Cross-Points in the Neumann-Neumann Method

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1 Introduction

The Neumann-Neumann method (NNM), first introduced in [1] in the case of two subdomains, is among the most popular non-overlapping domain decomposition methods. However, when used as a stationary solver at the continuous level, it has been observed that the method faced well-posedness issues in the presence of cross-points, see [2]. Here, our goal is to analyze in detail the behaviour of the NNM near cross-points on a simple, but rather instructive, bidimensional configuration.

Let $\Omega \subset \mathbb{R}^2$ be the square $(-1,1) \times (-1,1)$, divided into four non-overlapping square subdomains Ω_i, $i \in \mathcal{I} := \{1,2,3,4\}$, see Figure 1. This leads to one interior cross-point (red dot), and four boundary cross-points (black dots). We denote the interfaces between adjacent subdomains by $\Gamma_{ij} := \mathrm{int}(\partial\Omega_i \cap \partial\Omega_j)$, the skeleton of the partition by $\Gamma := \bigcup_{i,j} \overline{\Gamma}_{ij}$, and $\partial\Omega_i^0 := \partial\Omega_i \cap \partial\Omega$. We consider the Laplace problem with Dirichlet boundary conditions on Ω, that is: find u solution to

$$-\Delta u = f \ \text{ in } \Omega, \quad u = g \ \text{ on } \partial\Omega, \tag{1}$$

where $f \in L^2(\Omega)$ and $g \in H^{\frac{3}{2}}(\partial\Omega)$, ensuring that $u \in H^2(\Omega)$.

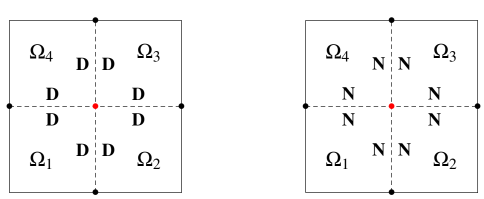

Fig. 1 Transmission conditions of the standard NNM for u (left) and ψ (right).

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Given an initial couple (u^0, ψ^0), and a relaxation parameter $\theta \in \mathbb{R}$, each iteration $k \geq 1$ of the NNM applied to (1) can be split into two steps:

- *(Dirichlet step)* Solve for all $i \in \mathcal{I}$,

$$-\Delta u_i^k = f \text{ in } \Omega_i \,, \quad u_i^k = g \text{ on } \partial\Omega_i^0 \,,$$
$$u_i^k = u_i^{k-1} - \theta\left(\psi_i^{k-1} + \psi_j^{k-1}\right) \text{ on } \Gamma_{ij},\ \forall j \in \mathcal{I} \text{ s.t. } \Gamma_{ij} \neq \emptyset \,.$$

- *(Neumann step)* Compute the correction ψ^k, that is, solve for all $i \in \mathcal{I}$,

$$-\Delta \psi_i^k = 0 \text{ in } \Omega_i \,, \quad \psi_i^k = 0 \text{ on } \partial\Omega_i^0 \,,$$
$$\partial_{n_i}\psi_i^k = \partial_{n_i} u_i^k + \partial_{n_j} u_j^k \text{ on } \Gamma_{ij},\ \forall j \in \mathcal{I} \text{ s.t. } \Gamma_{ij} \neq \emptyset \,.$$

For the method to be well defined, it is assumed in the rest of this paper that the initial couple (u^0, ψ^0) is compatible with the Dirichlet boundary condition, i.e. it satisfies: $u^0 \in H^2(\Omega)$, $\psi^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u^0 \mid_{\partial\Omega\cap\Gamma} = g \mid_{\Gamma}$.

2 Convergence analysis of the Neumann-Neumann method

Definition 1 A measurable function $h : \Omega \to \mathbb{R}$ is said to be *even symmetric* (resp. *odd symmetric*) if for a.e. $(x, y) \in \Omega$, $h(-x, -y) = h(x, y)$ (resp. $-h(x, y)$). Moreover, any measurable function h can be uniquely decomposed into $h = h_e + h_o$ where h_e is even symmetric and h_o is odd symmetric.

Following this notion, as in [3], we introduce the so-called *even symmetric* and *odd symmetric parts* of problem (1): find u_e and u_o solutions to

$$-\Delta u_e = f_e \text{ in } \Omega, \quad u_e = g_e \text{ on } \partial\Omega, \tag{2a}$$
$$-\Delta u_o = f_o \text{ in } \Omega, \quad u_o = g_o \text{ on } \partial\Omega. \tag{2b}$$

If u denotes the solution to (1), it is known (see [3]) that the unique solutions u_e and u_o to these subproblems are precisely the even symmetric part and the odd symmetric part of u. In what follows, we will perform the convergence analysis of the NNM separately for the errors associated with the even and odd symmetric subproblems, as they lead to completely different behaviours of the method.

Case of the even symmetric part. The next Theorem states that the NNM is convergent when applied to the even symmetric part of (1).

Theorem 1 *Taking* (u_e^0, ψ_e^0) *as initial couple for the NNM applied to* (2a) *produces a sequence* $\left\{u_e^k\right\}_k$ *that converges geometrically to the solution* u_e *with respect to the* L^2*-norm and the broken* H^1*-norm for any* $\theta \in (0, \frac{1}{2})$*. Moreover, the convergence factor is given by* $|1 - 4\theta|$*, which also proves that the method becomes a direct solver for the specific choice* $\theta = \frac{1}{4}$.

Proof As in [3] for the Dirichlet-Neumann method, let us study the first iterations of the NNM in terms of the local errors $e_{e,i}^k := u_e|_{\Omega_i} - u_{e,i}^k$.
• *Iteration $k = 1$, Dirichlet step:* In each Ω_i, $i \in \mathcal{I}$, the errors satisfy

$$-\Delta e_{e,i}^1 = 0 \text{ in } \Omega_i \,, \quad e_{e,i}^1 = 0 \text{ on } \partial\Omega_i^0 \,,$$
$$e_{e,i}^1 = e_{e,i}^0 + \theta\left(\psi_{e,i}^0 + \psi_{e,j}^0\right) \text{ on } \Gamma_{ij},\ \forall j \in \mathcal{I} \text{ s.t. } \Gamma_{ij} \neq \emptyset \,.$$

Since (u_e^0, ψ_e^0) is compatible with the even symmetric part of the Dirichlet boundary condition, $e_{e,i}^1$ exists and is unique in $H^1(\Omega_i)$. Using the even symmetry properties of e_e^0 and ψ_e^0, one can deduce that the $e_{e,i}^1$, for $i \in \{2, 3, 4\}$, can be expressed in terms of $e_{e,1}^1$ as follows:

$$e_{e,2}^1(x, y) = e_{e,1}^1(-x, y) \,, \qquad \text{for a.e. } (x, y) \in \Omega_2 \,,$$
$$e_{e,3}^1(x, y) = e_{e,1}^1(-x, -y) \,, \qquad \text{for a.e. } (x, y) \in \Omega_3 \,,$$
$$e_{e,4}^1(x, y) = e_{e,1}^1(x, -y) \,, \qquad \text{for a.e. } (x, y) \in \Omega_4 \,.$$

• *Iteration $k = 1$, Neumann step:* We compute the correction $\psi_{e,i}^1$ in each subdomain Ω_i. For instance, taking $i = 1$, we get in Ω_1

$$-\Delta\psi_{e,1}^1 = 0 \text{ in } \Omega_1 \,, \quad \psi_{e,1}^1 = 0 \text{ on } \Gamma_1 \,,$$
$$\partial_{n_1}\psi_{e,1}^1 = -\left(\partial_{n_1} e_{e,1}^1 + \partial_{n_2} e_{e,2}^1\right) = -2\partial_{n_1} e_{e,1}^1 \text{ on } \Gamma_{12} \,,$$
$$\partial_{n_1}\psi_{e,1}^1 = -\left(\partial_{n_1} e_{e,1}^1 + \partial_{n_4} e_{e,4}^1\right) = -2\partial_{n_1} e_{e,1}^1 \text{ on } \Gamma_{41} \,.$$

Thus, uniqueness of $\psi_{e,1}^1$ in $H^1(\Omega_1)$ yields $\psi_{e,1}^1 = -2e_{e,1}^1$ in Ω_1. A similar reasoning applies to each $\psi_{e,i}^1$, $i \in \{2, 3, 4\}$, therefore the recombined correction simply reads: $\psi_e^1 = -2e_e^1$ in $\Omega \setminus \Gamma$.
• *Iteration $k \geq 2$:* At iteration $k = 2$, the transmission condition for the Dirichlet step in Ω_i on each Γ_{ij} is given by, $e_{e,i}^2 = e_{e,i}^1 + \theta\left(\psi_{e,i}^1 + \psi_{e,j}^1\right) = (1 - 4\theta)e_{e,i}^1$. Uniqueness of $e_{e,i}^2$ in $H^1(\Omega_i)$ enables us to conclude that $e_{e,i}^2 = (1 - 4\theta)e_{e,i}^1$ in Ω_i. Since this holds in each subdomain, the exact same reasoning as for iteration $k = 1$ applies, and we get after the Neumann step $e_e^2 = (1 - 4\theta)e_e^1$ and $\psi_e^2 = -2(1 - 4\theta)e_e^1$ in $\Omega \setminus \Gamma$. By induction, we obtain for any $k \geq 3$, $e_e^k = (1 - 4\theta)^{k-1}e_e^1$ in $\Omega \setminus \Gamma$. This leads to the following estimates for the error on the whole domain Ω in the L^2-norm and the broken H^1-norm:

$$\| u_e^k - u_e \|_{L^2(\Omega)} = \sum_{i \in \mathcal{I}} \| e_{e,i}^k \|_{L^2(\Omega_i)} \leq C|1 - 4\theta|^{k-1},$$
$$\sum_{i \in \mathcal{I}} \| u_{e,i}^k - u_{e,i} \|_{H^1(\Omega_i)} \leq C'|1 - 4\theta|^{k-1},$$

where C, C' are strictly positive constants depending on the data and the geometry of the domain decomposition. □

Case of the odd symmetric part. As for the Dirichlet-Neumann method, the NNM does not converge in general when applied to the odd symmetric part of (1).

Theorem 2 *The NNM applied to* (2b) *is not well-posed. More specifically, taking* (u_o^0, ψ_o^0) *as initial couple, there exists an integer* $k_0 > 0$ *such that the solution to the problem obtained at the* k_0*-th iteration is not unique. In addition, all possible solutions* $u_o^{k_0}$ *are singular at the cross-point, with a leading singularity of type* $(\ln r)^2$.

Theorem 3 *If we let the NNM go beyond the ill-posed iteration* k_0 *from Theorem 2, we end up with a sequence* $\{u_o^k\}_{k\geq k_0}$ *of non-unique iterates. Moreover, for each* $k \geq k_0$*, all possible* u_o^k *are singular at the cross-point, with a leading singularity of type* $(\ln r)^{2(k-k_0)+2}$.

Proof The proofs of these results rely on the exact same arguments as those in the proofs of [3, Theorem 7 and 8]. □

The previous results show that, at some point in the iterative process, the NNM method will lead to solving an ill-posed problem. This will generate a singular solution, and the generated singularity will then propagate through the following iterations.

3 Toward a modified Neumann-Neumann method

The conclusions from the previous section suggest that the transmission conditions of the standard NNM are naturally well adapted to the even symmetric part of the problem. Indeed, in this context, one may express at each iteration k all local errors $e_{e,i}^k$ in terms of only one, say $e_{e,1}^k$, by symmetry. This motivates the search for different transmission conditions such that a similar symmetry property holds for the odd symmetric part of the problem.

Fixing the odd symmetric case. In order to fix the well-posedness issue in the odd symmetric case, and obtain the symmetry property mentioned above, we propose a new distribution of Dirichlet and Neumann transmission conditions, as shown in Figure 2.

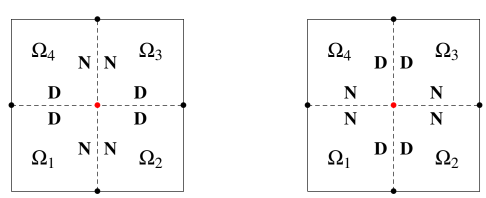

Fig. 2 Transmission conditions of the mixed NNM for u (left) and ψ (right).

Let us introduce Γ_D^1, Γ_N^1, Γ_D^2, Γ_N^2 the sets containing all parts of the interface Γ where transmission conditions of Dirichlet or Neumann type are imposed for u (superscript 1) and for ψ (superscript 2), that is :

$$\Gamma_D^1 := \{\Gamma_{23}, \Gamma_{41}\}, \quad \Gamma_N^1 := \{\Gamma_{12}, \Gamma_{34}\}, \quad \Gamma_D^2 := \{\Gamma_{12}, \Gamma_{34}\}, \quad \Gamma_N^2 := \{\Gamma_{23}, \Gamma_{41}\}.$$

Given an initial couple (u^0, ψ^0) and relaxation parameter θ, each iteration $k \geq 1$ of the proposed *mixed* Neumann-Neumann method can be split into two steps:

- *(First step)* Solve for all $i \in \mathcal{I}$

$$\begin{aligned} -\Delta u_i^k &= f \text{ in } \Omega_i, \quad u_i^k = g \text{ on } \partial\Omega_i^0, \\ u_i^k &= u_i^{k-1} - \theta\left(\psi_i^{k-1} + \psi_j^{k-1}\right) \text{ on } \Gamma_{ij},\ \forall j \in \mathcal{I} \text{ s.t. } \Gamma_{ij} \in \Gamma_D^1, \\ \partial_{n_i} u_i^k &= \partial_{n_i} u_i^{k-1} + (-1)^i \theta\left(\partial_{n_i}\psi_i^{k-1} + \partial_{n_j}\psi_j^{k-1}\right) \text{ on } \Gamma_{ij},\ \forall j \in \mathcal{I} \text{ s.t. } \Gamma_{ij} \in \Gamma_N^1. \end{aligned}$$

- *(Second step)* Compute the correction ψ^k, that is, solve for all $i \in \mathcal{I}$

$$\begin{aligned} -\Delta \psi_i^k &= 0 \text{ in } \Omega_i, \quad \psi_i^k = 0 \text{ on } \partial\Omega_i^0, \\ \psi_i^k &= u_i^k - u_j^k \text{ on } \Gamma_{ij},\ \forall j \in \mathcal{I} \text{ s.t. } \Gamma_{ij} \in \Gamma_D^2, \\ \partial_{n_i}\psi_i^k &= \partial_{n_i} u_i^k + \partial_{n_j} u_j^k \text{ on } \Gamma_{ij},\ \forall j \in \mathcal{I} \text{ s.t. } \Gamma_{ij} \in \Gamma_N^2. \end{aligned}$$

With this choice of transmission conditions, we are able to prove that the proposed mixed NNM is convergent when applied to the odd symmetric part of (1).

Theorem 4 *Taking (u_o^0, ψ_o^0) as initial couple for the mixed NNM applied to* (2b) *produces a sequence $\{u_o^k\}_k$ that converges geometrically to the solution u_o with respect to the L^2-norm and the broken H^1-norm for any $\theta \in (0, \frac{1}{2})$. Moreover, the convergence factor is given by $|1 - 4\theta|$, which also proves that the method becomes a direct solver for the specific choice $\theta = \frac{1}{4}$.*

Proof We follow the same steps as in the proof of Theorem 1.
• *Iteration $k = 1$, Dirichlet step:* In each Ω_i, $i \in \mathcal{I}$, the odd errors satisfy

$$\begin{aligned} -\Delta e_{o,i}^1 &= 0 \text{ in } \Omega_i, \quad e_{o,i}^1 = 0 \text{ on } \partial\Omega_i^0, \\ e_{o,i}^1 &= e_{o,i}^0 + \theta\left(\psi_{o,i}^0 + \psi_{o,j}^0\right) \text{ on } \Gamma_{ij},\ \forall j \in \mathcal{I} \text{ s.t. } \Gamma_{ij} \in \Gamma_D^1, \\ \partial_{n_i} e_{o,i}^1 &= \partial_{n_i} e_{o,i}^0 - (-1)^i \theta\left(\partial_{n_i}\psi_{o,i}^0 + \partial_{n_j}\psi_{o,j}^0\right) \text{ on } \Gamma_{ij},\ \forall j \in \mathcal{I} \text{ s.t. } \Gamma_{ij} \in \Gamma_N^1. \end{aligned}$$

These problems are well-posed since (u_o^0, ψ_o^0) is compatible with the odd symmetric part of the boundary condition. This time, using the mixed conditions enforced along Γ together with the odd symmetry properties of e_o^0 and ψ_o^0, we can deduce that

$$\begin{aligned} e_{o,2}^1(x,y) &= -e_{o,1}^1(-x,y)\,, && \text{for a.e. } (x,y)\in\Omega_2\,,\\ e_{o,3}^1(x,y) &= -e_{o,1}^1(-x,-y)\,, && \text{for a.e. } (x,y)\in\Omega_3\,,\\ e_{o,4}^1(x,y) &= e_{o,1}^1(x,-y)\,, && \text{for a.e. } (x,y)\in\Omega_4\,. \end{aligned}$$

Indeed, for the first equality, taking $(x, y) \in \Omega_2$, we have on Γ_{23} and Γ_{12}

$$\begin{aligned} e_{o,2}^1(x,0) &= e_{o,2}^0(x,0) + \theta\left(\psi_{o,2}^0(x,0) + \psi_{o,3}^0(x,0)\right)\\ &= -e_{o,1}^0(-x,0) - \theta\left(\psi_{o,4}^0(-x,0) + \psi_{o,1}^0(-x,0)\right) = -e_{o,1}^1(-x,0)\,,\\ (\partial_{n_2} e_{o,2}^1)(0,y) &= -(\partial_x e_{o,2}^0)(0,y) - \theta\left((\partial_x \psi_{o,2}^0)(0,y) + (\partial_x \psi_{o,1}^0)(0,y)\right)\\ &= -(\partial_x e_{o,1}^0)(0,y) - \theta\left((\partial_x \psi_{o,1}^0)(0,y) + (\partial_x \psi_{o,2}^0)(0,y)\right)\\ &= -(\partial_{n_1} e_{o,1}^1)(0,y) = -(\partial_{n_2} e_{o,1}^1(-\cdot,\cdot))(0,y)\,. \end{aligned}$$

Then uniqueness of the solution to the subproblem in Ω_2 yields $e_{o,2}^1 = -e_{o,1}^1(-\cdot,\cdot)$ a.e. in Ω_2. The two other equalities are obtained using similar arguments, see Figure 3 for an illustration of this symmetry property.

• *Iteration* $k = 1$*, Neumann step:* For $i = 1$, we get in Ω_1

$$\begin{aligned} -\Delta\psi_{o,1}^1 &= 0 \ \text{ in } \Omega_1\,, \quad \psi_{o,1}^1 = 0 \ \text{ on } \Gamma_1\,,\\ \psi_{o,1}^1 &= -e_{o,1}^1 + e_{o,2}^1 = -2e_{o,1}^1 \ \text{ on } \Gamma_{12}\,,\\ \partial_{n_1}\psi_{o,1}^1 &= -\left(\partial_{n_1} e_{o,1}^1 + \partial_{n_4} e_{o,4}^1\right) = -2\partial_{n_1} e_{o,1}^1 \ \text{ on } \Gamma_{41}\,. \end{aligned}$$

Therefore, $\psi_{o,1}^1 = -2e_{o,1}^1$ in Ω_1. Extending these arguments to the other subdomains yields a recombined correction $\psi_o^1 = -2e_o^1$ in $\Omega \setminus \Gamma$.

• *Iteration* $k \geq 2$*:* At iteration $k = 2$, the transmission conditions for the first step in Ω_1 are given by

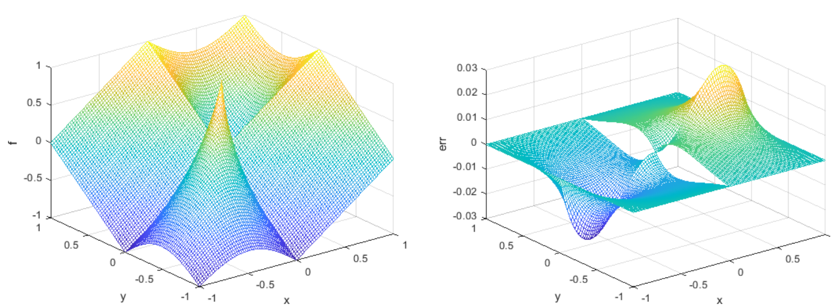

Fig. 3 Source term f (left), and absolute error at iteration 1 for $\theta = 0.25$ (right), in Example 2.

$$e^2_{o,1} = e^1_{o,1} + \theta\left(\psi^1_{o,1} + \psi^1_{o,4}\right) = (1-4\theta)e^1_{o,1} \ \text{ on } \Gamma_{41}\,,$$
$$\partial_{n_1} e^2_{o,1} = \partial_{n_1} e^1_{o,1} + \theta\left(\partial_{n_1}\psi^1_{o,1} + \partial_{n_2}\psi^1_{o,2}\right) = (1-4\theta)\partial_{n_1} e^1_{o,1} \ \text{ on } \Gamma_{12}\,.$$

This implies that $e^2_{o,1} = (1-4\theta)e^1_{o,1}$ in Ω_1. Using the same arguments in the other subdomains and performing the second step leads to $e^2_o = (1-4\theta)e^1_o$ and $\psi^2_o = -2(1-4\theta)e^1_o$ in $\Omega \setminus \Gamma$. As in the proof of Theorem 1, we obtain by induction that, for any $k \geq 3$, $e^k_o = (1-4\theta)^{k-1}e^1_o$ in $\Omega \setminus \Gamma$. The desired error estimates are then deduced from the last relation. □

The new NNM. Here are the different steps of our *new* NNM to solve (1) starting from an initial couple (u^0, ψ^0) compatible with the Dirichlet boundary condition, and a relaxation parameter $\theta \in (0, 1/2)$.

1. Decompose the data into their even/odd symmetric parts to get (2a) and (2b).
2. Solve in parallel:
   - (2a) using the standard NNM starting from (u^0_e, ψ^0_e),
   - (2b) using the mixed NNM starting from (u^0_o, ψ^0_o).
3. Recompose the solution $u = u_e + u_o$.

Remark 1 It is actually enough to solve for u_e and u_o in $\Omega_1 \cup \Omega_2$, and then extend them to the whole domain Ω by symmetry. One iteration of the new NNM thus costs the same as one iteration of the original NNM.

4 Numerical experiments

In order to test our new NNM, we apply it to two simple benchmarks: one with even symmetric data (Example 1: $g = 0$ and $f = 1$) and one with odd symmetric data (Example 2: $g = 0$ and $f = x + y + k$ where $k = \sin(2\phi)$ in Ω_1, $k = -\sin(2\phi)$ in Ω_3 and $k = 0$ in $\Omega_2 \cup \Omega_4$, with ϕ being the angle in polar coordinates, see Figure 3). The discretization of (1) is performed using a standard five point finite difference scheme on a cartesian grid of meshsize $h = 0.01$. When two Dirichlet conditions meet at a corner, the value of g at this node is set to the average of the two values. In addition, when Dirichlet and Neumann conditions meet at a corner, we choose the Dirichlet one to be enforced at this node. The results obtained show that the method behaves as predicted by Theorem 1 and Theorem 4. For $\theta = \frac{1}{4}$, the method converges after two iterations, see the left column in Figure 4. And for $\theta \in (0, \frac{1}{2})$, $\theta \neq \frac{1}{4}$, it converges geometrically to the solution with the expected convergence factor, see the right column in Figure 4 where $\theta_1 = 0.23$ and $\theta_2 = 0.247$. These two graphs also indicate that the convergence behaviour does not depend on h since, in each case, the error curves for $h = 0.01$ and $h = 0.005$ are almost overlaid on each other.

In this short paper, we gave a complete analysis of the standard NNM in a simple configuration involving one cross-point. The even/odd decomposition showed that the NNM was able to treat very efficiently the even symmetric part of the solution, while it faced well-posedness and convergence issues when applied to the

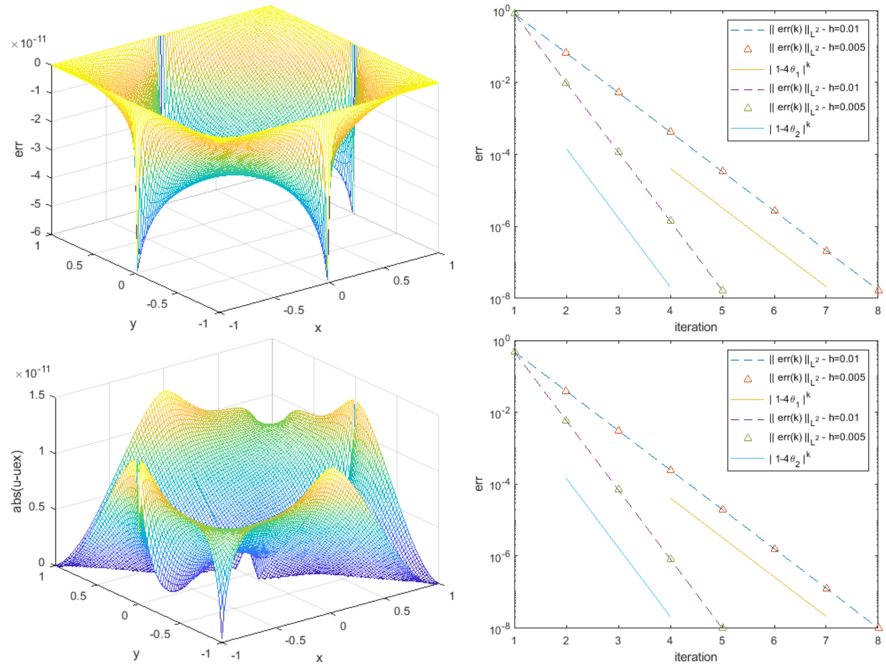

Fig. 4 Absolute error at iteration 2 for $\theta = 0.25$ (left column), and error curves for $\theta \in \{\theta_1, \theta_2\}$ and $h \in \{0.01, 0.005\}$ (right column), in Example 1 (top) and Example 2 (bottom).

odd symmetric part of the solution. Based on this observation, we proposed new mixed transmission conditions of Dirichlet/Neumann type to treat efficiently the odd symmetric part. We proved that the newly proposed NNM built upon a combination between the standard NNM and the new mixed method is convergent, and we validated this property by some numerical experiments. A natural extension of this work would be the 3D case of a cube divided into eight subcubes. It would also be interesting to generalize the notion of even/odd symmetry to the case of more general cross-points (not necessarily rectilinear, or with $N \neq 4$ subdomains).

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