A Computational Study of Algebraic Coarse Spaces for Two-Level Overlapping Additive Schwarz Preconditioners

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1 Introduction

Consider the scalar diffusion problem

$$\begin{aligned} -\nabla \cdot (\alpha(x)\nabla u(x)) &= f(x) \qquad && \text{in}\, \Omega \subset \mathbb{R}^d, \\ u &= u_D(x) \qquad && \text{on}\, \partial\Omega, \end{aligned} \tag{1}$$

for $d = 2, 3$ with the scalar coefficient function $\alpha : \mathbb{R}^d \to \mathbb{R}$ such that $0 < \alpha < \alpha_{max} < \infty$ for $\alpha_{max} \in \mathbb{R}^+$. Furthermore, α is highly heterogeneous and possibly containing large coefficient jumps.

Let the system of linear equations arising from the discretization of the boundary value problem in eq. (1) on a computational domain Ω be

$$Au = b. \tag{2}$$

Furthermore, let Ω be decomposed into N non-overlapping subdomains $\Omega_1, \ldots, \Omega_N$, which we can extend by layers of elements to obtain overlapping subdomains $\Omega_1', \ldots, \Omega_N'$. The corresponding two-level overlapping additive Schwarz preconditioner reads

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$$M_{OAS,2}^{-1} = \Phi A_0^{-1} \Phi^\top + \sum_{i=1}^{N} R_i^\top A_i^{-1} R_i, \tag{3}$$

where R_i are the restriction matrices to the overlapping subdomains Ω_i', the $A_i = R_i A R_i^\top$ are the corresponding local subdomain matrices, and $A_0 = \Phi^\top A \Phi$ is the coarse problem matrix, which is determined by the coarse basis functions, which are the columns of Φ. In general, the coarse problem is required for numerical scalability of domain decomposition methods; see [13] for more details. Additionally, the local overlapping subdomain matrices can be retrieved algebraically; see e.g. [5].

The simplest choice for Φ are finite element shape functions on a coarse triangulation [13]. However, such a coarse grid cannot be constructed algebraically based on A. Instead, we construct the coarse basis functions based on the nonoverlapping domain decomposition. As a first option, we employ the generalized Dryja–Smith–Widlund (GDSW) coarse space [3]. It is composed of energy-minimizing functions of trace functions defined on the interfaces of the nonoverlapping subdomains. The trace values are the restriction of the null space of the global Neumann problem to a decomposition of the interface into vertices, edges, and, in three dimensions, faces. Furthermore, we consider the reduced dimension GDSW (RGDSW) coarse space introduced in [4]. In RGDSW, only vertex functions are considered, resulting in a reduced number of interface components; the interface values are modified by the multiplication of the null space by a partition of unit function.

Related coarse spaces based on the multiscale finite element method (MsFEM) [8] have been studied in the context of overlapping Schwarz preconditioners [1]; MsFEM functions are also constructed as energy-minimizing extensions into the interior. They provide good results when applied to media with high coefficient jumps but are generally not algebraic; cf. [6] for a more robust variant. An algebraic formulation of MsFEM and multiscale finite volume method (MsFVM) was presented through a two-stage algebraic multiscale solver (TAMS) [15], later extended to an algebraic multiscale solver (AMS) for highly heterogeneous systems [14]. AMS was developed as an approximate solver to control the error of multiscale solutions to any desired level but has also been employed as a standalone linear solver [14].

In the following, we establish the relation between AMS and GDSW-type coarse basis functions to define a common framework. We compare the approaches numerically for some simple heterogeneous model problems on two dimensions.

2 Energy-minimizing coarse basis functions

Let $\Gamma := \bigcup_{i=1}^{N} \partial\Omega_i \setminus \partial\Omega$ be the interface of the nonoverlapping domain decomposition. We may order the degrees of freedom, such that the system matrix A and the prolongation operator Φ can be written as

$$A = \begin{pmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{pmatrix}, \ \Phi = \begin{pmatrix} \Phi_I \\ \Phi_\Gamma \end{pmatrix}, \tag{4}$$

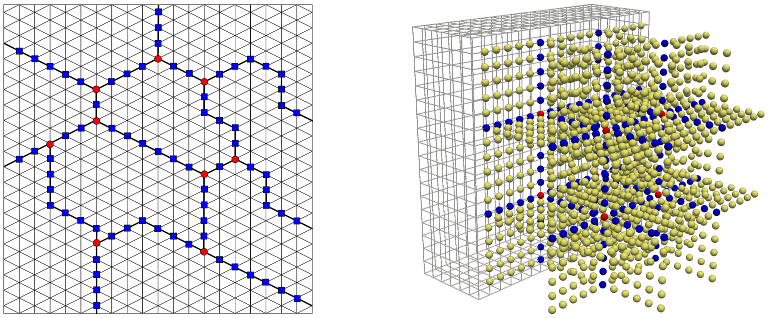

Fig. 1: Example of an interface partition of grid elements in two (left) and three (right) dimensions. The vertex, edge and face subcomponents are marked in red, blue and yellow, respectively.

where Γ indicates the degrees of freedom corresponding to interface finite element nodes and I the remaining degrees of freedom. As usual, if the boundary degrees of freedom have not been eliminated, we consider them as interior. The energy-minimizing extensions of interface values Φ_Γ are then defined as:

$$\Phi = H_\Gamma \Phi_\Gamma = \begin{pmatrix} -A_{II}^{-1} A_{I\Gamma} \\ I_{\Gamma\Gamma} \end{pmatrix} \Phi_\Gamma, \tag{5}$$

where $I_{\Gamma\Gamma}$ is the identity matrix of dimension $|\Gamma| \times |\Gamma|$. Equation (5) allows for the description of multiple coarse spaces based on different definitions of Φ_Γ.

2.1 GDSW and RGDSW coarse basis functions

For the GDSW coarse space [3], the interface is decomposed into vertices V, edges E, and, in three dimensions, faces F. An example of decomposition is illustrated in fig. 1. For each interface component, interface values of the corresponding basis function are given by the restriction of the null space of the problem to it. In particular, by the null space of the problem, we mean the null space of the system matrix A with Neumann boundary conditions on the whole boundary $\partial\Omega$.

For the RGDSW coarse space [4], the interface is partitioned based on nodal equivalence classes, thus reducing the number of interface components and, consequently, the dimension of the coarse space; in practice, the interface components overlap and mostly correspond to one of the vertices. Moreover, the interface values Φ_Γ are modified by multiplication with partition of unit functions. The option 1 described in [4] can be computed completely algebraically; the interface value in an interface finite element is defined by the inverse of the number of interface components the node belongs to. We will consider this algebraic variant in this paper.

We assume that the nonoverlapping domain decomposition is available for the construction of the interface components. Otherwise, we could still try to approximate reconstruct the interface algebraically; see [5].

2.2 AMS coarse basis functions

AMS [14] is a multiscale framework that has been developed for simulations of the Darcy flow in porous media. The AMS coarse basis functions corresponds to an algebraic variant of the MsFEM basis functions [14, 15]. When applied to a system assembled using a structured grid and a 5-point or 7-point stencil in two or three dimensions, respectively, the resulting basis is the same as for the MsFEM. Otherwise, the AMS gives an approximation of the latter. Some modifications are required in these cases to restrict the support of the coarse basis functions; see [11]; these changes do not affect the analysis of the two-level Schwarz method.

For the computation of the coarse basis, the grid nodes are again grouped into vertices (V), edges (E), faces (F) and interior (I) nodes. This is equivalent to the partition presented in fig. 1. Therefore, let us structure the system eq. (2) based on those groups of degrees of freedom:

$$\begin{pmatrix} A_{II} & A_{IF} & A_{IE} & A_{IV} \\ A_{FI} & A_{FF} & A_{FE} & A_{FV} \\ A_{EI} & A_{EF} & A_{EE} & A_{EV} \\ A_{VI} & A_{VF} & A_{VE} & A_{VV} \end{pmatrix} \begin{pmatrix} u_I \\ u_F \\ u_E \\ u_V \end{pmatrix} = \begin{pmatrix} b_I \\ b_F \\ b_E \\ b_V \end{pmatrix},$$

where, due to the symmetry of A, we have $A_{ij} = A_{ji}^\top$, $i, j \in \{I, F, E, V\}$.

We require u to be set as the solution of a reduced boundary condition problem on the boundary of the nonoverlapping subdomains. To enforce this assumption in the discrete system, we start by removing the blocks related to connections between internal and interface nodes when solving for edge and face nodes. This results in the system:

$$\begin{pmatrix} A_{II} & A_{IF} & A_{IE} & A_{IV} \\ 0 & \tilde{A}_{FF} & A_{FE} & A_{FV} \\ 0 & 0 & \tilde{A}_{EE} & A_{EV} \\ 0 & 0 & 0 & A_0 \end{pmatrix} \begin{pmatrix} u'_I \\ u'_F \\ u'_E \\ u'_V \end{pmatrix} = \begin{pmatrix} b_I \\ b_F \\ b_E \\ R_0 \end{pmatrix}. \tag{6}$$

It is important to point out that the solution to the linear system above gives only an approximation u' to the exact solution of eq. (2). Moreover, $A_0 u'_V = R_0$ corresponds to the coarse-scale problem, which is equivalent to the coarse problem solved in eq. (3). Specifically

$$A_0 = \Phi A \Phi^\top, \ R_0 = \Phi^\top b,$$

where the operator Φ will be defined later.

The blocks $\tilde{A}_{FF}$ and $\tilde{A}_{EE}$ are modified accordingly to take into account the elimination of the lower triangular part of the system:

$$\begin{aligned}\tilde{A}_{FF} &= A_{FF} + \operatorname{diag}\left(A_{FI}\mathbf{1}_I\right), \\ \tilde{A}_{EE} &= A_{EE} + \operatorname{diag}\left(A_{EI}\mathbf{1}_I\right) + \operatorname{diag}\left(A_{EF}\mathbf{1}_F\right),\end{aligned} \tag{7}$$

in which $\mathbf{1}_I$ and $\mathbf{1}_F$ are vectors of dimension $|I|$ and $|F|$, respectively, that contain only ones. The operation $A_{ij}\mathbf{1}_j$ corresponds to the sum of the rows of the block A_{ij}. In the step above, the diagonal blocks are modified to incorporate the removed off-diagonal blocks in such a way that the energy-minimality of constant functions is maintained. This ensures that corresponding reduced problems generate a partition of unity on the interface, which is an important result for the convergence analysis of the two-level Schwarz preconditioner [13].

The system in eq. (6) can be solved by backward substitution yielding:

$$\left(u'_I \; u'_F \; u'_E \; u'_V\right)^\top = \Phi u'_V + \mathbf{C},$$

where Φ is the AMS prolongation operator written as:

$$\Phi = \begin{pmatrix}\Phi_I \\ \Phi_F \\ \Phi_E \\ \Phi_V\end{pmatrix} = \begin{pmatrix}-A_{II}^{-1}\left(A_{IV}\Phi_V + A_{IE}\Phi_E + A_{IF}\Phi_F\right) \\ -\tilde{A}_{FF}^{-1}\left(A_{FV}\Phi_V + A_{FE}\Phi_E\right) \\ -\tilde{A}_{EE}^{-1}A_{EV}\Phi_V \\ I_{VV}\end{pmatrix}, \tag{8}$$

where I_{VV} is an identity matrix of dimension $|V| \times |V|$.

From eq. (8), it can be seen that the values of the AMS coarse basis functions are computed as energy-minimizing extensions in eq. (5). In particular, Φ_I in eq. (8) can be rewritten as:

$$\Phi_I = -A_{II}^{-1}A_{I\Gamma}\Phi_\Gamma = -A_{II}^{-1}\begin{pmatrix}A_{IF} & 0 & 0 \\ 0 & A_{IE} & 0 \\ 0 & 0 & A_{IV}\end{pmatrix}\begin{pmatrix}\Phi_F \\ \Phi_E \\ \Phi_V\end{pmatrix},$$

which is equivalent to eq. (5) when using that $\Gamma = F \cup E \cup V$.

The AMS coarse space thus only differs from the GDSW and RGDSW coarse spaces in the definition of the interface values of the basis functions. Similar to RGDSW, the AMS basis functions are vertex-centered. However, the values on the interface are computed recursively by solving reduced elliptic problems on the edges and faces; similar recursive computations are also performed in the non-algebraic MsFEM coarse basis functions [1, 6]. This allows the interface values to change according to the contrast in the coefficient function α in eq. (1), whereas for GDSW and RGDSW option 1 those are constant for diffusion problems.

3 Numerical examples

In this section, we provide numerical results to compare the algebraic coarse spaces discussed in section 2. For the sake of simplicity, we focus on two-dimensional

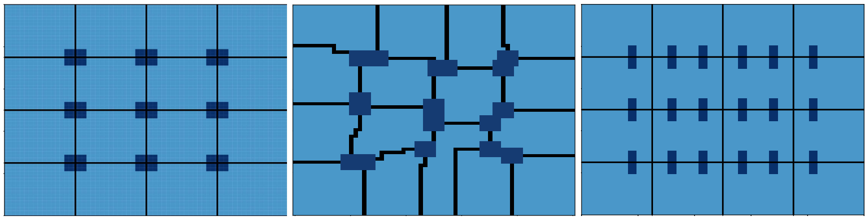

Fig. 2: Discontinuous coefficient functions used in the first (left and center) and second (right) examples. The dark blue regions correspond to the high coefficient inclusions with a value of $\alpha(x) = 10^8$, and $\alpha(x) = 1$ elsewhere. Illustration for a domain decomposition containing 16 subdomains.

cases of the model problem eq. (1) on $\Omega = (0,1)^2$, discretized using Q1 finite elements. Furthermore, we only consider zero Dirichlet boundary conditions and $f(x) = 1$. The resulting system eq. (2) is solved using the conjugate gradient (CG) method without full orthogonalization, preconditioned using the two-level Schwarz preconditioner in eq. (3), and a relative stopping criterion $||r^{(k)}||_2/||r^{(0)}||_2 < 10^{-8}$, where $r^{(0)}$ and $r^{(k)}$ are the initial and k-th unpreconditioned residuals, respectively. We employ both structured and unstructured domain decompositions. For structured domain decompositions, unless stated otherwise, we consider $H/h = 16$, where H and h are the subdomain and the grid size, respectively. The unstructured domain decompositions were computed using METIS [9]. In both cases, we employ two layers of elements for the overlap. In addition, we also present the results using a one-level overlapping additive Schwarz (OAS-1) preconditioner for reference.

The coefficient functions used in the examples are illustrated in fig. 2: high-coefficient inclusions at the coarse nodes in fig. 2 (left and center), and parallel high-coefficient channels crossing the edges in fig. 2 (right). In both cases, $\alpha(x) = 10^8$ in the dark blue regions and $\alpha(x) = 1$ elsewhere. For the unstructured domain decompositions, the coarse nodes are defined as described in [4].

The simulation results for the coefficient function in fig. 2 left and center are presented in figs. 3 and 4 for structured and unstructured domain decompositions, respectively. In these scenarios, the GDSW and the RGDSW coarse spaces are not robust with respect to the coefficient: the condition is in the order of the contrast and the iteration counts increase with the number of subdomains. On the other hand, the AMS coarse space is robust and scalable. This is in line with previous observations for coarse spaces based on MsFEM [6]. This family of coarse spaces remain robust as long as the high coefficient inclusions are connected to the coarse nodes.

In fig. 5, results for the coefficient function in fig. 2 (right) are shown, indicating contrast-dependent condition numbers for all approaches. As is well-known, coefficient jumps along the edge cannot be treated by the vertex-centered RGDSW and AMS coarse basis functions alone, while GDSW edge basis functions cannot deal with multiple jumps along the edges. However, GDSW and AMS perform better

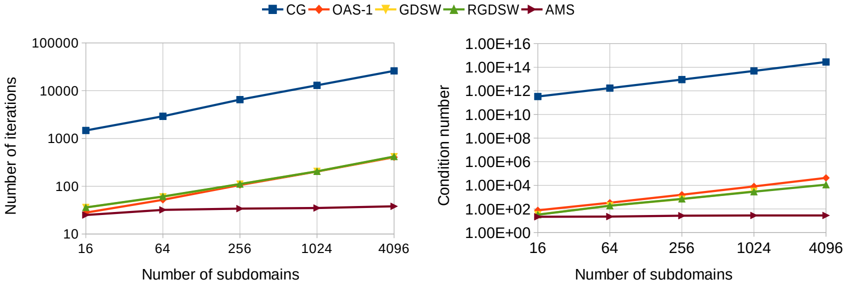

Fig. 3: Number of iterations (left) and condition number estimate (right) versus the number of subdomains for the coefficient function in fig. 2 (left) on a structured domain decomposition.

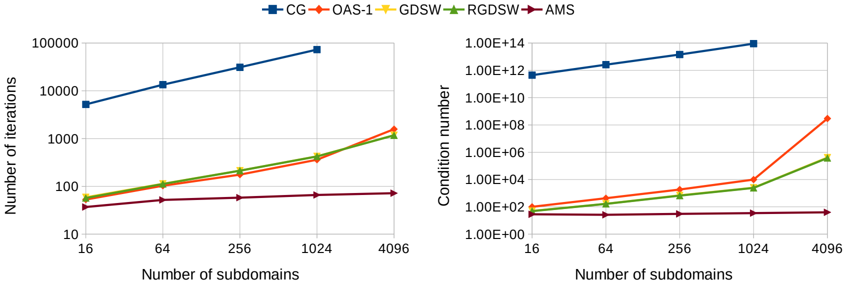

Fig. 4: Number of iterations (left) and condition number estimate (right) versus the number of subdomains for the coefficient function in fig. 2 (center) on an unstructured domain decomposition. The number of iterations for CG on 4096 subdomains is greater than 10^5 and the corresponding results are omitted.

than RGDSW in terms of the number of iterations, and even though iteration counts are high, they seem to stabilize with increasing numbers of subdomains.

In order to analyze the difference in the convergence behavior of RGDSW versus GDSW and AMS coarse spaces, we investigate the spectrum of the preconditioned system matrix for the three coarse spaces in fig. 6. We observe that the low eigenvalues are clustered around 10^{-7}, which remains the same when increasing the number of subdomains. Thus, the condition number also remains roughly constant. We observe that, while the clustering of the spectrum remains the same for the AMS and GDSW coarse spaces, the spectrum for the RGDSW coarse space is stretched significantly when increasing the number of subdomains. This could explain the different behavior in the convergence for increasing numbers of subdomains.

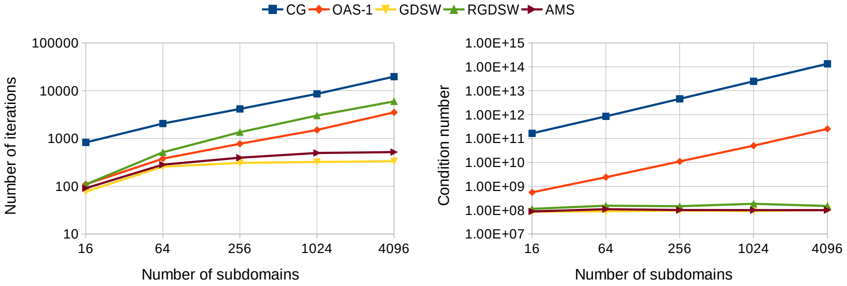

Fig. 5: Number of iterations (left) and condition number estimate (right) versus the number of subdomains for the coefficient function in fig. 2 (right).

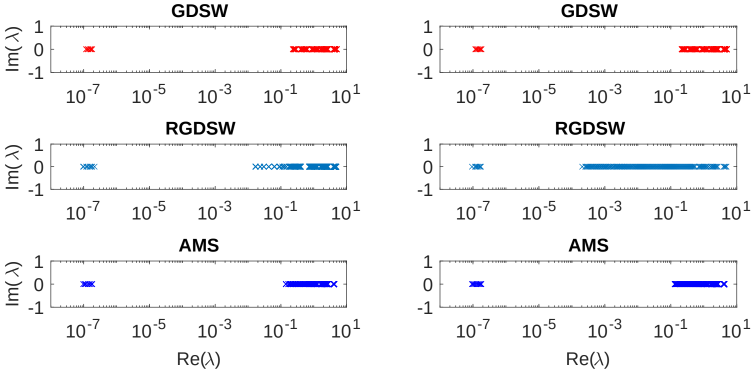

Fig. 6: Eigenvalues for each coarse space for the coefficient function in fig. 2 (right) and different subdomain sizes: **left**: $1/H = 8$; **right**: $1/H = 64$.

4 Conclusions

In this work, we have applied the algebraic formulation of the multiscale finite element method (MsFEM) provided by the algebraic multiscale solver (AMS) [14] as the coarse space for the two-level overlapping Schwarz preconditioner for scalar elliptic problems. In comparison with the algebraic GDSW [2, 3] and RGDSW [4] coarse spaces for some simple heterogeneous model problems on two dimensions, the AMS coarse space showed improved robustness when the high coefficient inclusions are on the coarse nodes. If the coefficient contrast crosses the subdomain edges, clusters of small eigenvalues in the spectrum of the preconditioned system hinder convergence. Nevertheless, we observe that the number of iterations for the AMS and GDSW coarse spaces tends to stabilize as more subdomains are added. In those cases, convergence could be further improved, for instance, by deflating the small eigenvalues [10] or by employing adaptive coarse spaces [7, 12].

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