SPAI Approximations of Inverses of Subdomain Matrices for AOSM

Martin J. Gander[0000−0001−8450−9223], Lahcen Laayouni[0009−0000−5378−2825], and Daniel B. Szyld[0000−0001−8010−0391]

1 Introduction

Algebraic Optimized Schwarz Methods (AOSM) were introduced in [5] to present an algebraic formulation of the Optimized Schwarz Methods (OSM) proposed in [2]. Over the past years, there was substantial research devoted to the better understanding of AOSM and further algebraic improvements: in [10], an algebraic discovery algorithm was developed for AOSM in order to use PDE results in the algebraic setting; see also [6] for transmission conditions based on probing; in [4], SPAI-based transmission conditions were first introduced for AOSM; and in [8] data-sparse transmission conditions. In [3], an alternating procedure based on SPAI was developed in order to algebraically optimize transmission conditions, and in [7] algebraic convergence bounds were developed for the parallel optimized Schwarz method and corresponding algebraic formulation given by Optimized Restricted Additive Schwarz.

A main difficulty in AOSM lies in determining optimized transmission conditions, which involve exact calculations of blocks B_{ij} in the inverses of subdomain matrices, and these calculations are expensive. Furthermore, the optimization process is a nonlinear problem. In [3], we proposed an alternating technique to tackle this issue. However, the factors involved in the alternating algorithm contain the blocks B_{ij}. This motivates the current study, in which we propose SPAI approximations to estimate the blocks B_{ij}. This is a one-time approximation, and we explore two different strategies:

Martin J. Gander
Section de Mathématiques, University of Geneva, Switzerland, e-mail: martin.gander@unige.ch

Lahcen Laayouni
School of Science and Engineering, Al Akhawayn University, Avenue Hassan II, 53000 P.O. Box 1630, Ifrane, Morocco e-mail: L.Laayouni@aui.ma

Daniel B. Szyld
Department of Mathematics, Temple University (038-16), 1805 N. Broad Street, Philadelphia, Pennsylvania 19122-6094, USA, e-mail: szyld@temple.edu

one based on using an adaptive technique, and the other based on using predefined sparsity. Our goal is to reduce the computational cost of calculating the optimized transmission conditions in AOSM using SPAI techniques. These techniques are parallelized, which provides the benefit of further reducing the time required to estimate the blocks B_{ij}. In Section 2 we give a short description of Algebraic Optimized Schwarz Methods. In Section 3 we present SPAI techniques used in approximating specific blocks B_{ij} for the Laplace and advection-reaction-diffusion problems, and we conclude in Section 4.

2 Algebraic Optimized Schwarz Methods (AOSM)

We are interested in solving linear systems $Au = f$ where the $n \times n$ matrix A has a block-banded structure,

$$A = \begin{pmatrix} A_{11} & A_{12} & & \\ A_{21} & A_{22} & A_{23} & \\ & A_{32} & A_{33} & A_{34} \\ & & A_{43} & A_{44} \end{pmatrix}, \tag{1}$$

with A_{ij} being blocks of size $n_i \times n_j$, for $i, j = 1, \ldots, 4$, and $n = \sum_i n_i$. Motivated by the application to discretizations of Partial Differential Equations (PDEs), we assume that $n_1 \gg n_2$ and $n_4 \gg n_3$, highlighting significant size differences between certain block dimensions; for PDEs, the large blocks A_{11} and A_{44} correspond to degrees of freedom within subdomain interiors, and the smaller blocks A_{22} and A_{33} correspond to interface (and possibly) overlap unknowns. We explore AOSMs, which are based on modifications inspired by OSM, e.g., Optimized Restricted Additive Schwarz (ORAS) and Optimized Restricted Multiplicative Schwarz (ORMS), whose error propagation operators are (see [5])

$$T_{ORAS} = I - \sum_{i=1}^{2} \tilde{R}_i^T \tilde{A}_i^{-1} R_i A, \quad \text{and} \quad T_{ORMS} = \prod_{i=2}^{1} (I - \tilde{R}_i^T \tilde{A}_i^{-1} R_i A). \tag{2}$$

Here, R_i and $\tilde{R}_i$ are the classical restriction matrices of RAS [1], i.e., with and without overlap, and the modified subdomain matrices leading to optimized methods [11] are of the form

$$\tilde{A}_1 = \begin{bmatrix} A_{11} & A_{12} & \\ A_{21} & A_{22} & A_{23} \\ & A_{32} & S_1 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} S_2 & A_{23} & \\ A_{32} & A_{33} & A_{34} \\ & A_{43} & A_{44} \end{bmatrix}, \tag{3}$$

with $S_1 = A_{33} + D_1$ and $S_2 = A_{22} + D_2$. Here, D_1 and D_2 are transmission matrices chosen for convergence improvement. It was shown in [5] that the asymptotic convergence factor of AOSM depends on the product of the two norms

$$\left\|(I + D_1 B_{33})^{-1}(D_1 B_{12} - A_{34} B_{13})\right\|, \quad \left\|(I + D_2 B_{11})^{-1}(D_2 B_{32} - A_{21} B_{31})\right\|, \tag{4}$$

where the B blocks involve columns of A_1^{-1} and A_2^{-1},

$$B_1 := \begin{bmatrix} B_{11} \\ B_{12} \\ B_{13} \end{bmatrix} := \begin{bmatrix} A_{22} & A_{23} & \\ A_{32} & A_{33} & A_{34} \\ & A_{43} & A_{44} \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, \; B_3 := \begin{bmatrix} B_{31} \\ B_{32} \\ B_{33} \end{bmatrix} := \begin{bmatrix} A_{11} & A_{12} & \\ A_{21} & A_{22} & A_{23} \\ & A_{32} & A_{33} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}. \tag{5}$$

One can easily derive the optimal blocks which make (4) zero,

$$D_{1,\text{Opt.}} = -A_{34}A_{44}^{-1}A_{43} \quad \text{and} \quad D_{2,\text{Opt.}} = -A_{21}A_{11}^{-1}A_{12}. \tag{6}$$

AOSM then converges in two iterations; thus, one cannot do better than this. However, computing $D_{j,\text{Opt.}}$ is very expensive because of A_{44}^{-1} and A_{11}^{-1}.

In [3], an approach was introduced to minimize the norms in (4) using an alternating technique. To obtain accurate results, exact blocks B_{ij} were calculated, which can be computationally expensive; in fact, the cost is comparable to the cost needed for the optimal blocks in (6), since the large matrices A_{11} and A_{44} are also involved in the inversion process for the matrices B_{ij}. Here, we therefore propose a new method based on SPAI approximations of these blocks.

3 SPAI approximations for AOSM

We present the new adaptive strategy for approximating the B_{ij} blocks first for the Poisson model problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega = (-1,1) \times (-1,1), \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

discretized using a standard five-point finite difference discretization and equal mesh sizes leading to $n_1 = 480$, $n_2 = 32$, $n_3 = 32$, $n_4 = 480$ such that $n = N^2 = n_1+n_2+n_3+n_4$. By linearity we consider $f \equiv 0$. In Figure 1 (top left), we show the 2-norm of the error measured as a function of the number of iterations of the algorithms Classical Schwarz, OO0 and OO2 which are PDE based optimized Schwarz methods from [2], and Alternating SPAI(1), Alternating SPAI(3), and Alternating SPAI(5) from [3] when the B_{ij} are calculated exactly. In our new study here, we are using Alternating SPAI(k) with $k = 1, 3$, and 5, where we approximate the blocks B_{ij} using also a SPAI technique, instead of using the exact B_{ij} as in [3]. We see that the optimized methods converge much faster than the classical Schwarz method. On the right and in the bottom row in Figure 1, we present the corresponding results when using SPAI approximations for the blocks B_{ij}. We use an adaptive strategy based on the approach described in [9]. The SPAI iterative process stops when a maximum fill-in is reached: 10, 50, or 500 entries per column. We see that to obtain an accurate estimation and an optimized Schwarz behavior, we need to consider using a large number of fill-ins, which is rather expensive, even though the SPAI computations are embarrassingly parallel. We therefore also consider SPAI

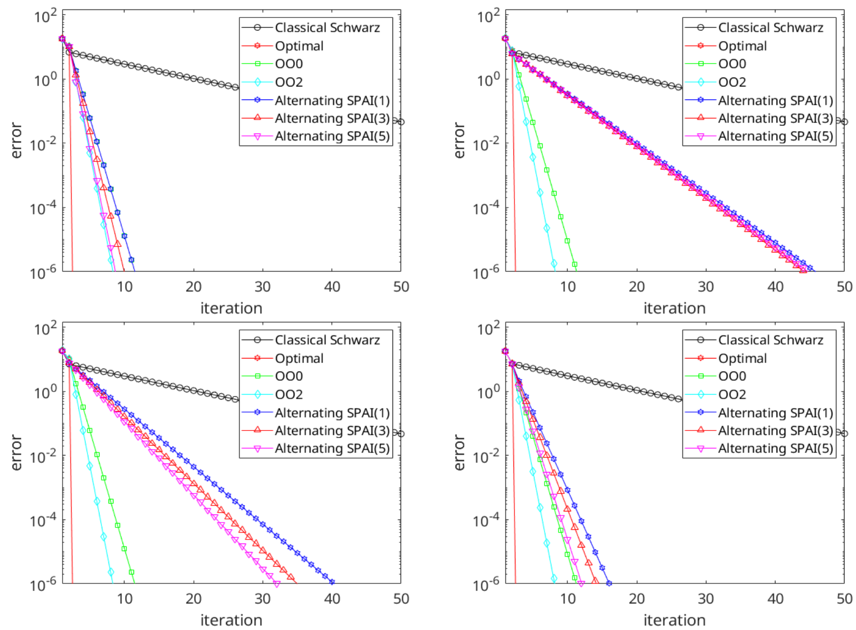

Fig. 1: Top left to bottom right: Number of iterations for all algorithms for the Laplace problem with exact B_{ij}, and then using adaptive SPAI approximations with 10, 50 and 500 fill-in.

approximations based on a predefined sparsity pattern inspired by the structure of the powers of the matrices whose inverse has to be approximated. We show in Figure 2 the results corresponding to the experiments in Figure 1 but now using as fill-in pattern in the alternating SPAI algorithm the non-zero entries in A, A^2, A^4 and A^8, where A denotes the sparse matrix that has to be inverted approximately. We see that increasing the sparsity pattern through the powers of the underlying matrix leads to better and better performance, approaching the performance of the alternating SPAI algorithm using the exact expressions for the matrices B_{ij} in the construction process shown in Figure 1 on the top left.

We next test our new approach on the advection-reaction-diffusion equation,

$$\eta u - \nabla \cdot (a \nabla u) + b \cdot \nabla u = f \quad \text{on } \Omega = (0,1) \times (0,1), \tag{8}$$

where $a = a(x,y) > 0$, $b = [b_1(x,y), b_2(x,y)]^T$, $\eta = \eta(x,y) \geq 0$, with

$$b_1 = y - \frac{1}{2}, \quad b_2 = -x + \frac{1}{2}, \quad \eta = x^2 \cos(x+y)^2, \quad a = 1 + (x+y)^2 e^{x-y}.$$

The right-hand side function $f = 0$, and we use zero Dirichlet boundary conditions. We use again a standard five-point centered finite difference discretization, leading to a matrix A of size 1024×1024, and decompose it according to the subdomain

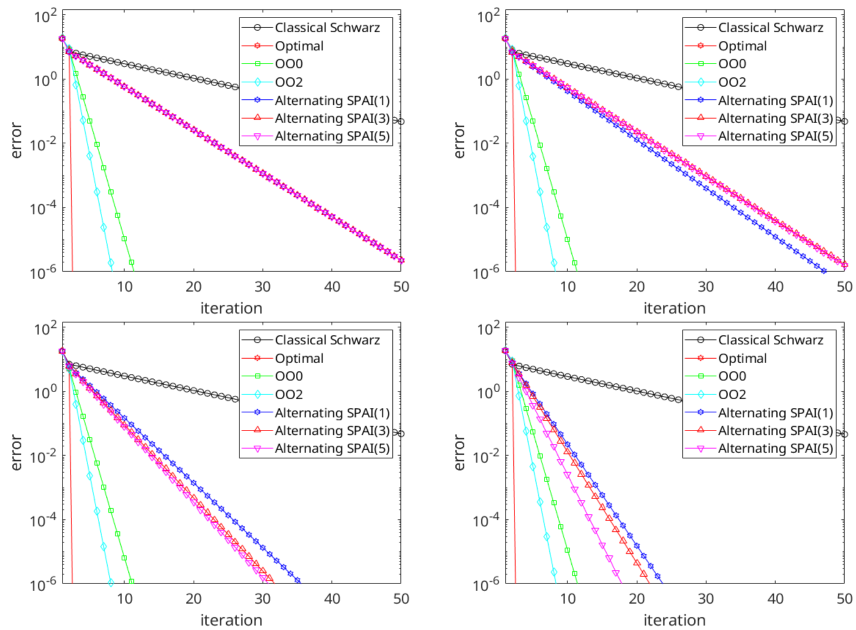

Fig. 2: Top left to bottom right: Number of iterations for all algorithms for the Laplace problem with SPAI approximations for B_{ij} with power $p = 1, 2, 4, 8$.

decomposition in (1) with $n_1 = 480$, $n_2 = 32$, $n_3 = 32$, and $n_4 = 480$. We show in Figures 3 and 4 the results corresponding to Figures 1 and 2 but now for the advection-reaction-diffusion case. We see that our new algebraic approach works very well for this substantially harder problem.

Both the adaptive strategy and the predefined strategy based on matrix powers performed well for the Laplacian and advection-reaction-diffusion problems. However, the latter is a promising technique because determining the graph of the powers of the matrices is computationally cheap.

If we approximate $D_{1,\text{Opt.}}$ and $D_{2,\text{Opt.}}$ in (6) directly by applying the SPAI techniques to A_{11}^{-1} and A_{44}^{-1} contained in them in the same way as we approximated B_{ij}, we obtain the results shown in Figure 5. We see that this also leads to quite good performance.

Note that all our SPAI approximations are embarrassingly parallel. In Figure 6, we present a comparison of the different methods with and without parallelization, where we used the `parfor` function in MATLAB with 4 workers. Once we have obtained all the required blocks B_{ij}, D_1, and D_2, the time required to iterate through the different optimized Schwarz methods is quite similar, as shown in Table 1. However, alternating SPAI methods are still promising due to their potential for further parallelization.

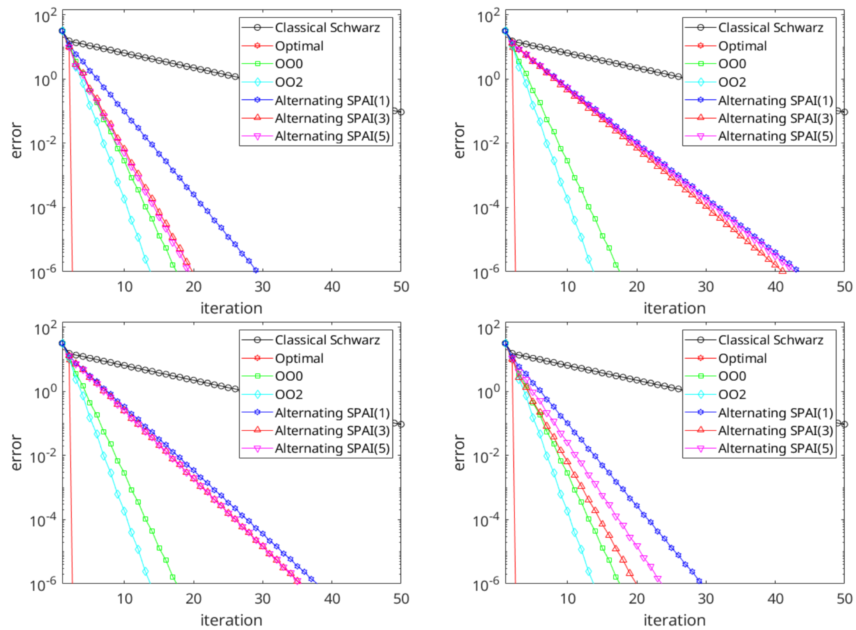

Fig. 3: Top left to bottom right: Number of iterations for all algorithms for the Advection Reaction Diffusion problem with exact B_{ij}, and then using SPAI with 10, 50 and 500 fill in.

N	OO0	OO2	Alternating SPAI(1)	Alternating SPAI(3)	Alternating SPAI(5)
2^5	8.1587×10^{-1}	9.5261×10^{-2}	8.1595×10^{-2}	8.8271×10^{-2}	9.3634×10^{-2}
2^6	9.3092×10^{0}	1.7168×10^{0}	1.4914×10^{0}	1.5024×10^{0}	1.5477×10^{0}
2^7	1.4336×10^{2}	2.8494×10^{1}	4.5415×10^{1}	4.5472×10^{1}	4.5017×10^{1}

Table 1: Timing results for the various algorithms as a function of N.

4 Concluding remarks

We explored various approximations using SPAI techniques in order to derive AOSMs. We tested them both on a Laplace problem and on a harder advection-reaction-diffusion problem, and the resulting methods performed well, and benefit further from the embarrassingly parallel nature of SPAI.

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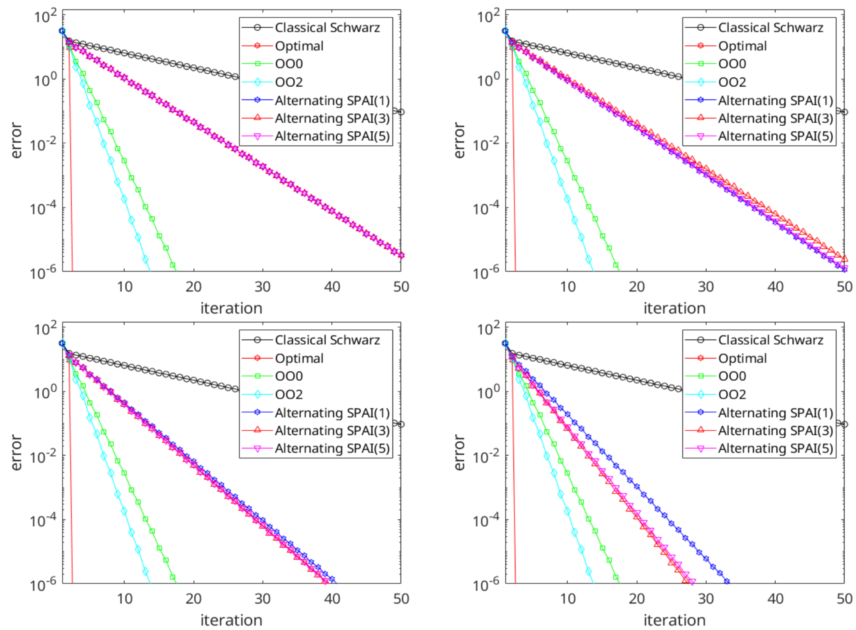

Fig. 4: Top left to bottom right: Number of iterations for all algorithms for the advection-reaction-diffusion problem with SPAI approximations for B_{ij} with power $p = 1, 2, 4, 8$.

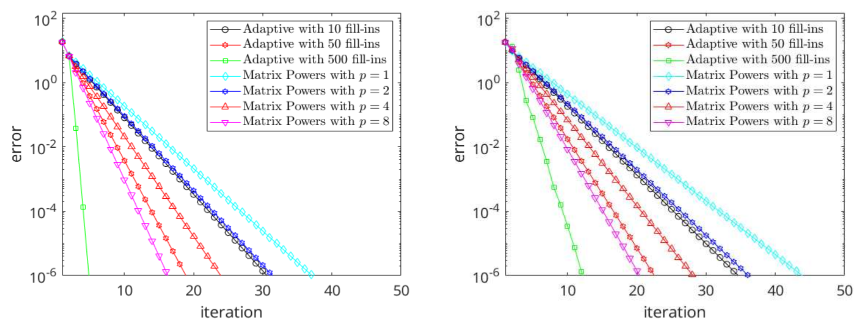

Fig. 5: Number of iterations when approximating $D_{1,\text{Opt.}}$ and $D_{2,\text{Opt.}}$ from (6) directly using the SPAI techniques. Left: Laplace problem. Right: advection-reaction-diffusion problem.

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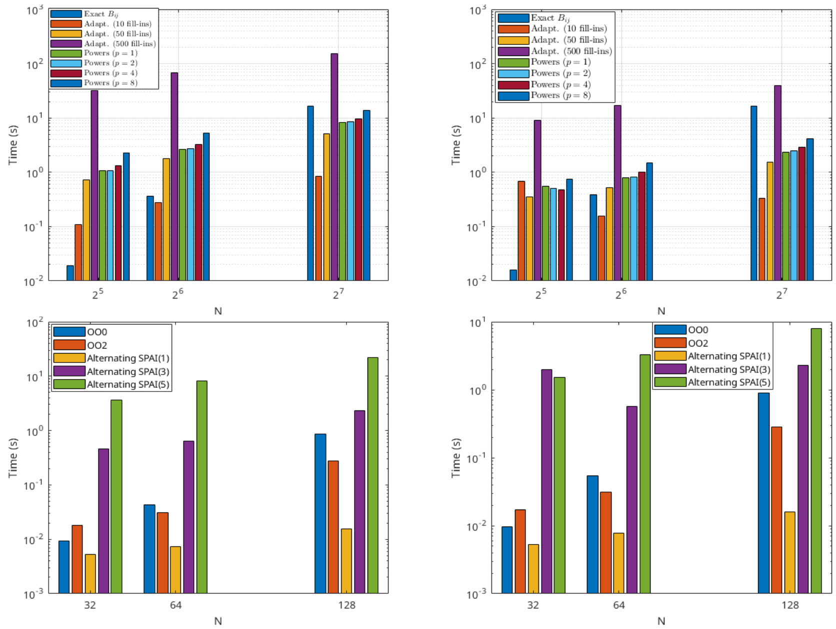

Fig. 6: Time in seconds for different N to estimate the blocks B_{ij} (top row) and D_1 and D_2 (bottom row) for different methods for Laplacian problem without (left) and with parallelization (right).

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