

# Adaptive Nonoverlapping Preconditioners for the Helmholtz Equation: h-FEM

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## 1 Helmholtz Equation and Its Finite Element Discretizations

The time-dependent wave equation is a fundamental partial differential equation that arises in various scientific and engineering applications, including acoustics, electromagnetics, and seismic imaging. In this proceeding, we focus on the preconditioning of the Helmholtz equation, which represents the time-harmonic case of the wave equation. Traditional domain decomposition methods (DDMs) face two significant challenges when applied to preconditioning the Helmholtz equation: first, the local Dirichlet problem may not be solvable; second, constructing a global problem is difficult. We propose a new substructuring approach and introduce a novel DDM that partially addresses these challenges.

Let  $\Omega \subset \mathbb{R}^{2,3}$  be a bounded convex polygonal/polyhedral domain. We study the following variational formulation of the Helmholtz equation with impedance boundary conditions. Given  $f \in (H^1(\Omega))'$  and  $g \in H^{-1/2}(\Gamma)$ , find  $u \in H^1(\Omega)$  such that

$$\begin{aligned} b(u, v) &:= \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx - k^2 \int_{\Omega} u \bar{v} dx + ik \int_{\partial\Omega} u \bar{v} ds \\ &= \int_{\Omega} f \bar{v} dx + \int_{\partial\Omega} g \bar{v} ds := l(v). \end{aligned} \quad \text{for all } v \in H^1(\Omega) \quad (1)$$

The existence and uniqueness of the weak solution to (1) are established in [1]. We begin by considering the conforming Galerkin finite element approximation of the Helmholtz equation's variational formulation (1). Let  $\mathcal{T}_h$  be a shape-regular partition of the physical domain  $\bar{\Omega}$  with mesh size  $h > 0$ . We define conforming finite element spaces  $V_h$  on  $\mathcal{T}_h$  and consider the following discrete problem:

$$b(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h. \quad (2)$$

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In the case of linear finite elements, under the very restrictive condition that “ $hk^2$  is sufficiently small,” as shown in [2], the discrete problem is well-posed and admits quasi-optimal error estimates. If only discrete stability and error estimates are required, the mesh condition can be relaxed to “ $h^2k^3$  sufficiently small” , see [3]. In this proceeding, we require the following discrete inf-sup condition.

**Assumption 1** (Discrete Inf-Sup Condition). *Assume that for the finite element spaces  $\mathcal{T}_h$ , there exists a constant  $\gamma > 0$ , such that the following discrete inf-sup condition holds:*

$$\inf_{u \in V_h \setminus \{0\}} \sup_{v \in V_h \setminus \{0\}} \frac{|b(u, v)|}{\|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}} \geq \gamma,$$

where the Helmholtz energy norm defined as:

$$\|v_h\|_{\mathcal{H}} = \|v_h\|_{H^1(\Omega)} + k\|v_h\|_{L^2(\Omega)}.$$

## 2 A New Substructuring Structure: The Schur Complement

We first introduce a domain partition in  $\Omega$  based on a fine-scale partition  $\mathcal{T}_h$ . Let  $\{\Omega_i\}_{i=1}^N$  be nonoverlapping open subdomains of size  $O(H)$  whose boundaries match across the fine-scale partition. These subdomains also satisfy:  $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$  and  $\Omega_i \cap \Omega_j = \emptyset$ ,  $i \neq j$ . For each subdomain, we define the interface and interior as  $\Gamma_i$  and  $I_i$ , respectively. The global interface  $\Gamma$  and the global interior  $I$  are then defined as:  $\Gamma_i = \partial\Omega_i$ ,  $\Gamma = \cup_{i=1}^N \Gamma_i$ , and  $I = \cup_{i=1}^N I_i$ . Focusing on the discrete problem (2), the corresponding linear system can be expressed as:

$$Bu_h = L, \quad (3)$$

where  $u_h$  represents both the finite element function and its corresponding column vector of nodal values.

We first consider the local assembly of the matrix  $B$ . The local sesquilinear form  $b^{(i)}(\cdot, \cdot)$  is defined as follows:

$$\begin{aligned} b^{(i)}(u^{(i)}, v^{(i)}) &= \int_{\Omega_i} (\nabla u^{(i)} \cdot \nabla \overline{v^{(i)}} - k^2 u^{(i)} \overline{v^{(i)}}) dx + ik \int_{\partial\Omega_i \cap \partial\Omega} u^{(i)} \overline{v^{(i)}} ds, \\ &:= v^{(i)H} A^{(i)} u^{(i)} - k^2 v^{(i)H} M^{(i)} u^{(i)} + ik v^{(i)H} N^{(i)} u^{(i)} \end{aligned}$$

where  $A^{(i)}$  and  $M^{(i)}$  are the stiffness and mass matrices, respectively, and  $N^{(i)}$  corresponds to the impedance boundary condition, which is a zero matrix when  $\partial D \cap \partial D_i = \emptyset$ . The matrix  $B^{(i)}$  is then expressed as:  $\begin{bmatrix} B_{\Gamma\Gamma}^{(i)} & B_{\Gamma I}^{(i)} \\ B_{I\Gamma}^{(i)} & B_{II}^{(i)} \end{bmatrix}$ , where the superscript denotes the subdomain numbering, and the subscripts indicate the vectors associated with the nodal points in  $\Gamma_i$  and  $I_i$ , respectively.

The linear system (3) can be rewritten as:

$$Bu_h = \sum_{i=1}^N R^{(i)T} B^{(i)} R^{(i)} u_h = \sum_{i=1}^N \begin{bmatrix} R_{\Gamma\Gamma}^T B_{\Gamma\Gamma}^{(i)} R_{\Gamma\Gamma} & R_{\Gamma\Gamma}^T B_{\Gamma I}^{(i)} R_{I\Gamma} \\ R_{I\Gamma}^T B_{\Gamma\Gamma}^{(i)} R_{\Gamma\Gamma} & R_{I\Gamma}^T B_{II}^{(i)} R_{I\Gamma} \end{bmatrix} \begin{bmatrix} u_\Gamma \\ u_I \end{bmatrix} = \begin{bmatrix} L_\Gamma \\ L_I \end{bmatrix}, \quad (4)$$

where the extension operators  $R^{(i)T} : V_h(\bar{\Omega}_i) \rightarrow V_h$ ,  $R_{\Gamma\Gamma}^T : V_h(\Gamma_i) \rightarrow V_h(\Gamma)$ , and  $R_{I\Gamma}^T : V_h(I_i) \rightarrow V_h(I)$  are zero extensions to the corresponding functional spaces. The restriction operators  $R^{(i)} : V_h \rightarrow V_h(\bar{\Omega}_i)$ ,  $R_{\Gamma\Gamma} : V_h(\Gamma) \rightarrow V_h(\Gamma_i)$ , and  $R_{I\Gamma} : V_h(I) \rightarrow V_h(I_i)$  are the transposes of the corresponding extension operators, which restrict a nodal vector in a larger space to a nodal vector in a smaller space.

Next, we introduce a novel substructuring approach that involves a global interface problem and  $N$  local problems. The main idea is to extract very small eigenmodes from each subdomain  $\Omega_i$  and incorporate them into the global problem, ensuring that the local problems to be well-posed. To achieve this, we begin by considering the following real generalized eigenvalue problem in each subdomain:

$$B_{II}^{(i)} \xi_j^{(i)} = \lambda_j^{(i)} H_{II}^{(i)} \xi_j^{(i)}, \quad (5)$$

where  $B_{II}^{(i)} = A_{II}^{(i)} - k^2 M_{II}^{(i)}$  is a real symmetric matrix, and  $H_{II}^{(i)} = A_{II}^{(i)} + k^2 M_{II}^{(i)}$  is a real symmetric positive definite matrix, and the eigenvectors  $\xi_j^{(i)}$  are orthonormal with respect to  $H_{II}^{(i)}$ .

The motivation for choosing  $H_{II}^{(i)}$  as the right-hand side matrix is that it is symmetric positive definite and represents the local Helmholtz energy norm in  $\Omega_i$ , which is useful for convergence analysis. We select only those eigenvalues whose magnitudes are very small, using a small threshold  $\beta > 0$  such that  $|\lambda_1^{(i)}| \leq |\lambda_2^{(i)}| \leq \dots \leq |\lambda_{k_i}^{(i)}| < \beta$ . We construct  $Q_s^{(i)} = [\xi_1^{(i)}, \dots, \xi_{k_i}^{(i)}] \in \mathbb{R}^{n_i \times k_i}$  as the eigenspace corresponding to all eigenvalues with absolute values less than or equal to  $\beta$ . The remaining eigenspaces are denoted by  $Q_l^{(i)}$ .

Consequently, any  $u^{(i)} \in V_h(\Omega_i)$  can be represented as:

$$u^{(i)} = \begin{bmatrix} u_\Gamma \\ u_I \end{bmatrix} = \begin{bmatrix} I_\Gamma & 0 & 0 \\ 0 & Q_s^{(i)} & Q_l^{(i)} \end{bmatrix} \begin{bmatrix} u_\Gamma \\ \alpha_s^{(i)} \\ \alpha_l^{(i)} \end{bmatrix}.$$

Here,  $\alpha_l^{(i)} = [\alpha_{k_i+1}^{(i)}, \dots, \alpha_{n_i}^{(i)}]^T \in \mathbb{C}^{n_i - k_i}$  and  $\alpha_s^{(i)} = [\alpha_1^{(i)}, \dots, \alpha_{k_i}^{(i)}]^T \in \mathbb{C}^{k_i}$ . In this context,  $\alpha_l^{(i)}$  and  $\alpha_s^{(i)}$  represent the coefficients of  $u_l^{(i)}$  under the sets of orthogonal bases  $Q_l^{(i)}$  and  $Q_s^{(i)}$ , respectively. The linear system (3) can then be rewritten as:

$$B_{\Gamma\Gamma} u_\Gamma + \sum_{i=1}^N R_{\Gamma\Gamma}^T B_{\Gamma\Gamma}^{(i)} Q_l^{(i)} \alpha_l^{(i)} + \sum_{i=1}^N R_{\Gamma\Gamma}^T B_{\Gamma\Gamma}^{(i)} Q_s^{(i)} \alpha_s^{(i)} = L_\Gamma, \quad (6a)$$

$$B_{II} u_\Gamma + \sum_{i=1}^N R_{I\Gamma}^T B_{II}^{(i)} Q_l^{(i)} \alpha_l^{(i)} + \sum_{i=1}^N R_{I\Gamma}^T B_{II}^{(i)} Q_s^{(i)} \alpha_s^{(i)} = L_I. \quad (6b)$$

Next, we formulate the system into one global interface problem and  $N$  local problems. For the local problems, we consider:

$$\alpha_{II}^{(i)} = (Q_l^{(i)T} B_{II}^{(i)} Q_l^{(i)})^{-1} Q_l^{(i)T} R_{I,I} L_I. \quad (7)$$

In practice, the dimensions of  $Q_s^{(i)}$ , which is  $k_i$ , are usually small, while  $n_i - k_i$  is large. Consequently, it is advisable to avoid directly computing  $Q_l^{(i)}$  in the numerical implementation. Instead of obtaining  $\alpha_{II}^{(i)}$  directly, we calculate  $u_i = Q_l^{(i)} \alpha_{II}^{(i)} \in V_h(I_i)$ , yielding:

$$u_i = Q_l^{(i)} (Q_l^{(i)T} B_{II}^{(i)} Q_l^{(i)})^{-1} Q_l^{(i)T} R_{I,I} L_I.$$

Thus, to obtain  $u_i$ , we solve the following local saddle point system:

$$\begin{bmatrix} B_{II}^{(i)} & H_{II}^{(i)} Q_s^{(i)} \\ Q_s^{(i)T} H_{II}^{(i)T} & 0 \end{bmatrix} \begin{bmatrix} u_i \\ \lambda_s^{(i)} \end{bmatrix} = \begin{bmatrix} R_{I,I} L_I \\ 0 \end{bmatrix}.$$

where  $\lambda_s^{(i)} \in \mathbb{C}^{k_i}$  is a Lagrange multiplier. The well-posedness of the saddle point problem can be established by demonstrating the uniqueness of the solution, utilizing the fact that  $Q_l^{(i)}$  is orthogonal to  $Q_s^{(i)}$  with respect to the  $H_{II}^{(i)}$ -norm. Numerically, this problem can be solved using  $LDL^T$  factorization.

Next, we derive the global interface problem. Multiply  $Q_l^{(i)T}$  by (6b) to obtain an explicit expression of  $\alpha_l^{(i)}$  in terms of  $u_i$  and  $u_\Gamma^{(i)}$ , for  $1 \leq i \leq N$ , and substitute in (6a). Also multiplying  $Q_s^{(i)T} R_{I,I}$  in (6b) again to eliminate each  $\alpha_l^{(i)}$ , the interface problem can be expressed as follows:

$$\begin{aligned} \sum_{i=1}^N R_{\Gamma\Gamma}^T \hat{B}_{\Gamma\Gamma}^{(i)} R_{\Gamma\Gamma} u_\Gamma + \sum_{i=1}^N R_{\Gamma\Gamma}^T B_{\Gamma\Gamma}^{(i)} Q_s^{(i)} \alpha_s^{(i)} &= L_\Gamma - \sum_{i=1}^N R_{\Gamma\Gamma}^T B_{\Gamma\Gamma}^{(i)} u_i, \\ Q_s^{(i)T} B_{II}^{(i)} R_{\Gamma\Gamma} u_\Gamma + Q_s^{(i)T} B_{II}^{(i)} Q_s^{(i)} \alpha_s^{(i)} &= R_{I,I}^T Q_s^{(i)T} R_{I,I} L_I, \quad \text{for } 1 \leq i \leq N. \end{aligned}$$

Here, we use the fact that  $Q_s^{(i)T} B_{II}^{(i)} u_i = 0$ , and  $\hat{B}_{\Gamma\Gamma}^{(i)} = B_{\Gamma\Gamma}^{(i)} - B_{\Gamma\Gamma}^{(i)} B_L^{(i)} B_{II}^{(i)}$ , where  $B_L^{(i)} = Q_l^{(i)} (Q_l^{(i)T} B_{II}^{(i)} Q_l^{(i)})^{-1} Q_l^{(i)T}$ . The global problem can then be presented in the following matrix form:

$$\sum_{i=1}^N \begin{bmatrix} R_{\Gamma\Gamma}^T \hat{B}_{\Gamma\Gamma}^{(i)} R_{\Gamma\Gamma} & R_{\Gamma\Gamma}^T B_{\Gamma\Gamma}^{(i)} Q_s^{(i)} R_S^{(i)} \\ R_S^{(i)T} Q_s^{(i)T} B_{II}^{(i)} R_{\Gamma\Gamma} & R_S^{(i)T} Q_s^{(i)T} B_{II}^{(i)} Q_s^{(i)} R_S^{(i)} \end{bmatrix} \begin{bmatrix} u_\Gamma \\ \alpha_s \end{bmatrix} = \sum_{i=1}^N \begin{bmatrix} L_\Gamma^{(i)} - R_{\Gamma\Gamma}^T B_{\Gamma\Gamma}^{(i)} u_i \\ R_S^{(i)T} Q_s^{(i)T} R_{I,I} L_I \end{bmatrix}, \quad (8)$$

where  $\alpha_s = [\alpha_s^{(1)}, \dots, \alpha_s^{(N)}]^T$  is the collection of indices for the small eigenvalues,  $R_S^{(i)}$  is the restriction operator for selecting  $\alpha_s^{(i)}$  from  $\alpha_s$ , and  $R_S^{(i)T}$  is its transpose.

As mentioned earlier, the computation of  $\hat{B}_{\Gamma\Gamma}^{(i)}$  using  $B_L^{(i)}$  can be avoided by leveraging the saddle point problem with Lagrange multipliers. This construc-

tion is purely algebraic. We now transform this formulation into a two-level non-overlapping additive Schwarz method. We define the new interface space and new local spaces as follows:

**Definition 1** (New space) Let

$$V_0 := V_h(\Gamma) \oplus \sum_{i=1}^N R_S^{(i)T} \alpha_s^{(i)}, \text{ and } V_i := \text{Range}(Q_i^{(i)}),$$

then the direct sum decomposition holds:

$$V_h(\Omega) = R_0^T V_0 \oplus R_1^T V_1 \oplus \dots \oplus R_N^T V_N.$$

Here,  $R_i^T : V_i \rightarrow V_h(\Omega)$  for  $1 \leq i \leq N$  represents zero extension to the nodal points in  $\Omega_h \setminus I_i$ . The extension operator  $R_0^T : V_0 \rightarrow V_h(\Omega)$  is defined as:

$$R_0^T u_0 = \begin{bmatrix} I_\Gamma & 0 \\ \sum_{i=1}^N -R_{I_i I}^T B_L^{(i)} B_{II}^{(i)} R_{I_i \Gamma} & \sum_{i=1}^N R_{I_i I}^T Q_S^{(i)} R_S^{(i)} \end{bmatrix} u_0,$$

where  $I_\Gamma$  is the identity matrix with respect to  $\Gamma$ . We also denote the transpose of  $R_0^T$  as  $R_0 : V_h(\Omega) \rightarrow V_0$ . Both  $R_0^T$  and  $R_0$  are real matrices. We also define the local component of the coarse space as  $V_0^{(i)} = V_h(I_i) \oplus \alpha_s^{(i)}$ .

We follow the procedure of two-level additive Schwarz methods to construct the local and coarse solvers. First, we consider the local problem. We define the local sesquilinear form  $b_i(\cdot, \cdot)$  on the local space  $V_i$  as:

$$b_i(u_i, v_i) = b(R_i^T u_i, R_i^T v_i) \quad \forall u_i, v_i \in V_i, \quad 1 \leq i \leq N.$$

Next, we define the projection-like operator  $P_i : V_h \rightarrow V_h$  given by  $P_i = R_i^T \tilde{P}_i$ , where  $\tilde{P}_i : V_h \rightarrow V_i$  is defined as the local solver for the following local problem:

$$b_i(\tilde{P}_i u_h, v_i) = b(u_h, R_i^T v_i) \quad \forall v_i \in V_i.$$

We note that the matrix form of the above local problem is equivalent to (7). Let us denote  $B_i$  as its corresponding matrix form, then  $B_i = Q_i^{(i)T} B_{II}^{(i)} Q_i^{(i)}$ , thus  $P_i$  is well-posed.

Next, we define the global sesquilinear form  $b_0(\cdot, \cdot)$  on the interface space  $V_0$  as:

$$b_0(u_0, v_0) = b(R_0^T u_0, R_0^T v_0) \quad \forall u_0, v_0 \in V_0,$$

and we denote the corresponding matrix form as  $B_0$ , which is the left-hand side matrix in (8). Next, we consider the projection-like operator  $P_0 : V_0 \rightarrow V_h$  given by  $P_0 = R_0^T \tilde{P}_0$ , where  $\tilde{P}_0 : V_h(\Omega) \rightarrow V_0$  is defined for the following global interface problem:

$$b_0(\tilde{P}_0 u_h, v_0) = b(u_h, R_0^T v_0) \quad \forall v_0 \in V_0.$$

We note that the matrix form of the above coarse problem is exactly (8), and we denote  $B_0$  as its corresponding matrix form. The well-posedness of  $P_0$  depends on the invertibility of the global sesquilinear form  $b_0(\cdot, \cdot)$ . By choosing an appropriate norm, we can show that the inf-sup condition of  $b_0(\cdot, \cdot)$  is greater than or equal to  $\gamma$ , which is the discrete inf-sup constant in Assumption 1.

Using the construction outlined above, we obtain an exact solver for the linear system

$$B^{-1} = R_0^T (B_0)^{-1} R_0 + \sum_{i=1}^N R_i^T Q_i^{(i)} (B_i)^{-1} Q_i^{(i)T} R_i. \quad (9)$$

### 3 A New Domain Decomposition Method

From (9), we know the local problem is well-posed and can be parallelized. This section proposes a construction for the coarse problem and ensures its parallelizability. We follow the framework of Nonoverlapping Additive Schwarz (NOSAS) methods based on generalized eigenvalue problems (see [4]). However, since  $B_0$  is a complex matrix, we must handle the real and imaginary parts separately.

For the real part, we consider the generalized eigenvalue problems in each subdomain using only the real part of  $B_0$ :

$$\Re B_0^{(i)} \xi_{Re_j}^{(i)} = \lambda_{Re_j}^{(i)} C_0^{(i)} \xi_{Re_j}^{(i)} \quad (j = 1, \dots, N_i), \quad (10)$$

where  $N_i$  is the number of DOFs on the interface  $\Gamma_i$ , and the eigenvectors  $\xi_{Re_j}^{(i)}$  are orthonormal with respect to  $C_0^{(i)}$ . Here  $C_0^{(i)}$  is a block diagonal matrix constructed as follows:

$$C_0^{(i)} = \begin{bmatrix} \hat{C}_{\Gamma\Gamma}^{(i)} & 0 \\ 0 & Q_s^{(i)T} H_{II}^{(i)} Q_s^{(i)} \end{bmatrix}, \quad (11)$$

with  $\hat{C}_{\Gamma\Gamma}^{(i)}$  being the block diagonal version of  $C_{\Gamma\Gamma}^{(i)} = H_{\Gamma\Gamma}^{(i)} - H_{\Gamma L}^{(i)} H_L^{(i)} H_{L\Gamma}^{(i)}$  by breaking the connection between subdomain vertices and edges, where  $H_L^{(i)} = Q_i^{(i)} Q_i^{(i)T}$ .

We then set a threshold  $\eta \in (0, 1)$  and select eigenvalues that are either smaller than  $\eta$  or larger than 2. The reason for selecting eigenvalues larger than 2 is that large eigenvalues can deteriorate the iterations of the preconditioned system. We denote the corresponding eigenvector space as  $Q_{Re}^{(i)} = [\xi_{Re_1}^{(i)}, \xi_{Re_2}^{(i)}, \dots, \xi_{Re_{\hat{K}_i}}^{(i)}]$ , and let  $D_{Re}^{(i)} = \text{diagonal}(1 - \lambda_{Re_1}^{(i)}, 1 - \lambda_{Re_2}^{(i)}, \dots, 1 - \lambda_{Re_{\hat{K}_i}}^{(i)})$ . Then, we define the following operators:  $\Pi_{Re}^{(i)} = Q_{Re}^{(i)} Q_{Re}^{(i)T} C_0^{(i)}$ , and  $\Pi_{DRe}^{(i)} = Q_{Re}^{(i)} D_{Re}^{(i)} Q_{Re}^{(i)T} C_0^{(i)}$ .

To develop parallelization property, we should also consider the imaginary part of  $B_0^{(i)}$ , which is exactly  $\Im B_{\Gamma\Gamma}^{(i)}$ . Let  $\hat{B}_{\Gamma\Gamma}^{(i)}$  simply be the diagonal or block diagonal form of  $\Im B_{\Gamma\Gamma}^{(i)}$ . Then, we can define a sesquilinear form  $b_P(\cdot, \cdot) : V_0 \times V_0 \rightarrow \mathbb{C}$  as follows:

$$\begin{aligned}
b_P(u_0, v_0) &= v_0^H \sum_{i=1}^N R_{\Gamma,s}^{(i)T} (\Pi_{Re}^{(i)T} \mathfrak{R} B_0^{(i)} \Pi_{Re}^{(i)} + (I - \Pi_{Re}^{(i)T}) C_0^{(i)} (I - \Pi_{Re}^{(i)}) + \mathbf{i} R_{\Gamma,0}^{(i)T} \hat{B}_{\Gamma\Gamma}^{(i)} R_{\Gamma,0}^{(i)}) R_{\Gamma,s} u_0 \\
&= v_0^H \sum_{i=1}^N R_{\Gamma,s}^{(i)T} (C_0^{(i)} - C_0^{(i)} \Pi_{DRe}^{(i)} + \mathbf{i} R_{\Gamma,0}^{(i)T} \hat{B}_{\Gamma\Gamma}^{(i)} R_{\Gamma,0}^{(i)}) R_{\Gamma,s} u_0,
\end{aligned}$$

where  $R_{\Gamma,s}^{(i)T} : V_0^{(i)} \rightarrow V_0$  and  $R_{\Gamma,0}^{(i)T} : V_h(\Gamma_i) \rightarrow V_0^{(i)}$  are zero extension operators, with  $R_{\Gamma,s}^{(i)}$  and  $R_{\Gamma,0}^{(i)}$  being their transposes. The resulting preconditioner can be written as:

$$P^{-1} = R_0^T (B_P)^{-1} R_0 + \sum_{i=1}^N R_i^T Q_l^{(i)} (B_i)^{-1} Q_l^{(i)T} R_i. \quad (12)$$

Finally, we show the scalability of  $B_P^{-1}$ . Let  $C = \sum_{i=1}^N R_{\Gamma,s}^{(i)T} (C_0^{(i)} + \mathbf{i} R_{\Gamma,0}^{(i)T} \hat{B}_{\Gamma\Gamma}^{(i)} R_{\Gamma,0}^{(i)}) R_{\Gamma,s}^{(i)}$ ,

$U = \sum_{i=1}^N R_{\Gamma,s}^{(i)T} C_0^{(i)} Q_{Re}^{(i)} R_{\lambda_i}$ , where  $R_{\lambda_i}$  is choose the eigenfunctions in  $i$ -th subdomain

from all chosen eigenspace, and  $D = \sum_{i=1}^N R_{\lambda_i}^T D^{(i)} R_{\lambda_i}$ . Then the matrix of  $b_P(\cdot, \cdot)$  can

be written as:  $B_P = C - U D U^T$ .

To solve the coarse problem efficiently and in parallel, we use the Woodbury matrix identity:

$$B_P^{-1} = (C - U D U^T)^{-1} = C^{-1} + C^{-1} U (D^{-1} - U^T C^{-1} U)^{-1} U^T C^{-1},$$

where  $C$  is a block diagonal matrix, making it completely parallelizable. The size of  $(D^{-1} - U^T C^{-1} U)^{-1}$  corresponds to the number of selected eigenfunctions, and it can be precomputed in advance.

## 4 Numerical Experiments

We present numerical results for the problem (1) with  $f = 0$  and various wavenumbers  $k$ . The computational domain is a square  $\Omega = (0, 1)^2$  with an impedance boundary condition modeled as a plane wave  $e^{ik \cdot \mathbf{X}}$  on  $\partial\Omega$ . The direction vector is  $\mathbf{V} = \langle \cos(\pi/8), \sin(\pi/8) \rangle$  and  $\mathbf{X}$  represents the coordinate of the wave. The discretization and domain partition is defined in Section 1. We used the Generalized Minimal Residual (GMRES) method to solve the preconditioned system, recording the number of iterations required to reduce the relative residual error to  $10^{-6}$  in the  $l^2$  norm. For these experiments,  $\hat{C}_{\Gamma\Gamma}^{(i)}$  was chosen as the block diagonal of  $C_{\Gamma\Gamma}^{(i)}$ , and  $\hat{B}_{\Gamma\Gamma}^{(i)}$  was taken as the diagonal part of  $\mathfrak{S} B_{\Gamma\Gamma}^{(i)}$ . The results are summarized in Table 1 and Table 2.

The parameter  $\beta = 0.01$  was chosen to guarantee the solvability of local problems. Our numerical experiments demonstrate that as the wavenumber  $k$  increases, the mesh size  $h$  must decrease to maintain proper resolution for the Helmholtz equa-

$k = 30$	$H = \frac{1}{4}$	$H = \frac{1}{8}$	$H = \frac{1}{16}$	$H = \frac{1}{32}$	$k = 40$	$H = \frac{1}{8}$	$H = \frac{1}{16}$	$H = \frac{1}{32}$	$H = \frac{1}{64}$
$h = 1/64$	11 (13)	15 (7)	9 (4)	9 (3)	$h = 1/128$	11 (9)	12 (5)	12 (4)	8 (3)
$h = 1/128$	10 (15)	15 (8)	12 (5)	11 (4)	$h = 1/256$	10 (11)	11 (7)	14 (5)	13 (4)
$h = 1/256$	10 (16)	15 (8)	15 (5)	14 (4)	$h = 1/512$	10 (12)	12 (8)	18 (5)	16 (4)

Table 1: GMRES iterations with preconditioner  $B_p^{-1}$  ( $\beta = 0.01$ ,  $\eta = 0.6$ ) using P1 finite elements. The numbers in parentheses is the selected eigenfunctions per subdomain in (10).

$k = 50$	$H = \frac{1}{8}$	$H = \frac{1}{16}$	$H = \frac{1}{32}$	$H = \frac{1}{64}$	$k = 60$	$H = \frac{1}{8}$	$H = \frac{1}{16}$	$H = \frac{1}{32}$	$H = \frac{1}{64}$
$h = 1/64$	10 (14)	10 (7)	11 (5)	10 (4)	$h = 1/128$	8 (17)	10 (9)	15 (5)	14 (4)
$h = 1/128$	10 (15)	11 (8)	15 (5)	14 (4)	$h = 1/256$	8 (20)	8 (12)	14 (8)	17 (5)
$h = 1/256$	10 (15)	12 (9)	14 (7)	16 (5)	$h = 1/512$	9 (20)	9 (12)	14 (8)	18 (7)

Table 2: GMRES iterations with preconditioner  $B_p^{-1}$  ( $\beta = 0.01$ ,  $\eta = 0.7$ ) using P2 finite element. Parentheses is the selected eigenfunctions per subdomain in (10).

tion, which in turn increases iteration counts. This growth in iterations can be effectively mitigated by increasing the threshold parameter  $\eta$ . For fixed values of  $k$  and  $h$ , reducing the subdomain size  $H$  leads to increased iteration counts and larger coarse problem dimensions, as our approximation strategy for near-zero eigenvalues becomes less effective with decreasing  $H$ . Thus, optimal selection of both  $H$  and  $\eta$  is crucial for Helmholtz preconditioning. The optimal choice of  $H$  depends on computational constraints: the processing capacity per compute node and the maximum locally solvable eigenvalue problem size. The threshold  $\eta$  is determined empirically, with our implementation selecting eigenvalues that are either smaller than  $\eta$  or larger than 2, which differs from elliptic preconditioners that target only very small eigenvalues. When including all subdomain eigenfunctions, the preconditioner becomes an exact solver.

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