension of $r(x_j)$, where $(E(R_{i\delta h}^{(i-1)} r)(x_j) = r(x_j)$, vanishes at $x_{j+1+\ell}$ and $x_{j-\ell}$ and is linear in the subintervals $(x_{j-\ell}, x_j)$ and $(x_j, x_{j+1+\ell})$. We have

$$(R_{(i-1)\delta} r, R_{i\delta h}^{(i-1)} r) \leq |u_{(i-1)\delta}|_{H^1(x_{j-\ell}, x_{j+1+\ell})} |E(R_{i\delta h}^{(i-1)} r)|_{H^1(x_{j-\ell}, x_{j+1+\ell})}.$$

And using the same arguments as above, we have

$$|E(R_{i\delta h}^{(i-1)} r)|^2_{H^1(x_{j-\ell}, x_{j+1+\ell})} = \frac{1}{h^2}\left(\frac{1}{\ell} + \frac{1}{\ell+1}\right) hr^2(x_j).$$

Hence,

$$(R_{(i-1)\delta} r, R_{i\delta h}^{(i-1)} r) \leq h^{-1}\left(\frac{1}{\ell} + \frac{1}{\ell+1}\right)^{1/2} |u_{(i-1)\delta}|_{H^1(x_{j-\ell}, x_{j+1+\ell})} \|R_{i\delta h} r\|.$$

Now let us bound $(\tilde{R}_{(i-1)h} r, R_{i\delta h}^{(i-1)} r)$. Define $u_{(i-1)h} = A_{(i-1)\delta}^{-1} \tilde{R}_{(i-1)h} r$ and see that $u_{(i-1)h}$ is also harmonic on the subintevals $(x_{j-\ell}, x_j)$ and $(x_j, x_{j+1+\ell})$. Using the same arguments as above we obtain

$$(R_{(i-1)h} r, R_{i\delta h}^{(i-1)} r) \leq h^{-1}\left(\frac{1}{\ell} + \frac{1}{\ell+1}\right)^{1/2} |u_{(i-1)h}|_{H^1(x_{j-\ell}, x_{j+1+\ell})} \|R_{i\delta h} r\|.$$

Now let us bound $(R_{i\delta}r, R_{i\delta h}r$ and let $u_{i\delta} = A_{i\delta}^{-1}R_{i\delta}r)$. Using similar arguments

$$(R_{i\delta}r, R_{i\delta h}r) \leq h^{-1}\left(\frac{1}{\ell} + \frac{1}{\ell+1}\right)^{1/2} |u_{i\delta}|_{H^1(x_{j-\ell}, x_{j+1+\ell})} \|R_{i\delta h}r\|.$$

Hence, we obtain $2c_2 = c_3 = \left(\frac{1}{\ell} + \frac{1}{\ell+1}\right)$

**Step 3** A proper choice is $\gamma = 2/3$ which gives $\gamma c_1 c_2 = (1-\lambda)c_1 c_3 < 2/3$. □

**We now consider the RAS lower bound** for $B^{-1} = B_{\text{ras}}^{-1}$ for both $V_0^1$ and $V_0^2$. Independently if we use $V_0^1$ or $V_0^2$, we have nonzero residuals at $x_j$, $x_{j+1}$, $x_{j+m}$ and $x_{j+m+1}$. If $V_0^2$ is used, a nonzero residuals will show up also at $x_{j+[m/2]}$.

**Lemma 5.** *For any $r \in \mathbb{R}^n_{\text{ras}}$, there exists $\gamma_1 = O(1 + \frac{H}{h})^{-1}$ for which* (4) *holds.*

*Proof.* We follow the same strategy as in the proof of the previous lemma.

**Step 1** Assume $2 \leq i \leq N-1$. Decompose

$$R_{i\delta h} = R_{i\delta h}^{-} + R_{i\delta h}^{+},$$

where $R_{i\delta h}^{-}r$ and $R_{i\delta h}^{+}r$ vanish on $\Omega_{i\delta}$ except at the nodes $x_j$ and $x_{j+m+1}$, respectively. We have

$$(A_{i\delta}^{-1}R_{i\delta h}r, R_{i\delta h}r) = hu_{i\delta h}(x_j)r(x_j) + hu_{i\delta h}(x_{j+m+1})r(x_{j+m+1})$$

and the $|u_{i\delta h}(x_j)|$ and $|u_{i\delta h}(x_{j+m+1})|$ are now controlled by the energy on the intervals $(x_{j-\ell}, x_j)$ and $(x_{j+m+1}, x_{j+m+1+\ell})$, respectively. Using the same arguments as above we obtain

$$(A_{i\delta}^{-1}R_{i\delta h}r, R_{i\delta h}r) \leq h^2\,\ell\left(\|R_{i\delta h}^{-}r\|^2 + \|R_{i\delta h}^{+}r\|^2\right).$$

**Step 3** Assume $2 \leq i \leq N-1$. Denote $R_{i\delta h}^{(i-1)-} = R_{(i-1)\delta}R_{i\delta}^{T}R_{i\delta h}$. We have

$$\|R_{i\delta h}^{-}r\|^2 = r(x_j)^2 = \gamma(R_{(i-1)\delta}r, R_{i\delta h}^{(i-1)-}r) + (1-\gamma)(\tilde{R}_{(i-1)h}r, R_{i\delta h}^{(i-1)-}r).$$

The $R_{i\delta h}^{+}$ case can be treated similarly. A difference now with respect to the ASH analysis is also that $u_{i\delta}$ now is not discrete harmonic at $x_{j+1}$, therefore, $E(R_{i\delta h}^{(i-1)-}r)$ can be extended from $r(x_j)$ linearly on the interval $(x_{j-\ell}, x_j)$ however with just a zero extension on $(x_j, x_{j+1})$. Another difference is that we cannot include the term $(R_{i\delta}r, R_{i\delta h}^{-}r)$ because the estimates would overlap with estimates for $(R_{i\delta}r, R_{(i-1)\delta h}^{(i)+}r)$ on the interval $(x_j, x_{j+1})$. Fortunately, the region where $u_{(i-1)h}$ and $u_{(i-1)\delta}$ now are harmonic in the larger region from $x_{j-m+\lfloor m/2\rfloor}$ (the midpoint of $\Omega_{ih}$) to $x_j$. Denote $L_i^{-} = (x_{j-m+\lfloor m/2\rfloor}, x_{j+1+\ell})$. We obtain

$$h^2\|R^-_{i\delta h}r\|^2 \leq \gamma\left(\frac{1}{m-\lfloor m/2\rfloor}+1\right)|u_{(i-1)\delta}|^2_{H^1(L_i^-)} + (1-\gamma)\left(\frac{1}{m-\lfloor m/2\rfloor}+\frac{1}{1+\ell}\right)|u_{(i-1)h}|^2_{H^1(L_i^-)}.$$

Gathering Steps 1 and 2 together we obtain

$$\sum_{i=1}^{N}(A^{-1}_{i\delta}R_{i\delta h}r, R_{i\delta h}r) \leq \gamma\left(1+\frac{\ell}{\lfloor m/2\rfloor}\right)(B^{-1}_{\text{asm}}r,r) + (1-\gamma)\left(\frac{\ell}{1+\ell}+\frac{\ell}{\lfloor m/2\rfloor}\right)(B^{-1}_{\text{rash}}r,r).$$

**Step 3** Let us choose $\gamma = 1/(2+\ell)$, that is, when $\gamma\ell = (1-\gamma)\frac{\ell}{1+\ell}$. We obtain

$$(1+\ell/2+o(1))(B^{-1}_{\text{ras}}r,r) \geq (B^{-1}_{\text{asm}}r,r) + (B^{-1}_{\text{rash}}r,r),$$

where $o(1)$ is a tiny positive number when $m$ is large compared to $\ell$. The result follows from the lower bounds for ASM and RASH since $O(1+H/\delta) * (1+\delta/h+o(1)) = O(1+H/h)$. □

**We now consider the ASH upper bound**.

**Lemma 6.** *For all $r \in \mathbb{R}^n_{\text{ash}}$, there exists $\gamma_1 = O(1)$ for which* (5) *holds.*

*Proof.* Since a node does not belong to more than two extended subdomains, we have

$$(AB^{-1}_{\text{ash}}r, B^{-1}_{\text{ash}}r) \leq 3\sum_{i=1}^{N}\left(AR^T_{i\delta}A^{-1}_{i\delta}\tilde{R}_{ih}r, R^T_{i\delta}A^{-1}_{i\delta}\tilde{R}_{ih}r\right) + 3\left(AR^T_0A^{-1}_0R_0r, R^T_0A^{-1}_0R_0r\right)$$

and see that

$$\left(AR^T_0A^{-1}_0R_0r, R^T_0A^{-1}_0R_0r\right) = (R_0r, A^{-1}_0R_0r),$$
$$\left(AR^T_{i\delta}A^{-1}_{i\delta}\tilde{R}_{ih}r, R^T_{i\delta}A^{-1}_{i\delta}\tilde{R}_{ih}r\right) = (A^{-1}_{i\delta}\tilde{R}_{ih}r, \tilde{R}_{ih}r)$$

and using the same analysis of Step 2 of Lemma 4 with $\gamma = 1$, and the classical ASM upper bounds

$$(A^{-1}_{i\delta}\tilde{R}_{ih}r, \tilde{R}_{ih}r) \leq 2(A^{-1}_{i\delta}R_{i\delta}r, R_{i\delta})r + 2(A^{-1}_{i\delta}R_{i\delta h}r, R_{i\delta h}r) \leq (2+\ell(\frac{1}{\ell}+\frac{1}{1+\ell}))(A^{-1}_{i\delta}R_{i\delta}r, R_{i\delta}r).$$

□

**We now consider the RAS upper bound**.

**Lemma 7.** *For all $r \in \mathbb{R}^n_{\text{ras}}$, there exists $\gamma_2 = O(1+\ell)$ for which* (5) *holds.*

*Proof.* Following the initial steps of the proof of Lemma 6, we now need to estimate

$$\left(\tilde{R}_{ih}^T A_{i\delta}^{-1} R_{i\delta} r, A\tilde{R}_{ih}^T A_{i\delta}^{-1} R_{i\delta} r\right) = (\tilde{R}_{ih}^T u_{i\delta}, A\tilde{R}_{ih}^T u_{i\delta}) \quad \text{where} \quad u_{i\delta} = A_{i\delta}^{-1} R_{i\delta} r.$$

We have

$$\begin{aligned}(\tilde{R}_{ih}^T u_{i\delta}, A\tilde{R}_{ih}^T u_{i\delta}) =& |u_{i\delta}|^2_{H^1(x_{j+1},x_{j+m})} + \frac{1}{h} u_{i\delta}(x_{j+1})^2 + \frac{1}{h} u_{i\delta}(x_{j+m})^2 \\ \leq& (1+\ell)(u_{i\delta}, A_{i\delta} u_{i\delta}).\end{aligned}$$

The result follows from the classical ASM upper bound [4]. □

Due to space limitations and since the analysis for RASH follows the classical abstract Schwarz theory for positive symmetric definite operators, the proofs for the RASH lower and upper bounds are ommited.

**Lemma 8.** *For any $r \in \mathbb{R}^n$, there exists $\gamma_1 = O(1 + \frac{H}{\delta})^{-1}$ for which* (4) *holds.*

**Lemma 9.** *For all $r \in \mathbb{R}^n_{\text{ras}}$, there exists $\gamma_2 = O(1 + \ell)^2$ for which* (5) *holds.*

**Final Remark**: The techniques used in the proofs for the two-level ASH and RAS hold also for their one-level versions, where in Step 3 we replace the lower bounds for the ASM and RASH from $O(1 + H/\delta)$ by $O(1 + 1/H\delta)$.

## 5 Numerical section and conclusions and future directions

We consider $\Omega = (0, 1)$ and fix $H/h = 64$ and $1/H = 8$ and vary $\ell$. We now test numerically the optimal lower and upper bounds of Lemma 1 by finding the smallest eigenvalue of $\frac{1}{2}(B^{-1} + B^{-T})r = \lambda_1 A^{-1}$ and the largest eigenvalue of $B^{-T} AB^{-1} v = \lambda_2 A^{-1}$. Here $B^{-T}$ stands for the transpose of $B^{-1}$. The convergence rate of GMRES or the Richardson with optimal parameter is related to $\sqrt{1 - (\gamma_1/\sqrt{\gamma_2})^2}$, hence, we provide numerically $\gamma_1$ and $\sqrt{\gamma_2}$.

In Table 1, $\gamma_1$ and $\sqrt{\gamma_2}$ (in parenthesis) are provided for ASH, RAS, RASH and ASM with no coarse space. The generalized eigenvalue problems described above are solved on reduced spaces, that is, on the subspace $\mathbb{R}^n_{\text{ash}}$ for ASH and ASM methods, and on the subspace $\mathbb{R}^n_{\text{ras}}$ for RAS and RASH. As predicted by Lemma 2, ASH and ASM methods are the same method and satisfy the $O(1+1/(H\delta))^{-1}$ (since we have no coarse space) for the lower bound and the $O(1)$ for the upper bound. The theory for the RASH method is also sharp by Lemmas 8 and 9. Clearly, RASH is not a good method due to mostly the upper bound. We were successful in showing that $B^{-1}_{\text{ras}}$ is positive on the subspace $\mathbb{R}^n_{\text{ras}}$ however we can see from the Table 1 that the theoretical upper and lower bounds are not sharp by a $O(1 + \ell)$ factor. It is an open problem to improve both bounds.

In Table 2, we run the previous test except that we add the coarse space $V_0^2$. The conclusions are similar except that the lower bounds are related to $O(1 + H/\delta)^{-1}$.

The techniques introduced here allowed us to obtain the first results on convergence rate and positiveness of $B_{\text{ras}}^{-1}$ and $B_{\text{ash}}^{-1}$. We also understand why $B_{\text{rash}}^{-1}$ is not a good method. Some open problems are:
1) Is it possible to improve the lower and upper bounds for $B_{\text{ras}}^{-1}$?
2) Is it possible to extend the new theory to the space $\mathbb{R}^n$ rather than for the reduced spaces, and also for inexact local solvers?, and
3) The extension of the new theory to the two-dimensional case, with and without a coarse space, and with or without cross points.

**Table 1** No coarse space. The reduced systems: min $\lambda_1$ and in parenthesis max $\sqrt{\lambda_2}$

| prec | $\ell = 0$ | $\ell = 1$ | $\ell = 2$ | $\ell = 3$ |
|---|---|---|---|---|
| ASH | 0.0012(1.9988) | 0.0035(1.9965) | 0.0059(1.9941) | 0.0083(1.9917) |
| RAS | 0.0012(1.9988) | 0.0035(1.9965) | 0.0059(1.9941) | 0.0083(1.9919) |
| RASH | 0.0012(1.9988) | 0.0024(3.9931) | 0.0035(5.9830) | 0.0047(7.9690) |
| ASM | 0.0012(1.9988) | 0.1058(1.9965) | 0.1594(1.9941) | 0.0083(1.9917) |

**Table 2** Coarse space $V_0^2$. The reduced systems: min $\lambda_1$ and in parenthesis max $\sqrt{\lambda_2}$

| prec | $\ell = 0$ | $\ell = 1$ | $\ell = 2$ | $\ell = 3$ |
|---|---|---|---|---|
| ASH | 0.0491(2.1180) | 0.1058(2.2045) | 0.1594(2.2638) | 0.2100(2.3119) |
| RAS | 0.0491(2.1180) | 0.1058(2.2412) | 0.1592(2.3730) | 0.2097(2.5122) |
| RASH | 0.0491(2.1180) | 0.0767(4.0147) | 0.1028(6.0013) | 0.1274(7.9861) |
| ASM | 0.0491(2.1180) | 0.1058(2.2045) | 0.1594(2.2638) | 0.2100(2.3119) |

## References

1. Cai, X.-C., Dryja, M., and Sarkis, M. Restricted additive Schwarz preconditioners with harmonic overlap for symmetric positive definite linear systems. *SIAM Journal on Numerical Analysis* **41**(4), 1209–1231 (2003).
2. Cai, X.-C. and Sarkis, M. A restricted additive Schwarz preconditioner for general sparse linear systems. *SIAM Journal on Scientific Computing* **21**(2), 792–797 (1999).
3. Frommer, A. and Szyld, D. B. An algebraic convergence theory for restricted additive Schwarz methods using weighted max norms. *SIAM Journal of Numerical Analysis* **39**(2), 463–479 (2001).
4. Toselli, A. and Widlund, O. *Domain Decomposition Methods-Algorithms and Theory*, vol. 34. Springer Science & Business Media (2006).