

Adaptive and Frugal BDDC Coarse Spaces for Virtual Element Discretizations of a Stokes Problem with Heterogeneous Viscosity

Tommaso Bevilacqua^[0000-0002-0828-5018], Axel Klawonn^[0000-0003-4765-7387], and Martin Lanser^[0000-0002-4232-9395]

1 Introduction

The virtual element method (VEM) [2, 21] is a finite element discretization approach for partial differential equations (PDEs) which can deal with very general polygonal/polyhedral computational grids. The linear systems that arise from these discretizations of PDEs are then generally worse conditioned than in the case of standard finite element methods (FEMs) for which recent studies proposed robust BDDC methods [3, 4, 6, 11]. In this work we analyze a Stokes problem with high heterogeneity in the viscosity function; thus an adequately enriched coarse space is needed. In the adaptive framework considered here, this is done by solving a generalized eigenvalue problem on each subdomain edge and by adding the solutions to the coarse space in an appropriate way. In our numerical simulations we make use of two coarse spaces analyzed for standard low order FEM in the paper [13], identifying with *first* the approach present in Section 4.5 and with *second* the one in Section 5. The first approach, was originally introduced in [18] and already successfully used for the VEM in [5, 7]. The variant used here, that allows to use any kind of scaling, has been presented in [12] for FEM discretizations. The second one, extensively used in dual-primal finite element tearing and interconnecting (FETI-DP) and BDDC in [14, 16, 17, 20], has been also recently extended to the VEM for diffusion and linear elasticity problems in [10].

An alternative approach to enrich the coarse space denoted as *frugal*, has been introduced in [8] and already successfully used for the VEM for stationary diffusion and linear elasticity in [10]. This is a heuristic and cheaper technique, that does not involve the solution of eigenvalue problems. This often allows to construct robust

Tommaso Bevilacqua¹, Axel Klawonn^{1,2}, Martin Lanser^{1,2}

¹Department of Mathematics and Computer Science, Division of Mathematics, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany, e-mail: tommaso.bevilacqua@uni-koeln.de, axel.klawonn@uni-koeln.de, martin.lanser@uni-koeln.de, url: <https://www.numerik.uni-koeln.de>

²Center for Data and Simulation Science, University of Cologne, Germany, url: <https://www.cds.uni-koeln.de>

coarse spaces in a computationally efficient way when it is sufficient to approximate the largest, or smallest, eigenvalues depending on the chosen coarse space.

In the present work we extend the first and second adaptive coarse space approaches as well as the frugal coarse space to the virtual element discretization of a Stokes problem with a heterogeneous viscosity function.

2 Continuous problem and virtual element discretization

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain, with $\Gamma = \partial\Omega$, and consider the stationary Stokes problem with homogeneous Dirichlet boundary conditions: find (\mathbf{u}, p) s.t.

$$\begin{cases} -\nu(\mathbf{x}) \Delta \mathbf{u} - \nabla p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \Gamma. \end{cases} \quad (1)$$

Here \mathbf{u} and p are respectively the velocity and the pressure fields, $\mathbf{f} \in [H^{-1}(\Omega)]^2$ represents the external force and $\nu \in L^\infty(\Omega)$, $\nu(\mathbf{x}) > 0$, $\mathbf{x} \in \Omega$, is the heterogeneous viscosity function.

Introducing $\mathbf{V} := [H_0^1(\Omega)]^2$ and $Q := L_0^2(\Omega) = \{q \in L^2(\Omega) \text{ s.t. } \int_\Omega q = 0\}$, the standard variational formulation reads: find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ s.t.

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) & \text{for all } \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) = 0 & \text{for all } q \in Q, \end{cases} \quad (2)$$

where $a(\mathbf{v}, \mathbf{w}) := \int_\Omega \nu(\mathbf{x}) \nabla \mathbf{v} : \nabla \mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in \mathbf{V}$, $b(\mathbf{v}, q) := \int_\Omega \operatorname{div} \mathbf{v} q$ for all $\mathbf{v} \in \mathbf{V}$, $q \in Q$ and $(\mathbf{f}, \mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v}$ for all $\mathbf{v} \in \mathbf{V}$.

The discretization of problem (2) is based on a virtual element space which is designed to solve a Stokes problem. In the following we present the basic elements of this discretization; we refer to [22] for further details.

Let $\{\mathcal{T}_h\}_h$ be a sequence of triangulations of Ω into general polygonal elements K with $h_K := \operatorname{diameter}(K)$ and $h := \sup_{K \in \mathcal{T}_h} h_K$. We suppose that, for all h , each element $K \in \mathcal{T}_h$ satisfies, for some $\gamma > 0$ and $c > 0$, the following assumptions

- K is star-shaped with respect to a ball of radius greater or equal than γh_K ,
- the distance between any two vertices of K is greater or equal than ch_K ,

For $k \in \mathbb{N}$, we then define the spaces: $\mathbb{P}_k(K)$ the set of polynomials on K of degree smaller or equal than k , $\mathbb{B}_k(K) := \{v \in C^0(\partial K) \text{ s.t. } v|_e \in \mathbb{P}_k(e) \forall \text{ edge } e \in \partial K\}$, $G_k(K) := \nabla(\mathbb{P}_{k+1}(K)) \subseteq [\mathbb{P}_k(K)]^2$ and its L^2 -orthogonal complement $G_k(K)^\perp \subseteq [\mathbb{P}_k(K)]^2$. The local virtual element spaces are defined for $k \geq 2$, on each $K \in \mathcal{T}_h$ as

$$\begin{aligned} \mathbf{V}_h^K &:= \left\{ \mathbf{v} \in [H^1(K)]^2 \mid \mathbf{v}|_{\partial K} \in [\mathbb{B}_k(\partial K)]^2, \begin{cases} -\nu \Delta \mathbf{v} - \nabla s \in G_{k-2}(K)^\perp, \\ \operatorname{div} \mathbf{v} \in \mathbb{P}_{k-1}(K), \end{cases} s \in L^2(K) \right\}, \\ Q_h^K &:= \mathbb{P}_{k-1}(K), \end{aligned}$$

and the global virtual element spaces are $\mathbf{V}_h := \{\mathbf{v} \in [H_0^1(\Omega)]^2 \mid \mathbf{v}|_K \in \mathbf{V}_h^K, \forall K \in \mathcal{T}_h\}$ and $Q_h := \{q \in L_0^2(\Omega) \mid q|_K \in Q_h^K, \forall K \in \mathcal{T}_h\}$.

In the VEM framework the basis function are never explicitly computed since it would be necessary to solve the PDE given in \mathbf{V}_h^K for each element. Alternatively, they are defined using polynomial projection operators (see [1]) and suitable degrees of freedom (dofs). This allows to exactly compute $b(\cdot, \cdot)$ and to introduce the approximations $a_h(\cdot, \cdot)$ and (\mathbf{f}_h, \cdot) (see [22] for more details).

The discrete virtual element problem states: find (\mathbf{u}_h, p_h) s.t.

$$\begin{cases} a_h(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) & \text{for all } \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h, q_h) = 0 & \text{for all } q_h \in Q_h \end{cases}. \quad (3)$$

3 Domain decomposition, BDDC preconditioner, and coarse spaces

We decompose \mathcal{T}_h into N non-overlapping subdomains Ω_i with characteristic size H_i as $\tilde{\mathcal{T}}_h = \bigcup_{i=1}^N \tilde{\Omega}_i$ where $\tilde{\Omega}_i$ is the union of different polygons of the tessellation \mathcal{T}_h and we define $\Gamma = \bigcup_{i \neq j} \partial\Omega_i \cap \partial\Omega_j$ as interface among the subdomains. We assume a shape-regular decomposition in the sense of [3] Section 3.

We refer the edges of the subdomains Ω_i with \mathcal{E} , moreover \mathcal{E}_{ij} denotes the edge shared by the subdomains Ω^i and Ω^j . Following the notations introduced in [15] and [6], we split the velocity dofs into interface (Γ) and internal (I) dofs and we decompose the discrete velocity and pressure space \mathbf{V} and Q into $\mathbf{V} = \mathbf{V}_I \oplus \widehat{\mathbf{V}}_\Gamma$ and $Q = Q_I \oplus Q_0$, with $Q_0 := \prod_{i=1}^N \{q \in \Omega_i \mid q \text{ is constant in } \Omega_i\}$. $\widehat{\mathbf{V}}_\Gamma$ is the continuous space of the traces on Γ of functions in \mathbf{V} , $\mathbf{V}_\Gamma = \prod_{i=1}^N \mathbf{V}_\Gamma^{(i)}$ the product space, $\mathbf{V}_I = \bigoplus_{i=1}^N \mathbf{V}_I^{(i)}$, and $Q_I = \bigoplus_{i=1}^N Q_I^{(i)}$. For simplicity, we have omitted the subscript h and will do so throughout the text, as we exclusively refer to the finite-dimensional space. The discrete global saddle-point problem (3) can be written as: find $(\mathbf{u}_I, p_I, \mathbf{u}_\Gamma, p_0) \in (\mathbf{V}_I, Q_I, \widehat{\mathbf{V}}_\Gamma, Q_0)$ s.t.

$$\begin{bmatrix} A_{II} & B_{II}^T & \widehat{A}_{\Gamma I} & 0 \\ B_{II} & 0 & \widehat{B}_{I\Gamma} & 0 \\ \widehat{A}_{\Gamma I} & \widehat{B}_{I\Gamma}^T & \widehat{A}_{\Gamma\Gamma} & \widehat{B}_{0\Gamma}^T \\ 0 & 0 & \widehat{B}_{0\Gamma}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Gamma \\ 0 \end{bmatrix}, \quad (4)$$

where the blocks $\widehat{\cdot}$ related to the continuous interface velocity are assembled from the corresponding subdomain submatrices. By static condensation one eliminates the interior variables and obtains the global interface saddle point problem

$$\widehat{S} \widehat{u} = \begin{bmatrix} \widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T \\ \widehat{B}_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_\Gamma \\ 0 \end{bmatrix} = \widehat{\mathbf{g}}, \quad (5)$$

where \widehat{S}_Γ is the Schur complement of the submatrices constituted by the top 3×3 block of the left-hand side matrix in (4) and $\widehat{\mathbf{g}}$ the corresponding right-hand side. We introduce $\widetilde{\mathbf{V}}_\Gamma = \widetilde{\mathbf{V}}_\Pi \oplus \mathbf{V}_\Delta$, a partially assembled interface velocity space where $\widetilde{\mathbf{V}}_\Pi$ is the continuous coarse-level primal velocity space, which dofs are shared by neighboring subdomains and the complementary space $\mathbf{V}_\Delta = \prod_{i=1}^N \mathbf{V}_\Delta^{(i)}$. Also, \widetilde{S} is the Schur complement system that arises using $\widetilde{\mathbf{V}}_\Gamma$ instead of $\widehat{\mathbf{V}}_\Gamma$.

The primal dofs are given by the nodal evaluation of both velocity components at subdomain vertices, along with an extra dof per subdomain edge to enforce the *no-net-flux* condition $\int_{\partial\Omega_i} \mathbf{v}_\Delta^{(i)} \cdot \mathbf{n} = 0 \ \forall \mathbf{v}_\Delta \in \mathbf{V}_\Delta$, where \mathbf{n} is the outward normal of $\partial\Omega_i$. This ensures the BDDC-preconditioned system (5) is symmetric and positive definite on certain subspaces [15], allowing to use the conjugate gradient (CG) method.

In our study we use a generalized transformation of basis approach [9] such that each primal basis function corresponds to an explicit dof. Firstly, on each \mathcal{E}_{ij} , we assume that the velocity vector should fulfill N_{ij} constraints c_{ij}^l for $l = 1, \dots, N_{ij}$, i.e., $c_{ij}^{lT} \mathbf{v}_{\Gamma|\mathcal{E}}^{(i)} = c_{ij}^{lT} \mathbf{v}_{\Gamma|\mathcal{E}}^{(j)}$. Then, we compute the orthonormal transformation with a modified Gram-Schmidt algorithm. Finally, since the transformations are independent of each other, we construct the resulting block diagonal global transformation. The constraints c_{ij}^l are established via the *no-net-flux* condition and the techniques to enrich the coarse space. More details can be found in [9].

For each Ω_i we introduce the scaling matrices D . These can be chosen in different ways and can be either diagonal or not, but they always must provide a partition of unity, i.e., $\widetilde{R}_{D,\Gamma}^T \widetilde{R}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{R}_{D,\Gamma} = I$, where $\widetilde{R}_\Gamma, \widetilde{R}_{D,\Gamma} : \widetilde{\mathbf{V}}_\Gamma \rightarrow \widetilde{\mathbf{V}}_\Gamma$ are respectively a restriction operator, and its scaled version. We use the standard *multiplicity-scaling* and a variant of the *deluxe-scaling* to preserve the normal fluxes [23].

We then define the average operator $E_D = \widetilde{R} \widetilde{R}_D^T$, where \widetilde{R} and \widetilde{R}_D^T are the extension by identity of the two previous one to the space of piecewise constant pressures.

The preconditioner for solving the global saddle-point problem (5) is then $M^{-1} = \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D$. Theoretical estimates show that the condition number is bounded by the norm of the average operator [6, 15].

Adaptive and frugal coarse spaces

The coarse spaces are either enriched by an adaptive technique or a heuristic one. The general idea of all considered approaches is to detect the largest eigenvalues on each subdomain edge and then include the corresponding eigenvectors in the coarse space as primal constraints.

In the adaptive approach this is done solving generalized eigenvalue problems defined on each subdomain edge \mathcal{E}_{ij} and then construct an enriched primal space such that the condition number of the preconditioned system will be bounded from above by a selected $TOL \in [1, \infty)$ times a constant C independent of h, H and N , that is,

$$\kappa_2(M^{-1}S) \leq C \text{ TOL}.$$

Before describing the coarse spaces, for the subdomains $\Omega_l, l = i, j$ we need to respectively partition and define:

$$S_\Gamma^{(l)} = \begin{bmatrix} S_{\mathcal{E}_{ij}\mathcal{E}_{ij}}^{(l)} & S_{\mathcal{E}_{ij}^\varepsilon\mathcal{E}_{ij}}^{(i)T} \\ S_{\mathcal{E}_{ij}^\varepsilon\mathcal{E}_{ij}}^{(l)} & S_{\mathcal{E}_{ij}^\varepsilon\mathcal{E}_{ij}}^{(i)} \end{bmatrix}, \quad \text{and} \quad S_{ij} = \begin{bmatrix} S_\Gamma^{(i)} \\ S_\Gamma^{(j)} \end{bmatrix}, \quad (6)$$

and introduce the matrices $S_{E_{ij},0}^{(l)} := S_{\mathcal{E}_{ij}\mathcal{E}_{ij}}^{(l)}$ and $S_{E_{ij}}^{(l)} := S_{\mathcal{E}_{ij}\mathcal{E}_{ij}}^{(l)} - S_{\mathcal{E}_{ij}^\varepsilon\mathcal{E}_{ij}}^{(l)T} S_{\mathcal{E}_{ij}^\varepsilon\mathcal{E}_{ij}}^{(l-1)} S_{\mathcal{E}_{ij}^\varepsilon\mathcal{E}_{ij}}^{(l)}$.

The *first* approach consists of solving the eigenvalue problem

$$S_{E_{ij}}^{(i)} : S_{E_{ij}}^{(j)} \mathbf{x}_m = \mu_m (D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(j)}) \mathbf{x}_m, \quad (7)$$

with the product $A : B = (A^{-1} + B^{-1})^{-1}$, select the \mathbf{x}_m , $m = 1, \dots, k$ for which $\mu_m < \text{TOL}$ and compute the coefficient c_{ij} for the constraints as

$$c_{ij} := (D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(i)} D_{E_{ij}}^{(j)} + D_{E_{ij}}^{(j)T} S_{E_{ij},0}^{(j)} D_{E_{ij}}^{(j)}) \mathbf{x}_m.$$

For the *second* coarse space, we need to recall the jump matrix $B = [B^{(1)} \dots B^{(N)}]$, such that $B \mathbf{v}_\Gamma = 0$ if \mathbf{v}_Γ is continuous. For each \mathcal{E}_{ij} , we define $B_{E_{ij}} = [B_{E_{ij}}^{(i)} B_{E_{ij}}^{(j)}]$ as a submatrix of B , its scaled version $B_{D,E_{ij}} = [B_{D,E_{ij}}^{(i)} B_{D,E_{ij}}^{(j)}]$, and the restricted version of the P_D operator $P_{D_{ij}} = B_{D,E_{ij}}^T B_{E_{ij}}$. We then solve the eigenvalue problem

$$\bar{\Pi}_{ij} \Pi_{ij} P_{D_{ij}}^T S_{ij} P_{D_{ij}} \Pi_{ij} \bar{\Pi}_{ij} \mathbf{x}_m = \mu_m (\bar{\Pi}_{ij} (\Pi_{ij} S_{ij} \Pi_{ij} + \sigma (I - \Pi_{ij})) \bar{\Pi}_{ij} + \sigma (I - \bar{\Pi}_{ij})) \mathbf{x}_m.$$

$\Pi_{ij} : \mathbf{V}_\Gamma^{(i)} \times \mathbf{V}_\Gamma^{(j)} \rightarrow \tilde{\mathbf{V}}_{ij}$ and $\bar{\Pi}_{ij} : \mathbf{V}_\Gamma^{(i)} \times \mathbf{V}_\Gamma^{(j)} \rightarrow \text{Range}((\Pi_{ij} S_{ij} \Pi_{ij} + \sigma (I - \Pi_{ij})))$ are two orthogonal projection, with σ is a suitable positive constant, usually taken as the maximum of the entries of the diagonal S_{ij} (see [13] Section 5 for further details). We then select the \mathbf{x}_m , $m = 1, \dots, k$ for which $\mu_m \geq \text{TOL}$ and we compute the coefficient constraints as c_{ij}^m

$$c_{ij}^m := B_{D,E_{ij}}^T S_{ij} P_{D_{ij}} \mathbf{x}_m.$$

We introduce now the *frugal* coarse space. Since in general the condition number is determined by few large eigenvalue, the idea consists of building constraints for each edge of the interface Γ , without the need to solve an eigenvalue problems. Like in the linear elasticity case, when applying the BDDC method to the Stokes problem in two dimensions, we need three constraints for each edge to control the three (linearized) rigid-body motions (two translations and one rotation). Given two subdomains Ω_l , $l = i, j$ with diameter H_l , we have $\mathbf{r}_1 := [1 \ 0]$, $\mathbf{r}_2 := [0 \ 1]$, $\mathbf{r}_3 := [x_2 - \hat{x}_2 \ -x_1 + \hat{x}_1] / H_l$, where $\hat{\mathbf{x}} \in \Omega_l$ is the center of the rotation. Differently from the approach in [10], we do not rescale the rigid body modes and we define the "approximate" eigenvector

$$v(\mathbf{x})_{\mathcal{E}_{ij}}^{(m,l)} := \begin{cases} r(\mathbf{x})_m^{(l)}, & \mathbf{x} \in \mathcal{E}_{ij}, \\ 0, & \mathbf{x} \in \partial\Omega_l \setminus \mathcal{E}_{ij}, \end{cases} \quad (8)$$

for $m = 1, 2, 3$ and $v(\mathbf{x})_{\mathcal{E}_{ij}}^{(m)T} := [v(\mathbf{x})_{\mathcal{E}_{ij}}^{(m,i)T}, -v(\mathbf{x})_{\mathcal{E}_{ij}}^{(m,j)T}]$.

The three frugal edge constraints are then obtained by $c_{ij} := B_{D,E_{ij}}^T S_{ij} P_{D_{ij}} v(\mathbf{x})_{\mathcal{E}_{ij}}^{(m)T}$.

4 Numerical Results

We solve a lid-driven cavity benchmark problem on the unit square domain $\Omega = [0, 1] \times [0, 1]$, applying Dirichlet boundary conditions on the whole $\partial\Omega$ and using a VEM implementation of degree $k = 2$. The heterogeneity is introduced to physically represent a practical example where drops (or sinkers) of a high viscosity material are spread in the fluid, in particular this is modeled defining $\nu(\mathbf{x})$ as a continuous function that exhibits sharp gradients (Figure 1) [19]. These inclusions of equal size are placed randomly in the unit square domain so that they can overlap and intersect the boundary. For $\mathbf{x} \in \Omega$, the viscosity $\nu(\mathbf{x}) \in [\nu_{min}, \nu_{max}]$, $0 < \nu_{min} < \nu_{max} < \infty$, is defined as $\nu(\mathbf{x}) := (\nu_{max} - \nu_{min})(1 - \chi_n(\mathbf{x})) + \nu_{min}$. Here, $\chi_n(\mathbf{x}) \in C^\infty$ is an indicator function $\chi_n(\mathbf{x}) \in [0, 1]$ that accumulates n sinkers defined as $\chi_n(\mathbf{x}) := \prod_{i=1}^n 1 - \exp(-\delta \max(0, |\mathbf{c}_i - \mathbf{x}| - \frac{\Omega}{2})^2)$, where $\mathbf{c}_i \in \Omega$, $i = 1, \dots, n$ are the centers of the sinkers, $\Omega \geq 0$ is their diameter and $\delta > 0$ a parameter that controls the exponential decay. By choosing $\delta = 2000$, $\Omega = 0.05$, $\lambda_{min} = 10^{-3}$ and $\lambda_{max} = 10^3$, we ensure that the viscosity exhibits sharp gradients. The right hand side is defined as $\mathbf{f}(\mathbf{x}) := (0, \beta(\chi_n(\mathbf{x}) - 1))$, with $\beta = 10$ to simulate gravity that takes down the high viscosity material. In our experiments we use meshes with a Centroid Voronoi Tassellation (CVT) and Random meshes (RND), while the subdomain partitioning is performed by METIS. We compare the two adaptive coarse spaces, with $TOL = 100$, and the frugal one by applying the two different type of scaling mentioned before. Our numerical simulation have been performed with MATLAB R2023A© therefore no CPU time analysis is provided.

In the following tables, we report the number of iterations to solve the global interface saddle-point problem (5) with the PCG method, accelerated by a BDDC preconditioner, where we set the tolerance for the relative residual error to 10^{-6} .

Table 1 Test with nSink = 11 and increasing the number of subdomains. Mesh sizes: 1000 elements for 2×2 subdomains, 5000 elements for 4×4 and 12000 elements for 8×8 .

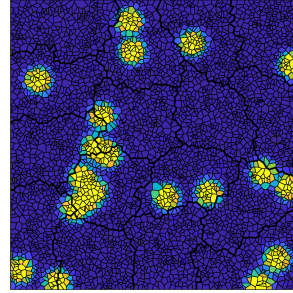
	multiplicity-scaling						deluxe-scaling						
	CVT			RND			CVT			RND			
	n_Π	it	k_2	n_Π	it	k_2	n_Π	it	k_2	n_Π	it	k_2	
<i>frugal</i>	2x2	24	133	1538.25	24	102	927.72	24	9	16.47	24	10	10.26
	4x4	168	62	417.12	168	93	3925.17	168	16	18.20	168	15	38.34
	8x8	842	52	124.72	830	53	204.71	842	16	10.89	830	14	5.52
<i>first</i>	2x2	49	58	106.54	56	48	78.09	11	11	9.92	12	11	10.99
	4x4	96	52	81.48	104	54	90.37	81	18	10.59	84	19	16.49
	8x8	418	54	102.54	424	53	98.16	386	21	25.95	380	21	16.93
<i>second</i>	2x2	47	57	107.74	33	50	94.27	9	11	10.11	9	11	11.00
	4x4	84	52	103.63	89	56	97.76	69	17	10.11	69	20	17.09
	8x8	387	57	97.97	396	56	96.37	361	20	10.14	362	21	19.83

Table 2 Test with increasing number of randomly placed sinkers. Both meshes are made of 4096 elements decomposed into 4×4 subdomains.

	nSink	multiplicity-scaling						deluxe-scaling					
		CVT			RND			CVT			RND		
		n_{Π}	it	k_2	n_{Π}	it	k_2	n_{Π}	it	k_2	n_{Π}	it	k_2
<i>frugal</i>	1	166	32	60.15	168	27	42.41	166	10	2.54	168	10	6.41
	5	166	50	259.22	168	85	920.27	166	12	35.92	168	12	24.34
	10	166	81	1535.32	168	118	1086.15	166	14	41.66	168	13	50.45
	20	166	106	2537.60	168	150	1523.15	166	18	96.97	168	13	59.60
<i>first</i>	1	70	37	85.74	70	28	42.42	70	15	4.97	70	17	4.72
	5	85	46	85.74	96	51	84.01	75	15	5.09	75	15	7.68
	10	99	54	96.28	117	57	98.86	80	16	9.56	78	15	8.32
	20	121	57	88.27	163	66	135.31	89	19	13.14	81	20	10.73
<i>second</i>	1	70	34	85.68	70	30	42.42	71	15	4.82	69	17	4.72
	5	80	46	86.96	93	52	91.34	75	14	5.49	74	15	9.97
	10	89	56	97.06	107	59	98.83	69	17	9.57	69	17	10.54
	20	106	57	94.27	150	64	98.81	73	20	15.79	71	21	10.84

We use the following notation: n_{Sub} = number of subdomains, n_{Sink} = number of sinkers, n_{Π} = number of primal constraints, it = iteration count (CG), k_2 = condition number, *frugal* = frugal coarse space, *first* = first adaptive technique, *second* = second adaptive technique.

We consider two different tests. We first set a configuration with $n_{\text{Sink}} = 11$ and we increase the number of the subdomains; see Table 1. For both the type of the mesh considered the adaptive coarse spaces with the multiplicity scaling respect our expectations, while the frugal approach exhibits a high condition number since the heuristic coarse space is not able to catch all the largest eigenvalues. Introducing the deluxe scaling we see that the number of primal constraints is drastically reduced in the adaptive coarse spaces. The frugal approach is then able to control the largest eigenvalues and performs well. In Table 2 we instead keep fixed the number of subdomains at 4×4 and we increase the number of the inclusions. Again, the adaptive coarse spaces are robust and also when introducing the deluxe scaling the frugal one shows a good improvement presenting itself as a valid alternative.

**Fig. 1** METIS decomposition of a RND mesh. $\nu = 1e3$ on yellow elements, $\nu = 1e - 3$ on blue ones.

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