Adaptive Nonoverlapping Preconditioners for the Helmholtz Equation: Linear HDG

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1 Linear HDG discretization

While standard finite element methods (FEMs) are widely used for the Helmholtz equation, they face well-known challenges at high wave numbers, including pollution errors and stringent mesh constraints to ensure stability. Certain discontinuous Galerkin (DG) methods address these limitations by providing inherent well-posedness without mesh constraints, while supporting arbitrary meshes and local polynomial adaptivity. However, traditional DG approaches incur excessive computational costs due to their large coupled systems of degrees of freedom (DOFs). The hybridizable DG (HDG) method [1] retains these DG advantages while dramatically reducing DOFs through static condensation. For elliptic problems, HDG achieves optimal convergence (e.g., $\mathscr{O}(h^2)$ for both solution and flux variables) by hybridizing unknowns onto the mesh skeleton and permits element-wise postprocessing to further enhance accuracy. For the Helmholtz equation with impedance boundary conditions, the stability of HDG has been proven without mesh constraints when using pure imaginary penalty parameters τ_K, as shown in [2]. When penalty parameters is a real number, and given that $u \in H^3(\Omega)$, k^3h^2 is sufficiently small, and $\tau_K = \mathscr{O}(k)$, [3] given the explicit wavenumber-dependent optimal convergence:

$$k\|u-u_h\|_{L^2(\Omega)} + \|\mathbf{q}-\mathbf{q}_h\|_{L^2(\Omega)} = \mathscr{O}(k^2h^2 + k^4h^3).$$

This combination of efficiency, adaptivity, and pollution control makes HDG a compelling alternative to standard FEM for high-frequency wave problems.

In this proceedings, we continue to show the construction of the adaptive nonoverlapping preconditioners for the Helmholtz equation using HDG discretizations. We begin with the mixed method formulation:

$$\begin{aligned} \mathbf{q} + \nabla u &= 0 && \text{in } \Omega \\ \nabla \cdot \mathbf{q} - k^2 u &= f && \text{in } \Omega \\ -\mathbf{q}\cdot\mathbf{n} + \mathbf{i}ku &= g && \text{on } \partial\Omega \end{aligned} \tag{1}$$

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The problem (1) has a unique solution $(\mathbf{q},u)\in\mathbf{H}(\mathrm{div},\Omega)\times H^1(\Omega)$, where

$$\mathbf{H}(\mathrm{div},\Omega):=\{\mathbf{q}\in L^2(\Omega)^d,\mathrm{div}\,\mathbf{q}\in L^2(\Omega)\}.$$

We proceed by describing the HDG discretization. Let $\mathscr{T}_h$ be a family of shape-regular triangulations of Ω with mesh size $O(h)$. Let $\mathscr{E}_h$, $\mathscr{E}_h^I$, and $\mathscr{E}_h^B$ denote the sets of all edges/faces of elements in $\mathscr{T}_h$, the inner edges/faces, and edges/faces on $\partial\Omega$, respectively. The HDG method provides a scalar approximation u_h to u, a vector approximation $\mathbf{q}_h$ to $\mathbf{q}$, and a scalar approximation λ_h to the trace of u on element faces, in the spaces $\mathbf{V}_h$, W_h, and M_h, respectively, where

$$\begin{aligned}\mathbf{V}_h&=\left\{\mathbf{p}\in L^2(\mathscr{T}_h)^d\mid\mathbf{p}|_K\in\mathbf{P}_k(K),\forall K\in\mathscr{T}_h\right\},\\W_h&=\left\{w\in L^2(\mathscr{T}_h)\mid w|_K\in P_k(K),\forall K\in\mathscr{T}_h\right\},\\M_h&=\left\{\mathbf{m}\in L^2(\mathscr{E}_h)\mid\mathbf{m}|_F\in P_k(F),\forall F\in\mathscr{E}_h\right\}.\end{aligned}$$

Here, $\mathbf{P}_k(K)=P_k(K)^d$ and $P_k(K)$ denotes the space of polynomials of degree at most k on K. In this proceedings, we focus on the linear HDG method. We remark that our preconditioner construction can be readily extended to HDG with P_2 elements. However, for the Helmholtz equation with higher-order HDG, since both $\mathbf{q}_h$ and u_h cannot be locally eliminated to obtain a formulation solely in terms of $\hat{u}_h$, the preconditioner construction requires consideration of degrees of freedom associated with both $\mathbf{q}_h$ and u_h. Consequently, this necessitates a different algorithmic approach.

We also need to define the numerical flux $\hat{\mathbf{q}}_h$, which is a double-valued vector function on mesh interfaces, as follows:

$$\hat{\mathbf{q}}_h\cdot\mathbf{n}=\mathbf{q}_h\cdot\mathbf{n}+\tau_K(u_h-\lambda_h)\qquad\text{on }\mathscr{E}_h.\tag{2}$$

where τ_K is the stabilizer, a nonnegative function on $\mathscr{E}_h$, which can be either a single-valued or double-valued function. Here, τ_K denotes the τ-value on the boundary ∂K.

With the numerical flux $\hat{\mathbf{q}}_h$ defined, the HDG discretization of problem (1) is: Find $(\mathbf{q}_h,u_h,\hat{u}_h)\in\mathbf{V}_h\times W_h\times M_h$ such that for all $(\mathbf{v},w,\mu)\in\mathbf{V}_h\times W_h\times M_h$:

$$\begin{aligned}-(\mathbf{q}_h,\mathbf{v})_{\mathscr{T}_h}+(u_h,\nabla\cdot\mathbf{v})_{\mathscr{T}_h}-<\hat{u}_h,\mathbf{v}\cdot\mathbf{n}>_{\mathscr{E}_h}&=\mathbf{0},\\-(\mathbf{q}_h,\nabla w)_{\mathscr{T}_h}-k^2(u_h,w)_{\mathscr{T}_h}+<\hat{\mathbf{q}}_h\cdot\mathbf{n},w>_{\mathscr{E}_h}&=(f,w)_{\mathscr{T}_h},\\<-\hat{\mathbf{q}}_h\cdot\mathbf{n}+\mathbf{i}k\hat{u}_h,\mu>_{\mathscr{E}_h^B}&=<g,\mu>_{\mathscr{E}_h^B},\\-<\hat{\mathbf{q}}_h\cdot\mathbf{n},\mu>_{\mathscr{E}_h^I}&=0,\end{aligned}\tag{3}$$

The weak formulation (3), together with the stabilizer (2), can be written in matrix form as:

$$\begin{bmatrix}A_{\mathbf{qq}}&A_{u\mathbf{q}}^T&A_{\lambda\mathbf{q}}^T\\A_{u\mathbf{q}}&A_{uu}&A_{\lambda u}^T\\A_{\lambda\mathbf{q}}&A_{\lambda u}&A_{\lambda\lambda}\end{bmatrix}\begin{bmatrix}\mathbf{q}_h\\u_h\\\hat{u}_h\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\F_W\\G_\mu\end{bmatrix},\tag{4}$$

where $(A_{\mathbf{qq}}\mathbf{p},\mathbf{r})_{\mathscr{T}_h}=-(\mathbf{p},\mathbf{r})_{\mathscr{T}_h}$, $(A_{u\mathbf{q}}\mathbf{r},w)_{\mathscr{T}_h}=(w,\nabla\cdot\mathbf{r})_{\mathscr{T}_h}$, $<A_{\lambda\mathbf{q}}\mathbf{r},\mu>_{\partial\mathscr{T}_h}=-<\mu,\mathbf{r}\cdot\mathbf{n}>_{\partial\mathscr{T}_h}$, $(A_{uu}w,v)_{\mathscr{T}_h}=-k^2(w,v)_{\mathscr{T}_h}+\tau_k<w,v>_{\partial\mathscr{T}_h}$, $<A_{\lambda u}w,\mu>_{\partial\mathscr{T}_h}=-\tau_k<w,\mu>_{\partial\mathscr{T}_h}$, $<A_{\lambda\lambda}\mu,\eta>_{\partial\mathscr{T}_h}=(\tau_K+\mathbf{i}k)<\mu,\eta>_{\partial\mathscr{T}_h}$, and F_W, G_μ represent the L^2 projections of f into W_h and g into M_h, respectively.

Next, we seek to rewrite (3) into a variational formulation involving only $\hat{u}_h$ by eliminating $\mathbf{q}_h$ and u_h. This elimination is feasible if the following local solvers are solvable, as discussed in [1].

The first local solver requires, given $m\in M_h$, finding $\mathbf{Q}m\in\mathbf{V}_h$ and $\mathbf{U}m\in W_h$ such that

$$\begin{aligned}
&-(\mathbf{Q}m,\mathbf{v})_K+(\mathbf{U}m,\nabla\cdot\mathbf{v})_K=<m,\mathbf{v}\cdot\mathbf{n}>_K,\\
&-(\mathbf{Q}m,\nabla w)_K-k^2(\mathbf{U}m,w)_K+<\hat{\mathbf{Q}}m\cdot\mathbf{n},w>_K=0,
\end{aligned}$$

where $\hat{\mathbf{Q}}m$ is the numerical trace of the flux $\mathbf{q}_h$.

The second local solver requires, given $f\in L^2(\Omega)$, finding $\mathbf{Q}f\in\mathbf{V}_h$ and $\mathbf{U}f\in W_h$ such that

$$\begin{aligned}
&-(\mathbf{Q}f,\mathbf{v})_K+(\mathbf{U}f,\nabla\cdot\mathbf{v})_K=0,\\
&-(\mathbf{Q}f,\nabla w)_K-k^2(\mathbf{U}f,w)_K+<\hat{\mathbf{Q}}f\cdot\mathbf{n},w>_K=(f,w)_k.
\end{aligned}$$

When τ_K is a pure imaginary number or $\tau_K=O(k)$ for HDG with P_1 or P_2 and k^3h^2 is sufficiently small, these local solvers are guaranteed to be solvable.

With the solvability of the local solvers established, we can reduce the HDG formulation (3) to the following linear system: Find $\hat{u}_h\in M_h$ such that

$$b(\hat{u}_h,\mu)=L(\mu),\quad\forall\mu\in M_h,\tag{5}$$

where

$$\begin{aligned}
b(\lambda,\mu)=\sum_{K\in\mathscr{T}_h}a_K(\lambda,\mu):=&(\mathbf{Q}\lambda,\mathbf{Q}\mu)_{K\in\mathscr{T}_h}-k^2(\mathbf{U}\lambda,\mathbf{U}\mu)_{K\in\mathscr{T}_h}\\
&+<\tau_K(\mathbf{U}\lambda-\lambda),(\mathbf{U}\mu-\mu)>_{F\in\mathscr{E}_h}+\mathbf{i}k<\lambda,\mu>_{F\in\mathscr{E}_h^B},\\
L(\mu)=\sum_{K\in\mathscr{T}_h}L_K(\mu):=&(f,\mathbf{U}\mu)_{K\in\mathscr{T}_h}.
\end{aligned}$$

We can also rewrite the matrix form of (4), incorporating the solvability of the local solvers. For each element K, we have:

$$B^{(K)}=A_{\lambda\lambda}^{(K)}-\begin{bmatrix}A_{\lambda q}^{(K)}&A_{\lambda u}^{(K)}\end{bmatrix}\begin{bmatrix}A_{qq}^{(K)}&A_{uq}^{(K)^T}\\A_{uq}^{(K)}&A_{uu}^{(K)}\end{bmatrix}^{-1}\begin{bmatrix}A_{\lambda q}^{(K)^T}\\A_{\lambda u}^{(K)^T}\end{bmatrix},\tag{6}$$

$$L^{(K)}=G_\mu^{(K)}-\begin{bmatrix}A_{\lambda q}^{(K)}&A_{\lambda u}^{(K)}\end{bmatrix}\begin{bmatrix}A_{qq}^{(K)}&A_{uq}^{(K)^T}\\A_{uq}^{(K)}&A_{uu}^{(K)}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0}\\F_w^{(K)}\end{bmatrix}.$$

This leads to the following linear system:

$$B\hat{u}_h = \sum_{K\in\mathscr{T}_h} B^{(K)}\hat{u}_h = \sum_{K\in\mathscr{T}_h} L^{(K)} = L. \tag{7}$$

Next, we introduce a domain decomposition in Ω based on a fine-scale partition $\mathscr{T}_h$. Let $\{\Omega_i\}_{i=1}^N$ be nonoverlapping open subdomains of size $O(H)$, whose boundaries align with the fine-scale partition. Based on this domain decomposition, the linear system (7) can be written as

$$B\hat{u}_h = \sum_{i=1}^N R^{(i)^T} B^{(i)} R^{(i)} \hat{u}_h = \sum_{i=1}^N \begin{bmatrix} R^T_{\Gamma_i\Gamma} B^{(i)}_{\Gamma\Gamma} R_{\Gamma_i\Gamma} & R^T_{\Gamma_i\Gamma} B^{(i)}_{\Gamma I} R_{I_iI} \\ R^T_{I_iI} B^{(i)}_{I\Gamma} R_{\Gamma_i\Gamma} & R^T_{I_iI} B^{(i)}_{II} R_{I_iI} \end{bmatrix} \begin{bmatrix} u_\Gamma \\ u_I \end{bmatrix} = \begin{bmatrix} L_\Gamma \\ L_I \end{bmatrix}, \tag{8}$$

where the superscripts denote the subdomain numbering, and the subscripts denote the vectors associated with the nodal points in Γ_i and I_i, respectively.

Finally, we note that the choice of τ_K, whether real or imaginary, impacts the construction of the preconditioner.

2 The Case When τ is a Real Number

We first consider the scenario where τ is a real number. In this case, the only complex component in equation (6) is $A^{(K)}_{\lambda\lambda}$ when element K interacts with the impedance boundary. Consequently, we can apply the construction from adaptive nonoverlapping preconditioners designed for the Helmholtz equation. We begin by defining a norm based on equation (4). Let $(\tilde{A}_{uu}w,v)_{\mathscr{T}_h} = k^2(w,v)_{\mathscr{T}_h} + \tau_k\langle w,v\rangle_{\partial\mathscr{T}_h}$. Substituting A_{uu} in equation (4) with $\tilde{A}_{uu}$ alters the sesquilinear form $b(\cdot,\cdot)$ by changing $-k^2(\mathbf{U}\lambda,\mathbf{U}\mu)_{K\in\mathscr{T}_h}$ to $k^2(\mathbf{U}\lambda,\mathbf{U}\mu)_{K\in\mathscr{T}_h}$. Therefore, let

$$H = Re(A_{\lambda\lambda}) - \begin{bmatrix} A_{\lambda q} & A_{\lambda u} \end{bmatrix} \begin{bmatrix} A_{qq} & A^T_{uq} \\ A_{uq} & \tilde{A}_{uu} \end{bmatrix}^{-1} \begin{bmatrix} A^T_{\lambda q} \\ A^T_{\lambda u} \end{bmatrix},$$

then it is easy to see that H is a real symmetric and positive definite matrix. In each subdomain, we consider the generalized eigenvalue problem $B^{(i)}_{II}\xi^{(i)}_j = \lambda^{(i)}_j H^{(i)}_{II}\xi^{(i)}_j$, and denote the $H^{(i)}_{II}$-orthonormal eigenvectors corresponding to small magnitudes of eigenvalues as $Q^{(i)}_s$, while the remaining eigenvectors are denoted as $Q^{(i)}_l$. Define $V_0 = M_h(\Gamma) \oplus \sum_{i=1}^N R^{(i)^T}_S \alpha^{(i)}_s$, where $\alpha^{(i)}_s$ is an index set and $R^{(i)^T}_S$ is the corresponding extension operator. Also, let $V_i := \text{Range}(Q^{(i)}_l)$. The extension operator $R^T_0 : V_0 \to V_h(\Omega)$ is defined as

$$R^T_0 u_0 = \begin{bmatrix} I_\Gamma & 0 \\ \sum_{i=1}^N -R^T_{I_iI} B^{(i)}_L B^{(i)}_{I\Gamma} R_{\Gamma_i\Gamma} & \sum_{i=1}^N R^T_{I_iI} Q^{(i)}_s R^{(i)}_S \end{bmatrix} u_0,$$

where $B^{(i)}_L = Q^{(i)}_l \left(Q^{(i)^T}_l B^{(i)}_{II} Q^{(i)}_l\right)^{-1} Q^{(i)^T}_l$.

Next, let $B_0^{(i)} = \begin{bmatrix} R_{\Gamma_i\Gamma}^T \left(B_{\Gamma\Gamma}^{(i)} - B_{\Gamma I}^{(i)} B_L^{(i)} B_{I\Gamma}^{(i)}\right) R_{\Gamma_i\Gamma} & R_{\Gamma_i\Gamma}^T B_{\Gamma I}^{(i)} Q_s^{(i)} R_S^{(i)} \\ R_S^{(i)^T} Q_s^{(i)^T} B_{I\Gamma}^{(i)} R_{\Gamma_i\Gamma} & R_S^{(i)^T} Q_s^{(i)^T} B_{II}^{(i)} Q_s^{(i)} R_S^{(i)} \end{bmatrix}$, and $C_0^{(i)} = \begin{bmatrix} \hat{C}_{\Gamma\Gamma}^{(i)} & 0 \\ 0 & Q_s^{(i)^T} H_{II}^{(i)} Q_s^{(i)} \end{bmatrix}$, where $\hat{C}_{\Gamma\Gamma}^{(i)}$ is the block diagonal version of $C_{\Gamma\Gamma}^{(i)} = H_{\Gamma\Gamma}^{(i)} - H_{\Gamma I}^{(i)} H_L^{(i)} H_{I\Gamma}^{(i)}$, with $H_L^{(i)} = Q_l^{(i)} Q_l^{(i)^T}$. We consider the generalized eigenvalue problem in each subdomain separately:

$$\Re B_0^{(i)} \xi_{Re_j}^{(i)} = \lambda_{Re_j}^{(i)} C_0^{(i)} \xi_{Re_j}^{(i)} \quad (j = 1, \cdots, N_i), \tag{9}$$

where N_i is the number of DOFs on the interface Γ_i, and the eigenvectors $\xi_{Re_j}^{(i)}$ are orthonormal with respect to $C_0^{(i)}$. For a chosen threshold $\eta \in (0,1)$, we choose the eigenvalues smaller than η and larger than 2, and denote its corresponding eigenvectors space as $Q_{Re}^{(i)} = [\xi_{Re_1}^{(i)}, \xi_{Re_2}^{(i)}, \cdots, \xi_{Re_{\hat{K}_i}}^{(i)}]$, and let $D_{Re}^{(i)} = \text{diagonal}(1 - \lambda_{Re_1}^{(i)}, 1 - \lambda_{Re_2}^{(i)}, \cdots, 1 - \lambda_{Re_{\hat{K}_i}}^{(i)})$. Then, we define the following operators: $\Pi_{Re}^{(i)} = Q_{Re}^{(i)} Q_{Re}^{(i)^T} C_0^{(i)}$, and $\Pi_{DRe}^{(i)} = Q_{Re}^{(i)} D_{Re}^{(i)} Q_{Re}^{(i)^T} C_0^{(i)}$. Note that for the HDG method, the imaginary part of $B_0^{(i)}$, which is precisely $\Im A_{\lambda\lambda}^{(i)}$, is already block diagonal, with each block associated with an element. We can then define a sesquilinear form $b_{P_1}(\cdot,\cdot) : V_0 \times V_0 \to \mathbb{C}$ as follows:

$$\begin{aligned} b_{P_1}(u_0, v_0) &= v_0^H \sum_{i=1}^{N} R_{\Gamma,s}^{(i)^T} \left(\Pi_{Re}^{(i)^T} \Re B_0^{(i)} \Pi_{Re}^{(i)} + (I - \Pi_{Re}^{(i)^T}) C_0^{(i)} (I - \Pi_{Re}^{(i)}) + \mathbf{i} R_{\Gamma,0}^{(i)^T} \Im B_{\Gamma\Gamma}^{(i)} R_{\Gamma,0}^{(i)}\right) R_{\Gamma,s} u_0 \\ &= v_0^H \sum_{i=1}^{N} R_{\Gamma,s}^{(i)^T} \Big(C_0^{(i)} - C_0^{(i)} \Pi_{DRe}^{(i)} + \mathbf{i} R_{\Gamma,0}^{(i)^T} \Im B_{\Gamma\Gamma}^{(i)} R_{\Gamma,0}^{(i)} \Big) R_{\Gamma,s}^{(i)} u_0, \end{aligned}$$

where $R_{\Gamma,s}^{(i)^T} : V_0^{(i)} \to V_0$, and $R_{\Gamma,0}^{(i)^T} : V_h(\Gamma_i) \to V_0^{(i)}$ are zero extension operators, with $R_{\Gamma,s}^{(i)}$ and $R_{\Gamma,0}^{(i)}$ being their respective transposes.

Then, the resulting preconditioner can be expressed as:

$$P_1^{-1} = R_0^T (B_{P_1})^{-1} R_0 + \sum_{i=1}^{N} R_i^T Q_l^{(i)} (B_i)^{-1} Q_l^{(i)^T} R_i, \tag{10}$$

where $B_i := Q_l^{(i)^T} B_{II}^{(i)} Q_l^{(i)}$.

3 The Case When τ is an Imaginary Number

When τ is an imaginary number, the matrices $A_{\lambda\lambda}^{(K)}$, $A_{\lambda u}^{(K)}$, and $A_{\lambda u}^{(K)^T}$ in (6) are purely imaginary, while $A_{uu}^{(K)}$ is a complex matrix. As a result, the matrix $B_{II}^{(i)}$ is complex. However, we can show that $B_{II}^{(i)}$ is invertible, which forms our local problem.

To define a new norm, we also replace A_{uu} in (4) with $\tilde{A}_{uu}$ and let

$$H = Re\Big(A_{\lambda\lambda} - \begin{bmatrix} A_{\lambda q} \; A_{\lambda u} \end{bmatrix} \begin{bmatrix} A_{qq} & A_{uq}^T \\ A_{uq} & \tilde{A}_{uu} \end{bmatrix}^{-1} \begin{bmatrix} A_{\lambda q}^T \\ A_{\lambda u}^T \end{bmatrix}\Big),$$

which is a real symmetric positive definite matrix.

For the coarse problem, we handle the real and imaginary parts separately. In each subdomain, we consider the following generalized eigenvalue problems for the real part: $\Re B_{\Gamma\Gamma}^{(i)} \xi_{Re_j}^{(i)} = \lambda_{Re_j}^{(i)} C_{\Gamma\Gamma}^{(i)} \xi_{Re_j}^{(i)}$, where $C_{\Gamma\Gamma}^{(i)}$ is the block diagonal matrix obtained from $H_{\Gamma\Gamma}^{(i)} - H_{\Gamma I}^{(i)} (H_{II}^{(i)})^{-1} H_{I\Gamma}^{(i)}$ by eliminating the connections between each element edge.

Similarly, we also consider the imaginary part generalized eigenvalue problems: $\Im B_{\Gamma\Gamma}^{(i)} \xi_{Im_j}^{(i)} = \lambda_{Im_j}^{(i)} \hat{B}_{\Gamma\Gamma}^{(i)} \xi_{Im_j}^{(i)}$, where $\hat{B}_{\Gamma\Gamma}$ is the block diagonal matrix obtained from $\Im B_{\Gamma\Gamma}^{(i)}$ by eliminating the connection between each element edge.

We proceed by choosing thresholds η_{Re} and η_{Im} for selecting eigenvectors. For both real part and imaginary part generalized eigenvalue problems, we choose eigenvectors corresponding to eigenvalues less than η_{Re} and greater than 2, and less than η_{Im} and greater than 2, respectively. We then construct the eigenfunction spaces Q_{Re} and Q_{Im}. We then define the local projection operators $\Pi_{Re} : V_0^{(i)} \to Q_{Re}^{(i)}$ and $\Pi_{Im} : V_0^{(i)} \to Q_{Im}^{(i)}$ as follows:

$$\Pi_{Re} = Q_{Re}^{(i)} (Q_{Re}^{(i)^T} C_{\Gamma\Gamma}^{(i)} Q_{Re}^{(i)})^{-1} Q_{Re}^{(i)^T} C_{\Gamma\Gamma}^{(i)}, \;\; \Pi_{Im} = Q_{Im}^{(i)} (Q_{Im}^{(i)^T} \hat{B}_{\Gamma\Gamma}^{(i)} Q_{Im}^{(i)})^{-1} Q_{Im}^{(i)^T} \hat{B}_{\Gamma\Gamma}^{(i)}.$$

Then, we define the local coarse sesquilinear form $b_{P_2}^{(i)}(\cdot,\cdot) : V_0^{(i)} \times V_0^{(i)} \to \mathbb{C}$ as follows:

$$\begin{aligned} b_{P_2}^{(i)}(u_0^{(i)}, v_0^{(i)}) &= v_0^{(i)^H} \big(\Pi_{Re}^{(i)^T} \Re B_{\Gamma\Gamma}^{(i)} \Pi_{Re}^{(i)} + (I - \Pi_{Re}^{(i)^T}) C_{\Gamma\Gamma}^{(i)} (I - \Pi_{Re}^{(i)})\big) u_0^{(i)} \\ &\quad + \mathbf{i} v_0^{(i)^H} \big(\Pi_{Im}^{(i)^T} \Im B_{\Gamma\Gamma}^{(i)} \Pi_{Im}^{(i)} + (I - \Pi_{Im}^{(i)^T}) \hat{B}_{\Gamma\Gamma}^{(i)} (I - \Pi_{Im}^{(i)})\big) u_0^{(i)} \\ &= v_0^{(i)^H} \Big(C_{\Gamma\Gamma}^{(i)} - C_{\Gamma\Gamma}^{(i)} \Pi_{DRe}^{(i)} + \mathbf{i}\hat{B}_{\Gamma\Gamma}^{(i)} - \mathbf{i}\hat{B}_{\Gamma\Gamma}^{(i)} \Pi_{DIm}^{(i)}\Big) u_0^{(i)}, \end{aligned}$$

where $\Pi_{DRe}^{(i)} = Q_{Re}^{(i)} D_{Re}^{(i)} Q_{Re}^{(i)^T} C_{\Gamma\Gamma}^{(i)}$ and $\Pi_{DIm}^{(i)} = Q_{Im}^{(i)} D_{Im}^{(i)} Q_{Im}^{(i)^T} \hat{B}_{\Gamma\Gamma}^{(i)}$, with $D_{Re}^{(i)}$ and $D_{Im}^{(i)}$ being the identity matrix minus the corresponding diagonal eigenvalue matrices, respectively.

Then we define the global coarse sesquilinear form $b_{P_2} = \sum_{i=1}^{N} b_{P_2}^{(i)}(R_\Gamma^{(i)} u_0, R_\Gamma^{(i)} v_0)$, and denote its matrix form as B_{P_2} .Together with the local problems, we have the following form of the preconditioner: ${P_2}^{-1} = R_0^T (B_{P_2})^{-1} R_0 + \sum_{i=1}^{N} R_i^T B_i^{-1} R_i$, where $B_i = R_i B R_i^T = B_{II}^{(i)}$.

Finally, we show the scalability of $B_{P_2}^{-1}$. By summing all local matrices, let $C = \sum_{i=1}^{N} R_{\Gamma}^{(i)^T}\left(C_{\Gamma\Gamma}^{(i)} + \mathbf{i}\Im\hat{B}_{\Gamma\Gamma}^{(i)}\right)R_{\Gamma}^{(i)}$, $U_1 = \sum_{i=1}^{N} R_{\Gamma}^{(i)^T} C_{\Gamma\Gamma}^{(i)} Q_{Re}^{(i)} R_{\lambda_i}$, $D_1 = \sum_{i=1}^{N} R_{\lambda_i}^T D_{Re}^{(i)} R_{\lambda_i}$, $U_2 = \sum_{i=1}^{N} R_{\Gamma}^{(i)^T} \hat{B}_{\Gamma\Gamma}^{(i)} Q_{Im}^{(i)} R_{\lambda_i}$, $D_2 = \mathbf{i}\sum_{i=1}^{N} R_{\lambda_i}^T D_{Im}^{(i)} R_{\lambda_i}$. Then, $B_{P_2} = C - U_1 D_1 U_1^T - U_2 D_2 U_2^T$. The Woodbury matrix identity can be used for implementation, that is,

$$B_{P_2}^{-1} = C^{-1} + C^{-1}[U_1, U_2]\left(\begin{bmatrix} D_1^{-1} & 0 \\ 0 & D_2^{-1} \end{bmatrix} - \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} C^{-1}[U_1, U_2]\right)^{-1} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} C^{-1}.$$

Here, C is a block diagonal matrix and its inverse can be computed in parallel. The size of the dense matrix inside the equation is the total number of selected eigenvalues from both the real and imaginary part eigenvalue problems, and it can be precomputed in advance.

4 Numerical Experiments

We will show the numerical results for the problem (3) in the domain $[0,1]^2$ with $f = 0$, considering various wavenumbers k, mesh sizes h, subdomain sizes H, and different eigenvalue thresholds. The exact solution is chosen as $u(x,y) = e^{ik(\cos\frac{\pi}{8}x + \sin\frac{\pi}{8}y)}$, and the linear HDG discretization follows the procedure outlined in Section 1. Tables 1 and 2 display the results for real τ with different thresholds η, where the parameter $\beta = 0.01$ was chosen to guarantee the solvability of local problems for real τ_K, while Table 3 presents the corresponding results for imaginary τ. We used the Generalized Minimal Residual (GMRES) method with the relative residual error tolerance of 10^{-6} in the ℓ^2 norm.

$k = 20$	$H = \frac{1}{2}$	$H = \frac{1}{4}$	$H = \frac{1}{8}$	$H = \frac{1}{16}$
$h = 1/32$	18 (15)	14 (7)	14 (4)	20 (2)
$h = 1/64$	12 (28)	14 (11)	12 (7)	11 (4)

$k = 30$	$H = \frac{1}{4}$	$H = \frac{1}{8}$	$H = \frac{1}{16}$	$H = \frac{1}{32}$
$h = 1/64$	12 (14)	12 (8)	13 (4)	22 (2)
$h = 1/128$	13 (22)	12 (12)	12 (6)	12 (4)

$k = 40$	$H = \frac{1}{8}$	$H = \frac{1}{16}$	$H = \frac{1}{32}$	$H = \frac{1}{64}$
$h = 1/128$	14 (11)	13 (7)	14 (4)	23 (2)
$h = 1/256$	19 (23)	11 (12)	14 (5)	13 (4)

$k = 50$	$H = \frac{1}{16}$	$H = \frac{1}{32}$	$H = \frac{1}{64}$	$H = \frac{1}{128}$
$h = 1/256$	13 (12)	17 (5)	15 (4)	24 (2)
$h = 1/512$	11 (23)	11 (12)	13 (5)	15 (4)

Table 1: GMRES iterations with preconditioner P_1^{-1} ($\tau_K = k$, $\beta = 0.01$, $\eta = 0.3$). Parentheses indicate the number of selected eigenfunctions per subdomain in (9).

We observe that our preconditioner performs significantly better for real τ than for imaginary τ, because for real τ we only need to solve real eigenvalue problems, and the imaginary part is already block diagonal due to the properties of HDG, so the imaginary part of our preconditioner equals the imaginary part of the original

$k = 20$	$H = \frac{1}{2}$	$H = \frac{1}{4}$	$H = \frac{1}{8}$	$H = \frac{1}{16}$
$h = 1/32$	8 (32)	8 (14)	9 (7)	9 (4)
$h = 1/64$	7 (53)	8 (25)	8 (12)	8 (6)

$k = 30$	$H = \frac{1}{4}$	$H = \frac{1}{8}$	$H = \frac{1}{16}$	$H = \frac{1}{32}$
$h = 1/64$	8 (26)	8 (13)	8 (7)	21 (3)
$h = 1/128$	7 (49)	7 (24)	8 (12)	7 (6)

$k = 40$	$H = \frac{1}{8}$	$H = \frac{1}{16}$	$H = \frac{1}{32}$	$H = \frac{1}{64}$
$h = 1/128$	8 (25)	8 (12)	8 (6)	22 (3)
$h = 1/256$	7 (48)	7 (24)	7 (12)	7 (6)

$k = 50$	$H = \frac{1}{16}$	$H = \frac{1}{32}$	$H = \frac{1}{64}$	$H = \frac{1}{128}$
$h = 1/256$	8 (24)	8 (12)	8 (6)	24 (3)
$h = 1/512$	7 (48)	7 (24)	7 (12)	7 (6)

Table 2: GMRES iterations with preconditioner P_1^{-1} ($\tau_K = k,\ \beta = 0.01,\ \eta = 0.6$). Parentheses indicate the number of selected eigenfunctions per subdomain in (9).

$k = 20$	$H = \frac{1}{2}$	$H = \frac{1}{4}$	$H = \frac{1}{8}$	$H = \frac{1}{16}$
$h = 1/32$	15 (15,13)	15 (7,6)	15 (4,4)	21 (2,2)
$h = 1/64$	15 (25,23)	15 (11,15)	14 (7,8)	14 (4,3)

$k = 30$	$H = \frac{1}{4}$	$H = \frac{1}{8}$	$H = \frac{1}{16}$	$H = \frac{1}{32}$
$h = 1/64$	14 (13,13)	13 (8,8)	15 (4,4)	26 (2,2)
$h = 1/128$	15 (22,31)	14 (12,15)	15 (6,8)	17 (4,3)

$k = 40$	$H = \frac{1}{8}$	$H = \frac{1}{16}$	$H = \frac{1}{32}$	$H = \frac{1}{64}$
$h = 1/128$	16 (11,15)	16 (7,8)	18 (4,3)	33 (2,2)
$h = 1/256$	21 (23,31)	15 (12,15)	21 (5,8)	22 (4,3)

$k = 50$	$H = \frac{1}{16}$	$H = \frac{1}{32}$	$H = \frac{1}{64}$	$H = \frac{1}{128}$
$h = 1/256$	15 (12,15)	24 (5,8)	23 (4,3)	44 (2,2)
$h = 1/512$	17 (23,31)	18 (12,15)	24 (5,8)	33 (4,4)

Table 3: GMRES iterations with preconditioner P_2^{-1} ($\tau_K = \frac{\mathbf{i}}{h},\ \eta_{\text{Re}} = 0.3,\ \eta_{\text{Im}} = 0.1$). Parentheses indicate the number of selected eigenfunctions per subdomain for the real and imaginary parts, respectively.

linear system. Similar to the FEM cases, our numerical experiments demonstrate that as the wavenumber k increases, the mesh size h must decrease to maintain the convergence rate, and we need to increase the threshold parameter η. For fixed values of k and h, reducing the subdomain size H leads to increased iteration counts and larger coarse problem dimensions because of the approximation property of our preconditioner. Thus, optimal selection of both H and η is crucial for our Helmholtz preconditioning. The optimal choice of H depends on computational constraints, and the threshold η is determined empirically, with our implementation selecting eigenvalues that are either smaller than η or larger than 2.

References

1. Cockburn, B., Gopalakrishnan, J., Lazarov, R.: Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. SIAM Journal on Numerical Analysis **47**(2), 1319–1365 (2009)
2. Cui, J., Zhang, W.: An analysis of HDG methods for the Helmholtz equation. IMA Journal of Numerical Analysis **34**(1), 279–295 (2014)
3. Zhu, B., Wu, H.: Preasymptotic error analysis of the HDG method for Helmholtz equation with large wave number. Journal of Scientific Computing **87**, 1–34 (2021)