Adaptive Three-Level BDDC Using Frugal Constraints

Axel Klawonn, Martin Lanser, and Janine Weber

1 Introduction

The convergence rate of both the FETI-DP (Finite Element Tearing and Interconnecting - Dual Primal) and the BDDC (Balancing Domain Decomposition by Constraints) domain decomposition methods strongly depend on the spectrum, i.e., the eigenvalues of the preconditioned system [7, 2]. To obtain a robust condition number estimate which is independent of the coefficient or material distribution, several adaptive coarse spaces have been developed which rely on the solution of local eigenvalue problems and use selected eigenvectors to enhance the coarse space; see, e.g., [9]. Besides the robustness of the considered domain decomposition method, its computational efficiency and parallel scalability is also of major interest. However, for an increasing number of subdomains, the exact solution of the coarse problem can become a scaling bottleneck within a parallel implementation. For the BDDC method, the coarse problem has the same structure as the original problem. Thus, it is straightforward to apply the BDDC preconditioner recursively either once or several times to the coarse problem, leading to a three-level [14] or a multilevel BDDC method [1, 10].

In the present work, we combine the three-level BDDC approach from [14] with the choice of adaptive constraints from [9] and, other than in the adaptive multilevel BDDC method in [12, 13], additionally with the frugal constraints from [3]. Since the computation of the frugal edge constraints is fairly cheap, we aim to reduce the computational effort of the adaptive three-level BDDC method by replacing the eigenvalue problems either on the subdomain or on the subregion level by frugal constraints while still retaining a satisfactory convergence behavior. We compare the robustness of the resulting different three-level BDDC methods using frugal

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and/or adaptive edge constraints on either the second and/or the third level for different heterogeneous stationary diffusion problems with high contrasts in two spatial dimensions.

2 Problem and three-level BDDC

We consider a stationary diffusion problem in two spatial dimensions, i.e., the weak formulation of

$$\begin{aligned} -\operatorname{div}(\rho \nabla u) &= 1 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here, $\rho\colon \Omega := [0,1]\times[0,1] \to \mathbb{R}$ denotes the coefficient function and in Section 4, we will consider various heterogeneous coefficient functions ρ.

In this paper, we numerically investigate different coarse spaces and approximate solvers for the BDDC domain decomposition method. Thus, we decompose the domain Ω into $N \in \mathbb{N}$ nonoverlapping subdomains $\Omega_i,\ i = 1,\dots,N$. For each of these subdomains, we then compute a conforming finite element triangulation and compute local stiffness matrices $K^{(i)}$ and local load vectors $f^{(i)},\ i = 1,\dots,N$. Due to space limitations, we refrain from explaining the classic two-level BDDC method in detail and focus instead on the description of the different approximate coarse spaces and a specific adaptive BDDC coarse space. For a detailed description of the two-level BDDC method, we refer to [2].

In a parallel implementation of the two-level BDDC method, the exact solution of the coarse problem in form of the primally coupled Schur complement matrix $\widetilde{S}^{-1}_{\Pi\Pi}$ can become a scaling bottleneck; see, e.g., [4, 5] for related parallel numerical experiments in a linear and a nonlinear framework, respectively. One possible approach to delay the related scaling bottleneck is to apply the BDDC preconditioner recursively and to compute only an approximation $\widehat{S}^{-1}_{\Pi\Pi}$ of the coarse problem $\widetilde{S}^{-1}_{\Pi\Pi}$, leading to a three-level BDDC method [14]. Here, the main idea is to introduce a third level of the domain decomposition by additionally decomposing the domain Ω into a number of nonoverlapping subregions $\Omega^j,\ j = 1,\dots,\overline{N}$. In particular, each of the subregions Ω^j comprises a given number of subdomains $\Omega_i,\ i = 1,\dots,N_j$. Then, all primal variables on the second level are again partitioned into inner, primal, and dual variables on the subregion level. We denote the respective index sets on the subregion level by $\overline{I}$, $\overline{\Pi}$, and $\overline{\Delta}$, respectively; see also Fig. 1 for an exemplary visualization. On the subregion level, analogously to the subdomain level, all inner and dual variables are eliminated, leading to a primal Schur complement system on the third level which is, generally, of much smaller size than the respective system on the second level depending on the number of subdomains per subregion. Hence, only a smaller coarse problem on the third level has to be solved compared to the classic, i.e., the two-level BDDC method. A complete mathematical description as well as the related theory and a condition number bound for the three-level BDDC method for stationary diffusion problems can be found in [14]. Parallel numerical results and a weak scaling study for the three-level BDDC method in comparison with other

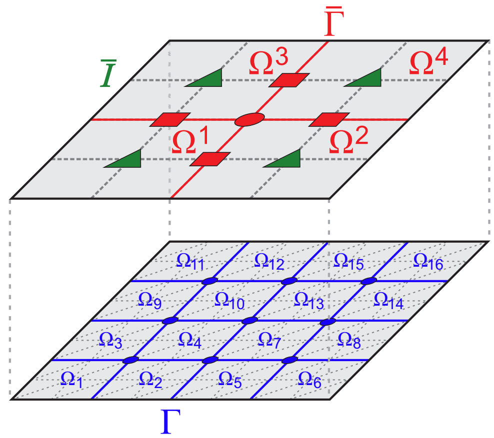

Fig. 1 Example of a three-level domain decomposition into 16 regular subdomains (bottom) and 4 regular subregions (top). We mark in blue the interface Γ between subdomains and in red the interface $\overline{\Gamma}$ between subregions. Primal nodes $\overline{\Pi}$ on the third level and primal nodes Π on the second level are visualized as red and blue circles, respectively. Inner or dual nodes on the third level, i.e., $\overline{I}$ or $\overline{\Delta}$, are visualized as green triangles or red squares, respectively. Figure taken from [15, Fig. 5.1].

approximate coarse solvers can, e.g., be found in [5]. Let us note that besides adding a third level of the domain decomposition, also further additional levels can be added leading to a recursive multilevel BDDC method; see, e.g., [1, 10].

3 Adaptive three-level BDDC combined with frugal edge constraints

In general, we are interested in BDDC coarse spaces which can efficiently be computed on a parallel computer and which are, preferably, robust for different heterogeneous problems. Unfortunately, the classic condition number bounds both for the FETI-DP and the BDDC method are only independent of the coefficient contrast under fairly restrictive assumptions on the coefficient distribution; see, e.g., [7, 8, 11] for a closer discussion for FETI-DP and BDDC, respectively. A similar theoretical condition number bound has also been derived for the three-level BDDC method; see [14]. As a remedy, different adaptive, i.e., problem-dependent coarse spaces have been developed. In the following, we will focus on a specific adaptive coarse space strategy which has been originally introduced in [9]. Here, the main idea is to solve a local generalized eigenvalue problem for each edge $\mathcal{E}_{ij}$ between two subdomains Ω_i and Ω_j which is of the general form: find $w_{ij} \in \left(\ker S_{ij}\right)^{\perp}$ such that

$$\langle P_{D_{ij}} v_{ij},\ S_{ij} P_{D_{ij}} w_{ij}\rangle = \mu_{ij} \langle v_{ij},\ S_{ij} w_{ij}\rangle \quad \forall v_{ij} \in \left(\ker S_{ij}\right)^{\perp}. \tag{2}$$

Here, $S_{ij} = \begin{pmatrix} S^{(i)} & \\ & S^{(j)} \end{pmatrix}$ is a local Schur complement matrix where $S^{(i)}$ and $S^{(j)}$ are the Schur complements of $K^{(i)}$ and $K^{(j)}$, respectively, and $P_{D_{ij}} = B_{D,\mathcal{E}_{ij}}^T B_{\mathcal{E}_{ij}}$ is a local jump operator; see [9] for more details. Assuming that R eigenvectors w_{ij}^r, $r = 1, \dots, R$ belong to eigenvalues which are larger than a user-defined tolerance TOL, we then enhance the BDDC coarse space with the edge constraints

$$B_{D,\mathcal{E}_{ij}} S_{ij} P_{D_{ij}} w_{ij}^r, \quad r = 1, \dots, R, \tag{3}$$

with $B_{D,\mathcal{E}_{ij}} = \left(B_{D,\mathcal{E}_{ij}}^{(i)},\ B_{D,\mathcal{E}_{ij}}^{(j)}\right)$ being a local submatrix of $\left(B_D^{(i)},\ B_D^{(j)}\right)$ obtained by taking the rows corresponding to the edge $\mathcal{E}_{ij}$. In particular, for two-dimensional problems and primal subdomain vertices, enhancing the BDDC coarse space with these adaptive constraints leads to a robust condition number which exclusively depends on the chosen tolerance TOL and some geometrical constants; see [6].

In order to benefit both from the robustness of the described adaptive coarse space as well as from the increased parallel scalability of a three-level BDDC method, we combine both approaches and also implement adaptive edge constraints on the subregion level within a three-level BDDC method. To compute the local eigenvalue problem for edges between two neighboring subregions Ω^i and Ω^j, both the local Schur complement matrices $S^{(i)}$ and $S^{(j)}$ as well as the local jump operator $P_{D_{ij}}$ in Eq. (2) are replaced by recursive versions with respect to the primal variables on the subregion level. This leads to an adaptive three-level BDDC approach; see also [12, 13] for previous work on adaptive multilevel BDDC. Due to the implementation of adaptive constraints within each level, here, a robust condition number estimate can be obtained.

As a drawback, we have to set up and solve local eigenvalue problems on both the subdomain and the subregion level, which can be computationally expensive, especially for three-dimensional problems. Hence, we propose a modified approach of the adaptive multilevel approach presented in [12, 13] by replacing the eigenvalue problems either on the second and/or the third level by frugal edge constraints as introduced in [3]. The resulting frugal edge constraints serve as a low-dimensional approximation of the adaptive coarse space defined in [9] and have been shown to be robust for a range of different heterogeneous coefficient or material distributions both in two and three spatial dimensions; see [3] for detailed experiments. For two-dimensional stationary diffusion problems, the frugal edge constraints on the subdomain level are defined as follows. We denote by $\omega(x)$ the support of the finite element basis functions associated with a finite element node $x \in (\Omega_i \cup \Omega_j)$. Then, for each x on $\partial\Omega_i$ or $\partial\Omega_j$, respectively, we compute $\widehat{\rho}^{(i)}(x) = \max_{y \in \omega(x) \cap \Omega_i} \rho(y)$ and $\widehat{\rho}^{(j)}(x) = \max_{y \in \omega(x) \cap \Omega_j} \rho(y)$. We define $v_{E_{ij}}^{(l)}$ on $\partial\Omega_l$ for $l = i, j$ by

$$v_{E_{ij}}^{(l)}(x) := \begin{cases} \widehat{\rho}^{(l)}(x), & x \in \partial\Omega_l \backslash \Pi^{(l)}, \\ 0, & x \in \Pi^{(l)} \end{cases} \tag{4}$$

and $v_{E_{ij}}^T := (v_{E_{ij}}^{(i)T}, -v_{E_{ij}}^{(j)T})$; see also Fig. 2 for an exemplary illustration. Here, $\Pi^{(l)}$ denotes the index set of all local primal variables. Finally, we obtain the frugal edge constraint by $c_{E_{ij}} := B_{D,\mathcal{E}_{ij}} S_{ij} P_{D_{ij}} v_{E_{ij}}$ in direct analogy to the adaptive edge constraints in Eq. (3). Let us note that on the subregion level, the subdomains take over the role of the finite elements on the subdomain level and thus, the ρ-coefficient is not uniquely defined for each subdomain, i.e., each element on the third level. Hence, for the additional construction of frugal edge constraints on the subregion level, we use the stiffness, i.e., the diagonal entries of the local subregion Schur complement matrices instead of using the maximum ρ-coefficient to construct $v_{E_{ij}}^{(l)}(x)$ in Eq. (4).

In Table 1, we summarize the different coarse spaces presented here as well as their main benefits and drawbacks. The overall goal of this paper is to combine the different BDDC methods discussed above to benefit from the robustness of the presented adaptive constraints, a reduced computational effort when using frugal constraints, and the increased parallel scalability of a three-level BDDC method.

In the following, we consider four different possibilities of how to combine the presented BDDC methods from Table 1. In particular, we consider three-level BDDC with:

i) Frugal constraints on the second and the third level.
ii) Frugal constraints on the second and adaptive constraints on the third level.
iii) Adaptive constraints on the second and the third level.
iv) Adaptive constraints on the second and frugal constraints on the third level.

Furthermore, variants i) and ii) can be slightly modified by using the stiffness instead of the maximum ρ-coefficient values for the construction of the frugal subdomain edge constraints. We denote these variants by i a/b) and ii a/b), respectively, in the experiments in Section 4.

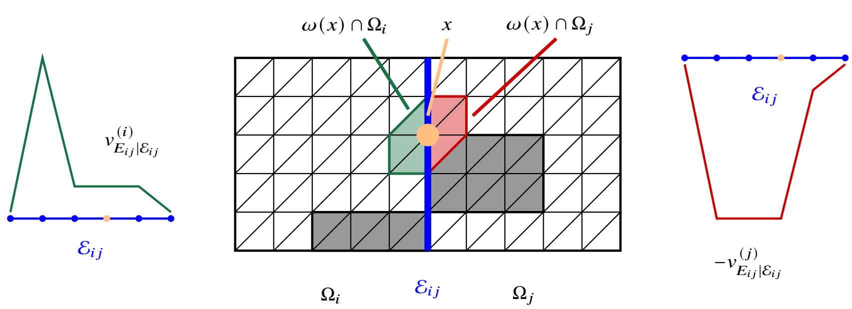

Fig. 2 Visualization of the construction of a frugal edge constraint in two dimensions for a given heterogeneous coefficient distribution. **Left/Right:** Maximum coefficient per finite element node of $\mathcal{E}_{ij}$ with respect to Ω_i and Ω_j, respectively, for the coefficient distribution in the middle. **Middle:** Exemplary heterogeneous coefficient distribution for two neighboring subdomains Ω_i and Ω_j sharing the edge $\mathcal{E}_{ij}$. High coefficients are marked in grey and low coefficients are marked in white. Figure taken from [15, Fig. 6.1].

Table 1 Non-exhaustive overview of benefits and drawbacks of the different BDDC algorithms considered in this paper.

Coarse space	Benefits	Drawbacks
Adaptive	Theoretical proof of robustness	Expensive setup
Frugal	Cheap setup	Limited robustness
3-Level, Classic	Increased parallel scalability	Robust only for moderate heterogeneities

4 Numerical results

In this section, we compare different BDDC methods using varying coarse spaces for different heterogeneous stationary diffusion problems in two dimensions. All shown computations were performed using MATLAB and a transformation-of-basis approach to implement the different coarse space enhancements. For all presented results, we choose all vertices as primal variables and we always use ρ-scaling unless explicitly mentioned otherwise. In Fig. 3, we show three different heterogeneous coefficient distributions which we use to evaluate the robustness of the four presented BDDC methods. Here, we always consider $\rho = 1e6$ in the dark blue pixels and $\rho = 1$ otherwise. Note that for the coefficient distribution in Fig. 3 (right), the ratio H/h has to be a multiple of five. Hence, we choose $H/h = 20$ for this case.

In Table 2 (top), we compare the iteration numbers and condition number estimates for the coefficient distribution in Fig. 3 (right) with coefficient jumps along and across both, subdomain and subregion edges. As we can observe from the results in Table 2, implementing adaptive constraints on both levels, i.e., algorithm iii) leads to the lowest iteration counts and lowest condition number estimates. However, all other BDDC variants using frugal edge constraints on the second and/or the third level also show condition numbers which are independent of the coefficient contrast and thus are robust. In particular, using frugal constraints on the subdomain level and computing adaptive constraints exclusively on the subregion level, i.e., algorithm ii), delivers results which are fairly similar to the fully adaptive approach. Hence, in this

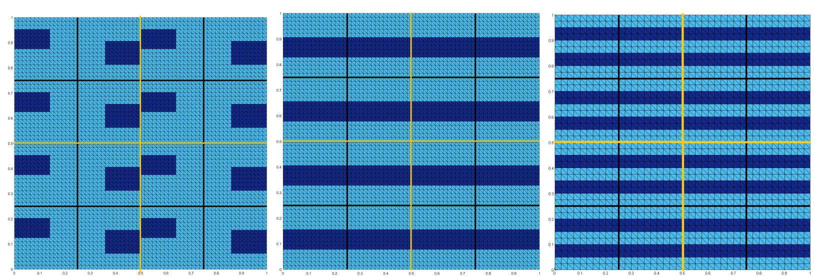

Fig. 3 Examples of three different heterogeneous coefficient functions. We set $\rho = 1e6$ in the dark blue pixels and $\rho = 1$ elsewhere. **Left:** Shifted boxes of a high coefficient with jumps along and across vertical edges; see Table 2 (top). **Middle:** One straight channel of a high coefficient crossing each vertical edge; see Table 2 (middle). **Right:** Two straight channels of a high coefficient crossing each vertical edge; see Table 2 (bottom).

Table 2 Iteration numbers (it) and condition numbers (cond) for a stationary diffusion problem with heterogeneous coefficient distributions as in Fig. 3. Decomposition of the domain into 4×4 subdomains and 2×2 subregions.

	2^{nd} level	3^{rd} level	it	cond
Shifted boxes; see Fig. 3 (left). $H/h = 16$**.**				
i a)	frugal, stiffness	frugal, stiffness	24	108.59
i b)	**frugal, rho-max**	**frugal, stiffness**	**18**	**35.58**
ii a)	frugal, stiffness	adaptive	19	24.78
ii b)	**frugal, rho-max**	**adaptive**	**18**	**30.11**
iii)	**adaptive**	**adaptive**	**18**	**21.73**
iv)	adaptive	frugal, stiffness	25	65.73
One straight channel; see Fig. 3 (middle). $H/h = 16$**.**				
i a)	frugal, stiffness	frugal, stiffness	17	11.11
i b)	**frugal, rho-max**	**frugal, stiffness**	**19**	**36.87**
ii a)	frugal, stiffness	adaptive	15	11.11
ii b)	**frugal, rho-max**	**adaptive**	**14**	**33.97**
iii)	**adaptive**	**adaptive**	**16**	**34.53**
iv)	adaptive	frugal, stiffness	20	35.93
Two straight channels; see Fig. 3 (right). $H/h = 20$**.**				
i a)	frugal, stiffness	frugal, stiffness	41	19 376
i b)	**frugal, rho-max**	**frugal, stiffness**	**42**	**54 582**
ii a)	frugal, stiffness	adaptive	34	19 355
ii b)	**frugal, rho-max**	**adaptive**	**28**	**55 009**
iii)	**adaptive**	**adaptive**	**22**	**42.92**
iv)	adaptive	frugal, stiffness	26	141.02

case, variant ii) would be our favored approach since it requires only the solution of smaller eigenvalue problems on the subregion level whereas the construction of frugal constraints on the subdomain level is computationally cheap.

For the coefficient distribution in Fig. 3 (middle) which is symmetric with respect to all edges and which has only a single channel crossing each subdomain edge, the numerical results for frugal and adaptive edge constraints are even more similar; see Table 2 (middle). This can be interpreted as an indicator that for this specific case, the computed frugal constraint is indeed a good approximation of the corresponding adaptive constraint. This will be further investigated in future research.

For the coefficient distribution in Fig. 3 (right) where we have more coefficient jumps along and across the subdomain and subregion edges, only adaptive three-level BDDC, i.e., algorithm iii) is robust with respect to the coefficient contrast; see Table 2 (bottom). However, also the remaining variants which use three-level BDDC with frugal constraints show satisfactory iteration numbers, indicating that we obtain only a few outliers within the spectrum of the preconditioned system.

As a conclusion, for completely arbitrary coefficient distributions with numerous jumps along and across the subdomain and subregion edges, only adaptive three-level BDDC ensures a robust condition number independent of the coefficient contrast. However, for rather moderate heterogeneities, also replacing the eigenvalue problems on either the second or the third level by frugal edge constraints can deliver a robust algorithm. With respect to computational efficiency, variant ii), i.e., frugal constraints on the subdomain level and adaptive constraints on the subregion level would be our favored choice due to the smaller size of the eigenvalue problems exclusively on the subregion level. For future research, we plan to fully integrate all proposed approaches into our parallel BDDC software and to test it more extensively with respect to parallel scalability for both, two- and three-dimensional problems.

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