Wavenumber Explicit Estimates of the Schwarz Preconditioner with Δ-GenEO Coarse Space for the Indefinite Helmholtz Problem

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1 Introduction

In this work we focus on the indefinite version of Helmholtz boundary value problems with highly variable coefficients defined on $\Omega \subset \mathbb{R}^d$, given by:

$$\begin{aligned} -\operatorname{div}(A\nabla u) - k^2 u = f \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega. \end{aligned} \tag{1}$$

We use the same setting as in [1], namely we work on a computational domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) which is supposed to be polygonal or Lipschitz polyhedral. The matrix A is positive definite, and we assume that (1) has a unique weak solution $u \in H^1(\Omega)$ for all $f \in L^2(\Omega)$. The global domain Ω is covered by a set of overlapping subdomains Ω_i, $i = 1, \ldots, N$, and the classical one-level additive Schwarz preconditioner is built from partial solutions on each subdomain. Since this preconditioner is, in general, not scalable as the number of subdomains grows, an additional global coarse solve is usually added to enhance scalability, as well as robustness with respect to coefficient heterogeneity or the increase in the wavenumber. As in [1] we use local generalised eigenvalue problems that are obtained by a shift of the left-hand side of (1). After discretisation by finite elements, the linear systems arising from (1) are symmetric but indefinite. For this reason, we use GMRES as the iterative solver, and the convergence analysis relies on the 'Elman theory' [4], which requires

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an upper bound for the norm of the preconditioned matrix and a lower bound on the distance of its field of values from the origin. As a result, the number of GMRES iterations to achieve a given error tolerance is a function of these two bounds.

Our main theoretical result provides rigorous and k-explicit upper bounds on the coarse mesh diameter H and on the ‘eigenvalue tolerance’ τ (for the local generalised eigenvalue problems) which ensure that GMRES enjoys robust and mesh-independent convergence when applied to the preconditioned problem. This rigorous explicit bound for (1) follows the methodology from [1] and provides an improvement of the k-dependent estimates. As a reminder, GenEO coarse spaces are usually based on the m_i (dominant) eigenfunctions corresponding to the smallest eigenvalues $\lambda_1^i \le \lambda_2^i \le \cdots \le \lambda_{m_i}^i$ of the generalised eigenvalue problem on the subdomain Ω_i. To obtain a robust rate of convergence for GMRES that depends only on Λ (the maximum number of times any point is overlapped by the subdomains Ω_i), we need some conditions on H and the eigenvalue tolerance $\tau := \min_{i=1}^N \lambda_{m_i+1}^i$. In the estimates $C_{\mathsf{stab}} > 0$ the stability constant for the problem (1) also appears. The hidden constants usually depend only on Λ.

More explicitly, in [1, Theorem 4.1] it is proved that a robust rate is achieved if

$$H \lesssim k^{-2} \qquad \text{and} \qquad (1 + C_{\mathsf{stab}})^2\, k^8 \lesssim \tau. \tag{2}$$

In the current paper, we show that the bounds (2) can be improved:

$$H \lesssim k^{-1} \qquad \text{and} \qquad (1 + C_{\mathsf{stab}})^2 k^4 \lesssim \tau. \tag{3}$$

Although, this is a first improvement, these bounds remain quite pessimistic, showing a strong dependence on the wavenumber k.

Remark 1 This work is only concerned with applying the Δ-GenEO coarse space to problem (1), which is equivalent to [1, Example 1.1] where we have $k^2 = \kappa$. Note that [1] deals with more general problems.

2 Problem formulation and discretisation

The weak formulation of (1) is to find $u \in H_0^1(\Omega)$ such that

$$b(u, v) := \int_\Omega (A\nabla u \cdot \nabla v - k^2 uv)\,\mathrm{d}x = (f, v) \qquad \text{for all } v \in H_0^1(\Omega), \tag{4}$$

where $f \in L^2(\Omega)$. We shall be making use of the positive definite bilinear form $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$,

$$a(u, v) = \int_\Omega A\nabla u \cdot \nabla v\,\mathrm{d}x.$$

If a and b are defined on a subdomain Ω' of Ω we employ the notations $a_{\Omega'}$ and $b_{\Omega'}$ and extend the definition to $H^1(\Omega')$. For any subdomain $\Omega' \subset \Omega$, we use $(\cdot,\cdot)_{\Omega'}$ to denote the $L^2(\Omega')$ inner product, with the norm denoted by $\|\cdot\|_{\Omega'}$. When $\Omega' = \Omega$, the inner product is written as $(\cdot,\cdot)$ with norm $\|\cdot\|$. The norm induced by the positive bilinear form a is denoted by $\|u\|_a = \sqrt{a(u,u)}$. We see that $b(u,v) = a(u,v) - k^2(u,v)$.

The following assumptions are used throughout this work.

Assumption 1 (Weak regularity)**.** *The coefficient A in problem* (1) *satisfies the following conditions.*

(i) $A : \Omega \to \mathbb{R}^{d\times d}$ *is symmetric and there exists $a_{\mathsf{max}} \geq 1$ s.t.*

$$|\boldsymbol{\xi}|^2 \leq A(x)\boldsymbol{\xi}\cdot\boldsymbol{\xi} \leq a_{\mathsf{max}}|\boldsymbol{\xi}|^2 \qquad \text{for all } x \in \Omega,\ \boldsymbol{\xi} \in \mathbb{R}^d. \tag{5}$$

The diameter, D_Ω, of the domain Ω satisfies $D_\Omega \leq 1$.

(ii) *For any $f \in L^2(\Omega)$, problem* (4) *has a unique solution $u \in H_0^1(\Omega)$ and there exists a constant $C_{\mathsf{stab}} > 0$ such that*

$$\|u\|_a \leq C_{\mathsf{stab}}\|f\| \qquad \text{for all } f \in L^2(\Omega). \tag{6}$$

Let $\mathcal{T}_h$ be any shape-regular triangular mesh over the domain Ω, and let $V^h \subset H_0^1(\Omega)$ be any conforming finite element space. The Galerkin approximation of (4) is to

$$\text{find } \ u_h \in V^h \ \text{ s.t. } \quad b(u_h, v) = (f, v) \qquad \text{for all } v \in V^h. \tag{7}$$

Let n be the dimension of V^h, and let a basis be given by $\{\phi_i\}_{i=1}^n$; then (7) can be represented by the linear system

$$\mathbf{B}\mathbf{u} = \mathbf{f}. \tag{8}$$

The matrix $\mathbf{B}$ is defined in terms of the basis functions as $(\mathbf{B})_{ij} = b(\phi_j, \phi_i)$, and $(\mathbf{f})_i = (f, \phi_i)$. It is also possible to define the matrix $\mathbf{A}$ using the same basis functions as $(\mathbf{A})_{ij} = a(\phi_j, \phi_i)$. The solvability of (7) is assured by the Schatz and Wang lemma from [5, Theorem 2]. This is required due to the indefiniteness of (7).

Lemma 1 (Schatz and Wang, 1996) *Let Assumption 1 hold. Then there exists an $h_0 > 0$ such that, for each $h \in (0, h_1)$, problem* (7) *has a unique solution $u_h \in V^h$. Moreover, let u be the unique solution of* (4)*. Then, for every $\varepsilon > 0$ there exists $h_1 = h_1(\varepsilon) > 0$ such that, for every $h \in (0, h_1)$,*

$$\|u - u_h\| \leq \varepsilon\|u - u_h\|_{H^1(\Omega)} \qquad \textit{and} \qquad \|u - u_h\|_{H^1(\Omega)} \leq \varepsilon\|f\|. \tag{9}$$

3 Domain decomposition: the Δ-GenEO coarse space

In order to construct the two-level Schwarz preconditioner, the first-level preconditioner needs to be formulated. This is achieved by first partitioning the global domain, Ω, into a set of N subdomains, $\{\Omega_i'\}_{i=1}^N$. It is being assumed that the global mesh, $\mathcal{T}_h$, is sufficiently fine to resolve the subdomains. Then each subdomain, Ω', is expanded by one or more layers of mesh elements, creating the desired overlapping set of N subdomains, $\{\Omega_i\}_{i=1}^N$. For the domains $i = 1, \dots, N$, it is possible to define the spaces $\widetilde{V}_i = \{v|_{\Omega_i} : v \in V^h\} \subset H^1(\Omega_i)$ and $V_i = \{v \in \widetilde{V}_i : v|_{\partial\Omega_i} = 0\} \subset H_0^1(\Omega_i)$.

For each subdomain, Ω_i, we denote its diameter by H_i, and we set $H := \max\{H_i\}$. For any $v_i \in V_i$, let $E_i v_i$ denote its zero extension to the whole of the domain Ω,

$$E_i : V_i \to V^h, \quad i = 1, \dots, N. \tag{10}$$

The $L^2(\Omega)$ adjoint of the E_i is called restriction operator, $R_i : V^h \to V_i$.

In order to achieve robustness when using a domain decomposition preconditioner, a coarse space $V_0 \subset V^h$ is needed. This improves the global exchange of information between the subdomains. Let $E_0 : V_0 \to V^h$ be the natural embedding, and let R_0 be the L^2 adjoint of E_0, i.e. $(v_0, R_0 w) = (E_0 v_0, w)$ for all $w \in V^h$ and $v_0 \in V_0$. The two-level additive Schwarz preconditioner can now be given in matrix form as,

$$\mathbf{M}_{AS,2}^{-1} = \sum_{i=0}^{N} \mathbf{E}_i \mathbf{B}_i^{-1} \mathbf{R}_i, \qquad \text{where } \mathbf{B}_i = \mathbf{R}_i \mathbf{B} \mathbf{E}_i; \tag{11}$$

here, $\mathbf{E}_i$ and $\mathbf{R}_i$ denote the matrix representations of E_i and R_i with respect to the basis functions $\{\phi_i\}_{i=1}^n$ and some basis in V_i. The preconditioned linear system from (8) reads as,

$$\mathbf{M}_{AS,2}^{-1} \mathbf{B} \mathbf{u} = \mathbf{M}_{AS,2}^{-1} \mathbf{f} \tag{12}$$

It is now possible to define the projectors used in the analysis. For each $i = 0, \dots, N$, the projectors, $T_i : V^h \to V_i$, are defined by

$$b_{\Omega_i}(T_i u, v) = b(u, E_i v) \qquad \text{for all } v \in V_i,\ u \in V^h, \tag{13}$$

where $\Omega_0 = \Omega$. Given the operators T_i, the operator $T : V^h \to V^h$ is defined as

$$T := \sum_{i=0}^{N} E_i T_i. \tag{14}$$

The two-level additive Schwarz preconditioner can be represented in terms of the projector operator, T, as follows. We recall here the result from [1].

Proposition 1 *For any $u, v \in V^h$, with corresponding nodal vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,*

$$\langle \mathbf{M}_{AS,2}^{-1} \mathbf{B} \mathbf{u}, \mathbf{v} \rangle_{\mathbf{A}} = a(Tu, v), \tag{15}$$

where $\langle \cdot, \cdot \rangle_{\mathbf{A}}$ is the inner product on $\mathbb{R}^n$, using the matrix $\mathbf{A}$.

In order to use the Δ-GenEO coarse space, we recall some definitions from [6].

Definition 1 (Degrees of freedom) Given a subdomain Ω_i, let $\overline{\mathrm{dof}}(\Omega_i)$ be the set of all degrees of freedom active in Ω_i (including the boundary) and $\mathrm{dof}(\Omega_i)$ be the set of internal degrees of freedom in Ω_i.

Definition 2 (Partition of unity) For any degree of freedom, $j \in \{1, \ldots, n\}$, let μ_j denote the number of subdomains for which j is an internal degree of freedom, i.e. $\mu_j := \#\big\{i : 1 \leq i \leq N,\ j \in \mathrm{dof}(\Omega_i)\big\}$. Then, for $i \in \{1, \ldots, N\}$, the local partition of unity operator, $\Xi_i : \widetilde{V}_i \rightarrow V_i$ is defined by

$$\Xi_i(v) := \sum_{j \in \mathrm{dof}(\Omega_i)} \frac{1}{\mu_j} v_j \phi_j^i \qquad \text{for } \ v = \sum_{j \in \overline{\mathrm{dof}}(\Omega_i)} v_j \phi_j^i \in \widetilde{V}_i.$$

Definition 3 (Local generalised eigenvalue problem and Δ-GenEO coarse space) For each $i = 1, \ldots, N$, we define the following generalised eigenvalue problem. Find $(p_\ell^i, \lambda_\ell^i) \in \widetilde{V}_j \setminus \{0\} \times \mathbb{R}$ such that

$$a_{\Omega_i}(p_\ell^i, v) = \lambda_\ell^i a_{\Omega_i}\big(\Xi_i(p_\ell^i), \Xi_i(v)\big) \qquad \text{for all } v \in \widetilde{V}_i.$$

For each $i \in \{1, \ldots, N\}$, let $\{p_\ell^i\}_{\ell=1}^{m_i}$ be the eigenfunctions corresponding to the m_i smallest eigenvalues, where m_i is to be chosen later such that $\lambda_{m_i+1}^i > 0$. The coarse space, V_0, is given by

$$V_0 := \mathrm{span}\big\{E_i \Xi_i(p_\ell^i) : l = 1, \ldots, m_i \text{ and } i = 1, \ldots, N\big\}. \tag{16}$$

Let also $\Lambda := \max_{T \in \mathcal{T}_h}\big(\#\{\Omega_i : 1 \leq i \leq N, T \subset \Omega_j\}\big)$ be the overlapping constant and $\tau := \min_{i=1}^N \lambda_{m_i+1}^i$ the eigenvalue threshold.

4 Statement and proof of the main result

In this section we give the main result and provide a few technical lemmas needed in the proof of this result, which will be given later on. For convenience we set $\Theta := \frac{1}{\tau}$.

Theorem 1 (GMRES convergence of the two-level preconditioned system) *There exists $h_1 > 0$ such that the following statements hold. Assume that $h \in (0, h_1)$, then select H and τ such that*

$$s := (1 + \Lambda^2\Theta) 16\sqrt{2} k^2 \Lambda^{\frac{3}{2}} \Theta^{\frac{1}{2}} (1 + C_{\mathsf{stab}}) < 1, \qquad t := 32 H^2 k^2 (1 + \Lambda^2\Theta)\Lambda < 1. \tag{17}$$

If GMRES is applied in the $\langle \cdot, \cdot \rangle_{\mathbf{A}}$-inner product to solve the preconditioned system given by (12)*, then after m iterations, the norm of the residual, $\mathbf{r}^{(m)}$, is bounded as*

$$\|\mathbf{r}^{(m)}\|_{\mathbf{A}}^2 \leq \left(1 - \frac{c_1^2}{c_2^2}\right)^m \|\mathbf{r}^{(0)}\|_{\mathbf{A}}^2, \tag{18}$$

where c_1 and c_2 are given by

$$c_1 := \big(4(1+\Lambda^2\Theta)\big)^{-1}\big(1-\max\{t,s\}\big), \qquad c_2 := 18+18\Lambda^2. \tag{19}$$

Corollary 1 *If* (17) *in Theorem 1 is satisfied, then this leads to the conditions*

$$H \lesssim k^{-1} \quad \textit{and} \quad (1+C_{\mathsf{stab}})^2 k^4 \lesssim \tau, \tag{20}$$

and c_1 and c_2 are both independent on problem parameters including the heterogeneity and the wavenumber k, leading to a robust GMRES convergence.

Remark 2 In practice, (20) will introduce a constraint in the size of the subdomains, depending on k and on the number of modes to be added in the coarse space. Note that condition (17) is different from the one obtained in [1], and allows for the improved conditions on H and τ.

The main theorem relies on a few technical results.

Lemma 2 (T_i is well defined and stable for $i = 1, \ldots, N$) *Suppose that $Hk \le 1$ and let $i \in \{1, \ldots, N\}$. Then T_i is well defined and, for all $u \in V^h$,*

$$\|T_i u\|_{a_{\Omega_i}} \le 2\|u\|_{a_{\Omega_i}} + k\sqrt{2}\|u\|_{\Omega_i}. \tag{21}$$

A detailed proof is found in [3, Lemmas 3.8 and 3.10]. This is a reformulation of [1, Lemmas 3.4 and 3.6] when applied to problem (1).

This lemma is necessary even if there are no improvements of the estimates at this stage. However, we will see that the sufficient conditions for solvability and stability of the coarse space operator T_0 from Lemma 3 is an improvement of their counterparts from [1]. These will be the key results having an influence in the dependence on k.

First, to ensure that T_0 is stable and well defined, a condition on Θ is required.

Lemma 3 (T_0 is stable and well defined) *Suppose that*

$$2k^2\Lambda\Theta^{\frac{1}{2}}(1+C_{\mathsf{stab}}) \le \frac{1}{2}. \tag{22}$$

Then T_0 is well defined and there exists $h_1 > 0$ such that, for $h \in (0, h_1)$,

$$\|T_0u - u\| \le 2\Lambda\Theta^{\frac{1}{2}}(1+C_{\mathsf{stab}})\|T_0u-u\|_a \qquad \textit{for all } u \in V^h \tag{23}$$

and

$$\|T_0u-u\| \le \sqrt{2}\|u\|_a \qquad \textit{for all } u \in V^h. \tag{24}$$

Proof. Under condition (22), the existence of $h_1 > 0$ such that the operator T_0 is well defined is ensured by [3, Lemma 3.11]. Consider the auxiliary problem:

$$\text{find } \; w_h \in V^h \; \text{ such that } \qquad b(w_h, v) = (T_0u - u, v) \quad \text{for all } v \in V^h. \tag{25}$$

We have $b(T_0u - u, z) = 0$ for all $z \in V_0$. Let $v = T_0u - u$ in (25), which yields

$$
\begin{aligned}
\|T_0u - u\|^2 = b(w_h - z, T_0u - u) &\le \|w_h - z\|_a \|T_0u - u\|_a + k^2\|w_h - z\| \, \|T_0u - u\| \\
&\le \|w_h - z\|_a \|T_0u - u\|_a + k^2\varepsilon\|w_h - z\|_{H^1(\Omega)} \|T_0u - u\| \\
&\le \|w_h - z\|_a \|T_0u - u\|_a + k^2\varepsilon^2\|T_0u - u\|^2,
\end{aligned}
$$

where we used Lemma 1 for the last two inequalities. Choosing $\varepsilon > 0$ such that $k^2\varepsilon^2 \le \frac{1}{2}$ and using [3, Lemma 3.5] we obtain

$$
\|T_0u - u\|^2 \le 2\Lambda\Theta^{\frac{1}{2}} \|T_0u - u\|_a \|w_h\|_a. \tag{26}
$$

If ε is small enough so that $a_{\mathsf{max}}\varepsilon \le 1$, we can combine (6) and the second inequality in (9) to get $\|w_h\|_a \le (1 + C_{\mathsf{stab}})\|T_0u - u\|$; together with (26) this proves (23).

We now come to the proof of (24). In this part we need the projection operator $P_0 : V^h \to V_0$, which is defined by $a(P_0u, v) = a(u, v)$ for all $v \in V_0$ and $u \in V^h$. From this definition and the definition of T_0 we have $a(T_0u, P_0u - u) = 0$ and $b(u - T_0u, P_0u - T_0u) = 0$. When combined with the link between the bilinear forms a and b, and the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
\|u - T_0u\|_a^2 &= a(u, u - P_0u) + k^2(u - T_0u, T_0u - P_0u) && (27) \\
&\le \|u\|_a \|u - P_0u\|_a + k^2\|u - T_0u\| \, \|P_0u - T_0u\|. && (28)
\end{aligned}
$$

From assumption 1 (i), the Friedrichs inequality and the projection property of P_0 we get

$$
\|u - P_0u\|_a \le \|u\|_a, \quad \|P_0u - T_0u\| \le \|P_0u - T_0u\|_a = \|P_0(u - T_0u)\|_a \le \|u - T_0u\|_a.
$$

These relations, together with (28) and (23), lead to

$$
\|u - T_0u\|_a^2 \le \|u\|_a^2 + k^2 2\Lambda\Theta^{\frac{1}{2}} (1 + C_{\mathsf{stab}}) \, \|T_0u - u\|_a^2 \, . \tag{29}
$$

By rearranging (29) and using assumption (22) we arrive at (24). □

Before we can prove Theorem 1, we need to state the following lemma.

Lemma 4 *Suppose that the assumptions in Theorem 1 are satisfied. Then, for all* $u \in V^h$,

$$
(Tu, u)_a \ge c_1\|u\|_a^2 \qquad \textit{and} \qquad \|Tu\|_a \le c_2\|u\|_a^2, \tag{30}
$$

where c_1 *and* c_2 *are as in Theorem 1.*

The details of this proof can be found in [3, Lemma 4.1]. This is a reformulation of [1, Lemma 4.1] for application to (1). It is here that we take advantage of the improvements to the estimates in Lemma 3.

Proof of Theorem 1. Using the first relation in (30) with (15) we obtain

$$
c_1\|\mathbf{u}\|_{\mathbf{A}}^2 = c_1\|u\|_a^2 \le a(Tu, u) = \langle \mathbf{M}_{AS,2}^{-1}\mathbf{B}\mathbf{u}, \mathbf{u} \rangle_{\mathbf{A}}.
$$

In a similar way we use the second relation in (30), again with (15), to get

$$\|\mathbf{M}_{AS,2}^{-1}\mathbf{B}\mathbf{u}\|_{\mathbf{A}}^2 = \|Tu\|_a^2 \leq c_2\|u\|_a^2 = c_2\|\mathbf{u}\|_{\mathbf{A}}^2,$$

which implies $\|\mathbf{M}_{AS,2}^{-1}\mathbf{B}\|_{\mathbf{A}}^2 \leq c_2$, and the result follows from Elman theory [4]. □

5 Conclusion

In this work, we have developed and analysed the Δ-GenEO coarse space in the context of the additive Schwarz preconditioner for solving the indefinite Helmholtz problem with highly variable coefficients. Through our analysis, we have derived k-explicit upper bounds for the coarse mesh diameter H and the eigenvalue tolerance τ. Specifically, our results indicate that, for the preconditioned problem to achieve robust convergence of GMRES, it is sufficient to ensure that (3) is true. These bounds improve upon the previously established results from [1]. While these improved estimates represent a significant advancement, they still exhibit a strong dependence on the wavenumber k. This dependence highlights the inherent difficulty in preconditioning indefinite problems at high frequencies.

Future work will aim to refine these bounds and explore alternative coarse space constructions to reduce the wavenumber dependence further. One such coarse space being explored in [2], uses the full indefinite Helmholtz problem as part of the generalised eigenvalue problem.

Acknowledgements M.F. acknowledges the support of EPSRC through a studentship.

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